*Electronic Journal of Qualitative Theory of Differential Equations*
2010, No. **32**, 1-17; **http://www.math.u-szeged.hu/ejqtde/**

# Iterated Order of Solutions of Linear Differential Equations with Entire Coefficients

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**Abstract.** In this paper, we investigate the iterated order of solutions of higher order homogeneous linear differential equations with entire coefficients. We improve and extend some results of Belaïdi and Hamouda by using the concept of the iterated order. We also consider nonhomogeneous linear differential equations.

2010 *Mathematics Subject Classification*: 34M10, 34M05, 30D35.
*Key words*: Differential equations, Meromorphic function, Iterated order.

## 1 Introduction and main results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [13]). In addition, we use the notations $\sigma(f)$ to denote the order of growth of a meromorphic function $f(z)$.

We define the linear measure of a set $E \subset [0, +\infty)$ by $m(E) = \int_0^{+\infty} \chi_E(t)dt$ and the logarithmic measure of a set $H \subset [1, +\infty)$ by $lm(H) = \int_1^{+\infty} \frac{\chi_H(t)}{t}dt$, where $\chi_F$ denote the characteristic function of a set $F$.

For the definition of the iterated order of a meromorphic function, we use the same definition as in $[14]$, $[5,$ p. $317]$, $[15,$ p. $129]$. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all $r$

sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

**Definition 1.1** Let $p \geq 1$ be an integer. Then the iterated $p-$order $\sigma_p(f)$ of a meromorphic function $f(z)$ is defined by

$$\sigma_p(f) = \limsup_{r \to +\infty} \frac{\log_p T(r,f)}{\log r}, \tag{1.1}$$

where $T(r,f)$ is the characteristic function of Nevanlinna. For $p = 1$, this notation is called order and for $p = 2$, hyper-order.

**Remark 1.1** The iterated $p-$order $\sigma_p(f)$ of an entire function $f(z)$ is defined by

$$\sigma_p(f) = \limsup_{r \to +\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r \to +\infty} \frac{\log_{p+1} M(r,f)}{\log r}, \tag{1.2}$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

**Definition 1.2** The finiteness degree of the order of a meromorphic function $f$ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is rational}, \\ \min\{j \in \mathbb{N} : \sigma_j(f) < \infty\}, & \text{if } f \text{ is transcendental} \\ & \text{with } \sigma_j(f) < \infty \text{ for some } j \in \mathbb{N}, \\ \infty, & \text{if } \sigma_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases} \tag{1.3}$$

**Definition 1.3** The iterated convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r \to +\infty} \frac{\log_p N(r, 1/f)}{\log r} \quad (p \geq 1 \text{ is an integer}), \tag{1.4}$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| < r\}$.

Similarly, the iterated convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r \to +\infty} \frac{\log_p \overline{N}(r, 1/f)}{\log r} \quad (p \geq 1 \text{ is an integer}), \tag{1.5}$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z:|z|<r\}$.

**Definition 1.4** The finiteness degree of the iterated convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$i_\lambda(f)=\begin{cases}0, & \text{if } n(r,1/f)=O(\log r),\\ \min\{j\in\mathbb{N}:\lambda_j(f)<\infty\}, & \text{if } \lambda_j(f)<\infty \text{ for some } j\in\mathbb{N},\\ \infty, & \text{if } \lambda_j(f)=\infty \text{ for all } j\in\mathbb{N}.\end{cases} \tag{1.6}$$

**Remark 1.2** Similarly, we can define the finiteness degree $i_{\overline{\lambda}}(f)$ of $\overline{\lambda}_p(f)$.

Let $n\geq 2$ be an integer and let $A_0(z),...,A_{n-1}(z)$ with $A_0(z)\not\equiv 0$ be entire functions. It is well-known that if some of the coefficients of the linear differential equation

$$f^{(n)}+A_{n-1}(z)f^{(n-1)}+...+A_1(z)f'+A_0(z)f=0 \tag{1.7}$$

are transcendental, then the equation (1.7) has at least one solution of infinite order. Thus, the question which arises is : What conditions on $A_0(z),...,A_{n-1}(z)$ will guarantee that every solution $f\not\equiv 0$ of (1.7) has an infinite order?

For the above question, there are many results for the second and higher order linear differential equations (see for example [2], [3], [4], [8], [11], [14], [15]). In 2001 and 2002, Belaïdi and Hamouda have considered the equation (1.7) and have obtained the following two results:

**Theorem A** [4] *Let $A_0(z),...,A_{n-1}(z)$ with $A_0(z)\not\equiv 0$ be entire functions such that for real constants $\alpha,\beta,\mu,\theta_1$ and $\theta_2$ satisfying $0\leq\beta<\alpha$, $\mu>0$ and $\theta_1<\theta_2$, we have*

$$|A_0(z)|\geq\exp\{\alpha|z|^\mu\} \tag{1.8}$$

*and*

$$|A_j(z)|\leq\exp\{\beta|z|^\mu\}\quad(j=1,2,...,n-1) \tag{1.9}$$

*as $z\to\infty$ in $\theta_1\leq\arg z\leq\theta_2$. Then every solution $f\not\equiv 0$ of the equation* (1.7) *has an infinite order.*

**Theorem B** [3] *Let* $A_0(z), ..., A_{n-1}(z)$ *with* $A_0(z) \not\equiv 0$ *be entire functions. Suppose that there exist a sequence of complex numbers* $(z_k)_{k\in\mathbb{N}}$ *with* $\lim_{k\to+\infty} z_k = \infty$ *and three real numbers* $\alpha, \beta$ *and* $\mu$ *satisfying* $0 \leq \beta < \alpha$ *and* $\mu > 0$ *such that*

$$|A_0(z_k)| \geq \exp\{\alpha |z_k|^{\mu}\} \tag{1.10}$$

*and*

$$|A_j(z_k)| \leq \exp\{\beta |z_k|^{\mu}\} \quad (j = 1, 2, ..., n-1) \tag{1.11}$$

*as* $k \to +\infty$. *Then every solution* $f \not\equiv 0$ *of the equation* (1.7) *has an infinite order.*

Let $n \geq 2$ be an integer and consider the linear differential equation

$$A_n(z) f^{(n)} + A_{n-1}(z) f^{(n-1)} + ... + A_1(z) f' + A_0(z) f = 0. \tag{1.12}$$

It is well-known that if $A_n \equiv 1$, then all solutions of this equation are entire functions but when $A_n$ is a nonconstant entire function, equation (1.12) can possess meromorphic solutions. For instance the equation

$$z^2 f^{'''} + 6z f^{''} + 6f^{'} - z^2 f = 0$$

has a meromorphic solution

$$f(z) = \frac{e^z}{z^2}.$$

Recently, L. Z. Yang [18], J. Xu and Z. Zhang [17] have considered equation (1.12) and obtained different results concerning the growth of its solutions, but the condition that the poles of every meromorphic solution of (1.12) must be of uniformly bounded multiplicity was missing in [17]. See Remark 3 in [9].

In the present paper, we improve and extend Theorem A and Theorem B for equations of the form (1.12) by using the concept of the iterated order. We also consider the nonhomogeneous linear differential equations. We obtain the following results:

**Theorem 1.1** *Let* $p \geq 1$ *be an integer and let* $A_0(z), ..., A_{n-1}(z), A_n(z)$ *with* $A_0(z) \not\equiv 0$ *and* $A_n(z) \not\equiv 0$ *be entire functions such that* $i_\lambda(A_n) \leq 1, i(A_j) =$

*$p$ $(j=0,1,...,n)$ and $\max\{\sigma_p(A_j): j=1,2,...,n\} < \sigma_p(A_0)=\sigma$. Suppose that for real constants $\alpha,\beta,\theta_1$ and $\theta_2$ satisfying $0\le\beta<\alpha$ and $\theta_1<\theta_2$ and for $\varepsilon>0$ sufficiently small, we have*

$$|A_0(z)| \ge \exp_p\left\{\alpha |z|^{\sigma-\varepsilon}\right\} \tag{1.13}$$

*and*

$$|A_j(z)| \le \exp_p\left\{\beta |z|^{\sigma-\varepsilon}\right\} \quad (j=1,2,...,n) \tag{1.14}$$

*as $z\to\infty$ in $\theta_1\le\arg z\le\theta_2$. Then every meromorphic solution $f\not\equiv 0$ whose poles are of uniformly bounded multiplicity of the equation* (1.12) *has an infinite iterated $p-$order and satisfies $i(f)=p+1, \sigma_{p+1}(f)=\sigma$.*

**Theorem 1.2** *Let $p\ge 1$ be an integer and let $A_0(z),...,A_{n-1}(z),A_n(z)$ with $A_0(z)\not\equiv 0$ and $A_n(z)\not\equiv 0$ be entire functions such that $i_\lambda(A_n)\le 1, i(A_j)=p$ $(j=0,1,...,n)$ and $\max\{\sigma_p(A_j): j=1,2,...,n\} < \sigma_p(A_0)=\sigma$. Suppose that there exist a sequence of complex numbers $(z_k)_{k\in\mathbb{N}}$ with $\lim_{k\to+\infty} z_k=\infty$ and two real numbers $\alpha$ and $\beta$ satisfying $0\le\beta<\alpha$ such that for $\varepsilon>0$ sufficiently small, we have*

$$|A_0(z_k)| \ge \exp_p\left\{\alpha |z_k|^{\sigma-\varepsilon}\right\} \tag{1.15}$$

*and*

$$|A_j(z_k)| \le \exp_p\left\{\beta |z_k|^{\sigma-\varepsilon}\right\} \quad (j=1,2,...,n-1) \tag{1.16}$$

*as $k\to+\infty$. Then every meromorphic solution $f\not\equiv 0$ whose poles are of uniformly bounded multiplicity of the equation* (1.12) *has an infinite iterated $p-$order and satisfies $i(f)=p+1$, $\sigma_{p+1}(f)=\sigma$.*

Let $A_0(z),...,A_{n-1}(z),A_n(z)$, $F(z)$ be entire functions with $A_0(z)\not\equiv 0$, $A_n(z)\not\equiv 0$ and $F\not\equiv 0$. Considering the nonhomogeneous linear differential equation

$$A_n(z)f^{(n)} + A_{n-1}(z)f^{(n-1)} + ... + A_1(z)f' + A_0(z)f = F, \tag{1.17}$$

we obtain the following result:

**Theorem 1.3** *Let* $A_0(z), ..., A_{n-1}(z), A_n(z)$ *with* $A_0(z) \not\equiv 0$ *and* $A_n(z) \not\equiv 0$ *be entire functions satisfying the hypotheses of Theorem 1.2 and let* $F \not\equiv 0$ *be an entire function of iterated order with* $i(F) = q$.
(i) *If* $q < p+1$ *or* $q = p+1$ *and* $\sigma_{p+1}(F) < \sigma_p(A_0) = \sigma$, *then every meromorphic solution* $f$ *whose poles are of uniformly bounded multiplicity of the equation* (1.17) *satisfies* $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p+1$ *and* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$ *with at most one exceptional solution* $f_0$ *satisfying* $i(f_0) < p+1$ *or* $\sigma_{p+1}(f_0) < \sigma$.
(ii) *If* $q > p+1$ *or* $q = p+1$ *and* $\sigma_p(A_0) < \sigma_{p+1}(F) < +\infty$, *then every meromorphic solution* $f$ *whose poles are of uniformly bounded multiplicity of the equation* (1.17) *satisfies* $i(f) = q$ *and* $\sigma_q(f) = \sigma_q(F)$.

# 2 Preliminary Lemmas

**Lemma 2.1** [10] *Let* $f(z)$ *be a meromorphic function. Let* $\alpha > 1$ *and* $\Gamma = \{(k_1, j_1), (k_2, j_2), ..., (k_m, j_m)\}$ *denote a set of distinct pairs of integers satisfying* $k_i > j_i \geq 0$. *Then there exist a set* $E_1 \subset (1, +\infty)$ *having finite logarithmic measure and a constant* $B > 0$ *that depends only on* $\alpha$ *and* $\Gamma$ *such that for all* $z$ *satisfying* $|z| = r \notin [0,1] \cup E_1$ *and all* $(k,j) \in \Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}(\log^\alpha r)\log T(\alpha r, f)\right]^{k-j}. \tag{2.1}$$

**Lemma 2.2** [10] *Let* $f(z)$ *be a meromorphic function. Let* $\alpha > 1$ *and* $\varepsilon > 0$ *be given constants. Then there exist a constant* $B > 0$ *and a set* $E_2 \subset [0, +\infty)$ *having finite linear measure such that for all* $z$ *satisfying* $|z| = r \notin E_2$, *we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(\alpha r, f) r^\varepsilon \log T(\alpha r, f)\right]^j \ (j \in \mathbb{N}). \tag{2.2}$$

**Lemma 2.3** [6, 7] *Let* $p \geq 1$ *be an integer and* $g(z)$ *be an entire function with* $i(g) = p+1$ *and* $\sigma_{p+1}(g) = \sigma$. *Let* $\nu_g(r)$ *be the central index of* $g(z)$. *Then*

$$\limsup_{r\to+\infty} \frac{\log_{p+1} \nu_g(r)}{\log r} = \sigma. \tag{2.3}$$

**Lemma 2.4** *Let* $p \geq 1$ *be an integer and let* $f(z) = g(z)/d(z)$ *be a meromorphic function, where* $g(z)$ *and* $d(z)$ *are entire functions satisfying*

$\sigma_p(f)=\sigma_p(g)=+\infty$, $i(d)<p$ *or* $i(d)=p$ *and* $\sigma_p(d)=\rho<+\infty$. *Then there exist a sequence of complex numbers* $\{z_k\}_{k\in\mathbb{N}}$ *and a set* $E_3$ *of finite logarithmic measure such that* $|z_k|=r_k\notin E_3$, $r_k\to+\infty$, $|g(z_k)|=M(r_k,g)$ *and for sufficiently large* $k$, *we have*

$$\frac{f^{(n)}(z_k)}{f(z_k)}=\left(\frac{\nu_g(r_k)}{z_k}\right)^n(1+o(1))\ \ (n\geq 1 \text{ is an integer}) \tag{2.4}$$

*and*

$$\lim_{k\to+\infty}\frac{\log_p \nu_g(r_k)}{\log r_k}=\sigma_p(g)=+\infty, \tag{2.5}$$

*where* $\nu_g(r)$ *is the central index of* $g$.

*Proof.* By induction, we obtain

$$f^{(n)}=\frac{g^{(n)}}{d}+\sum_{j=0}^{n-1}\frac{g^{(j)}}{d}\sum_{(j_1\cdots j_n)}C_{jj_1\cdots j_n}\left(\frac{d'}{d}\right)^{j_1}\cdots\left(\frac{d^{(n)}}{d}\right)^{j_n}, \tag{2.6}$$

where $C_{jj_1\cdots j_n}$ are constants and $j+j_1+2j_2+\ldots+nj_n=n$. Hence

$$\frac{f^{(n)}}{f}=\frac{g^{(n)}}{g}+\sum_{j=0}^{n-1}\frac{g^{(j)}}{g}\sum_{(j_1\cdots j_n)}C_{jj_1\cdots j_n}\left(\frac{d'}{d}\right)^{j_1}\cdots\left(\frac{d^{(n)}}{d}\right)^{j_n}. \tag{2.7}$$

From the Wiman-Valiron theory [12, 16], there exists a set $E\subset(1,+\infty)$ with finite logarithmic measure such that for a point $z$ satisfying $|z|=r\notin E$ and $|g(z)|=M(r,g)$, we have

$$\frac{g^{(j)}(z)}{g(z)}=\left(\frac{\nu_g(r)}{z}\right)^j(1+o(1))\ \ (j=1,2,\ldots,n), \tag{2.8}$$

where $\nu_g(r)$ is the central index of $g$. Substituting (2.8) into (2.7) yields

$$\frac{f^{(n)}(z)}{f(z)}=\left(\frac{\nu_g(r)}{z}\right)^n\Bigg[(1+o(1))$$

$$+\sum_{j=0}^{n-1}\left(\frac{\nu_g(r)}{z}\right)^{j-n}(1+o(1))\sum_{(j_1\cdots j_n)}C_{jj_1\cdots j_n}\left(\frac{d'}{d}\right)^{j_1}\cdots\left(\frac{d^{(n)}}{d}\right)^{j_n}\Bigg]. \tag{2.9}$$

By Lemma 2.1, there exist a constant $B>0$ and a set $E_1\subset(1,+\infty)$ having finite logarithmic measure such that for all $z$ satisfying $|z|=r\notin[0,1]\cup E_1$, we have

$$\left|\frac{d^{(m)}(z)}{d(z)}\right|\leq B\left[T(2r,d)\right]^{2m}\quad(m=1,2,...,n). \tag{2.10}$$

For any given $\varepsilon>0$ and sufficiently large $r$, we have

$$T(2r,d)\leq\exp_{p-1}\left\{(2r)^{\rho+\frac{\varepsilon}{2}}\right\}. \tag{2.11}$$

From (2.10) and (2.11) and $j_1+2j_2+...+nj_n=n-j$, we obtain for sufficiently large $r$, $|z|=r\notin[0,1]\cup E_1$

$$\left|\left(\frac{d'}{d}\right)^{j_1}\cdots\left(\frac{d^{(n)}}{d}\right)^{j_n}\right|\leq c\left[\left(\exp_{p-1}\left\{(2r)^{\rho+\frac{\varepsilon}{2}}\right\}\right)^2\right]^{(n-j)}$$

$$=c\left[\exp\left(2\exp_{p-2}\left\{(2r)^{\rho+\frac{\varepsilon}{2}}\right\}\right)\right]^{(n-j)}\leq c\left[\exp_{p-1}\left\{r^{\rho+\varepsilon}\right\}\right]^{(n-j)}, \tag{2.12}$$

where $c$ is a positive constant. Since $\sigma_p(g)=+\infty$, it follows that there exists a sequence $\{r'_k\}$ $(r'_k\to+\infty)$ satisfying

$$\lim_{k\to+\infty}\frac{\log_p\nu_g(r'_k)}{\log r'_k}=+\infty. \tag{2.13}$$

Setting the logarithmic measure of $E_3=[0,1]\cup E\cup E_1$, $lm(E_3)=\delta<+\infty$, there exists a point $r_k\in[r'_k,(\delta+1)r'_k]-E_3$. Since

$$\frac{\log_p\nu_g(r_k)}{\log r_k}\geq\frac{\log_p\nu_g(r'_k)}{\log\left[(\delta+1)r'_k\right]}=\frac{\log_p\nu_g(r'_k)}{(\log r'_k)\left[1+\frac{\log(\delta+1)}{\log r'_k}\right]}, \tag{2.14}$$

we deduce that

$$\lim_{k\to+\infty}\frac{\log_p\nu_g(r_k)}{\log r_k}=+\infty. \tag{2.15}$$

Then from (2.15) for a given arbitrary large $L>\rho+\varepsilon+1$,

$$\nu_g(r_k)>\exp_{p-1}\left\{r_k^L\right\} \tag{2.16}$$

holds for sufficiently large $r_k$. This and (2.12) lead

$$\left|\left(\frac{\nu_g(r_k)}{z_k}\right)^{j-n}\left(\frac{d'}{d}\right)^{j_1}\cdots\left(\frac{d^{(n)}}{d}\right)^{j_n}\right|$$

$$\leq c\left[\frac{r_k \exp_{p-1}\left\{r_k^{\rho+\varepsilon}\right\}}{\exp_{p-1}\left\{r_k^{L}\right\}}\right]^{(n-j)} \to 0,\ r_k \to +\infty \tag{2.17}$$

for $|z_k| = r_k$ and $|g(z_k)| = M(r_k, g)$. From $(2.15)$, $(2.9)$ and $(2.17)$, we obtain our result.

**Lemma 2.5** [6] *Let $p \geq 1$ be an integer. Suppose that $f(z)$ is a meromorphic function such that $i(f) = p$, $\sigma_p(f) = \sigma$ and $i_\lambda\left(\frac{1}{f}\right) \leq 1$. Then for any given $\varepsilon > 0$, there exists a set $E_4 \subset (1, +\infty)$ that has finite linear measure and finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0,1] \cup E_4$, $r \to +\infty$, we have*

$$|f(z)| \leq \exp_p\left\{r^{\sigma+\varepsilon}\right\}. \tag{2.18}$$

**Lemma 2.6** [14] *Let $p \geq 1$ be an integer and let $f(z)$ be a meromorphic function with $i(f) = p$. Then $\sigma_p(f) = \sigma_p(f')$.*

**Lemma 2.7** [6] *Let $p \geq 1$ be an integer and let $f(z)$ be a meromorphic solution of the differential equation*

$$f^{(n)} + B_{n-1}(z) f^{(n-1)} + \ldots + B_1(z) f' + B_0(z) f = F, \tag{2.19}$$

*where $B_0(z), \ldots, B_{n-1}(z)$ and $F \not\equiv 0$ are meromorphic functions such that*
(i) $\max\{i(F), i(B_j)\ \ (j = 0, \ldots, n-1)\} < i(f) = p+1$ *or*
(ii) $\max\{\sigma_{p+1}(F), \sigma_{p+1}(B_j)\ \ (j = 0, \ldots, n-1)\} < \sigma_{p+1}(f)$.
*Then $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p+1$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f)$.*

To avoid some problems caused by the exceptional set we recall the following lemmas.

**Lemma 2.8** [1] *Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $E_5$ of finite linear measure. Then for any $\mu > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\mu r)$ for all $r > r_0$.*

**Lemma 2.9** [11] *Let* $\varphi : [0, +\infty) \to \mathbb{R}$ *and* $\psi : [0, +\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r) \leq \psi(r)$ *for all* $r \notin E_6 \cup [0, 1]$, *where* $E_6 \subset (1, +\infty)$ *is a set of finite logarithmic measure. Let* $\eta > 1$ *be a given constant. Then there exists an* $r_1 = r_1(\eta) > 0$ *such that* $\varphi(r) \leq \psi(\eta r)$ *for all* $r > r_1$.

# 3 Proof of Theorem 1.1

Suppose that $f\ (\not\equiv 0)$ is a meromorphic solution whose poles are of uniformly bounded multiplicity of the equation (1.12). From (1.12), it follows that

$$|A_0(z)| \leq |A_n(z)| \left|\frac{f^{(n)}}{f}\right| + |A_{n-1}(z)| \left|\frac{f^{(n-1)}}{f}\right| + ... + |A_1(z)| \left|\frac{f'}{f}\right|. \tag{3.1}$$

By Lemma 2.2, there exist a constant $B > 0$ and a set $E_2 \subset [0, +\infty)$ having finite linear measure such that for all $z$ satisfying $|z| = r \notin E_2$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq Br\,[T(2r, f)]^{n+1} \quad (j = 1, 2, ..., n). \tag{3.2}$$

Hence from (1.13), (1.14), (3.1) and (3.2), it follows that

$$\exp_p\left\{\alpha\,|z|^{\sigma-\varepsilon}\right\} \leq Bnr\,[T(2r, f)]^{n+1} \exp_p\left\{\beta\,|z|^{\sigma-\varepsilon}\right\} \tag{3.3}$$

as $r \to +\infty$, $|z| = r \notin E_2$ and $\theta_1 \leq \arg z \leq \theta_2$. By Lemma 2.8 and (3.3), we obtain that $\sigma_p(f) = +\infty$ and $i(f) \geq p + 1$, $\sigma_{p+1}(f) \geq \sigma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\sigma_{p+1}(f) \geq \sigma$. Set

$$\delta = \max\{\sigma_p(A_j) : j = 1, 2, ..., n\} < \sigma_p(A_0) = \sigma < +\infty.$$

We can rewrite (1.12) as

$$f^{(n)} + \frac{A_{n-1}(z)}{A_n(z)} f^{(n-1)} + ... + \frac{A_1(z)}{A_n(z)} f' + \frac{A_0(z)}{A_n(z)} f = 0.$$

Obviously, the poles of $f(z)$ can only occur at the zeros of $A_n(z)$. Note that the multiplicity of the poles of $f$ is uniformly bounded, and thus we have $i_\lambda\left(\frac{1}{f}\right) \leq p$ and $\lambda_p(1/f) \leq \delta < \sigma < +\infty$. By Hadamard factorization theorem, we can write $f$ as $f(z) = g(z)/d(z)$, where $g(z)$ and $d(z)$ are entire functions satisfying $i(f) = i(g) = t \geq p+1$, $\sigma_t(f) = \sigma_t(g)$ and $i(d) \leq p$, $\sigma_p(d) = \lambda_p(1/f) < \sigma < +\infty$. Thus by Lemma 2.4, there exists a sequence of complex numbers $\{z_k\}_{k\in\mathbb{N}}$ and a set $E_3$ of finite logarithmic measure such that $|z_k| = r_k \notin E_3$, $r_k \to +\infty$, $|g(z_k)| = M(r_k, g)$ and for sufficiently large $k$, we have

$$\frac{f^{(j)}(z_k)}{f(z_k)} = \left(\frac{\nu_g(r_k)}{z_k}\right)^j (1+o(1)) \quad (j = 1, 2, ..., n). \tag{3.4}$$

By Remark 1.1, for any given $\varepsilon > 0$ and for sufficiently large $r$, we have

$$|A_j(z)| \leq \exp_p\left\{r^{\sigma+\varepsilon}\right\} \quad (j = 0, 1, ..., n-1). \tag{3.5}$$

By Lemma 2.5, for the above $\varepsilon > 0$, there exists a set $E_4 \subset [1, +\infty)$ that has finite linear measure and finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0,1] \cup E_4, r \to +\infty$, we have

$$|1/A_n(z)| \leq \exp_p\left\{r^{\sigma+\varepsilon}\right\}. \tag{3.6}$$

We can rewrite (1.12) as

$$-A_n(z)\frac{f^{(n)}}{f} = A_{n-1}(z)\frac{f^{(n-1)}}{f} + ... + A_1(z)\frac{f'}{f} + A_0(z). \tag{3.7}$$

Substituting (3.4) into (3.7), we obtain for the above $z_k$

$$-A_n(z_k)\left(\frac{\nu_g(r_k)}{z_k}\right)^n (1+o(1)) = A_{n-1}(z_k)\left(\frac{\nu_g(r_k)}{z_k}\right)^{n-1} (1+o(1))$$

$$+ ... + A_1(z_k)\left(\frac{\nu_g(r_k)}{z_k}\right)(1+o(1)) + A_0(z_k). \tag{3.8}$$

Hence from (3.5), (3.6) and (3.8), we have

$$\left(1/\exp_p\left\{r_k^{\sigma+\varepsilon}\right\}\right)\left|\frac{\nu_g(r_k)}{z_k}\right|^n |1+o(1)|$$

$$\leq \exp_p\left\{r_k^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r_k)}{z_k}\right|^{n-1} |1+o(1)|$$

$$+ ... + \exp_p\left\{r_k^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r_k)}{z_k}\right| |1+o(1)| + \exp_p\left\{r_k^{\sigma+\varepsilon}\right\}$$

$$\leq n \exp_p \left\{ r_k^{\sigma+\varepsilon} \right\} \left| \frac{\nu_g (r_k)}{z_k} \right|^{n-1} |1 + o(1)|, \tag{3.9}$$

where $|z_k| = r_k \notin [0,1] \cup E_3 \cup E_4$, $r_k \to +\infty$ and $|g(z_k)| = M(r_k, g)$. By Lemma 2.9 and (3.9), we get

$$\limsup_{k \to +\infty} \frac{\log_{p+1} \nu_g(r_k)}{\log r_k} \leq \sigma + \varepsilon. \tag{3.10}$$

Since $\varepsilon > 0$ is arbitrary, by (3.10) and Lemma 2.3, we obtain $i(f) = i(g) \leq p+1$ and $\sigma_{p+1}(f) = \sigma_{p+1}(g) \leq \sigma$. This and the fact that $\sigma_{p+1}(f) \geq \sigma$ yield $i(f) = p+1$ and $\sigma_{p+1}(f) = \sigma$.

# 4 Proof of Theorem 1.2

Suppose that $f$ ($\not\equiv 0$) is a meromorphic solution whose poles are of uniformly bounded multiplicity of the equation (1.12). From (1.12), it follows that

$$|A_0(z)| \leq |A_n(z)| \left| \frac{f^{(n)}}{f} \right| + |A_{n-1}(z)| \left| \frac{f^{(n-1)}}{f} \right| + ... + |A_1(z)| \left| \frac{f'}{f} \right|. \tag{4.1}$$

By Lemma 2.2, there exist a constant $B > 0$ and a set $E_2 \subset [0, +\infty)$ having finite linear measure such that for all $z$ satisfying $|z| = r \notin E_2$, we have

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq Br \left[ T(2r, f) \right]^{n+1} \quad (j = 1, 2, ..., n). \tag{4.2}$$

Hence from (1.15), (1.16), (4.1) and (4.2), we have

$$\exp_p \left\{ \alpha |z_k|^{\sigma-\varepsilon} \right\} \leq Bnr_k \left[ T(2r_k, f) \right]^{n+1} \exp_p \left\{ \beta |z_k|^{\sigma-\varepsilon} \right\} \tag{4.3}$$

as $k \to +\infty$, $|z_k| = r_k \notin E_2$. Hence from (4.3) and Lemma 2.8, we obtain that $\sigma_p(f) = +\infty$ and $i(f) \geq p+1$, $\sigma_{p+1}(f) \geq \sigma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\sigma_{p+1}(f) \geq \sigma$. By using the same arguments as in proof of Theorem 1.1, we obtain $i(f) \leq p+1$ and $\sigma_{p+1}(f) \leq \sigma$. Hence $i(f) = p+1$ and $\sigma_{p+1}(f) = \sigma$.

## 5 Proof of Theorem 1.3

First, we show that (1.17) can possess at most one exceptional meromorphic solution $f_0$ satisfying $i(f_0) < p+1$ or $\sigma_{p+1}(f_0) < \sigma$. In fact, if $f^*$ is another solution with $i(f^*) < p+1$ or $\sigma_{p+1}(f^*) < \sigma$ of the equation (1.17), then $i(f_0 - f^*) < p+1$ or $\sigma_{p+1}(f_0 - f^*) < \sigma$. But $f_0 - f^*$ is a solution of the corresponding homogeneous equation (1.12) of (1.17). This contradicts Theorem 1.2. We assume that $f$ is an infinite iterated $p-$order meromorphic solution whose poles are of uniformly bounded multiplicity of (1.17) and $f_1, f_2, ...f_n$ is a solution base of the corresponding homogeneous equation (1.12) of (1.17). Then $f$ can be expressed in the form

$$f(z) = B_1(z) f_1(z) + B_2(z) f_2(z) + ... + B_n(z) f_n(z), \tag{5.1}$$

where $B_1(z), ..., B_n(z)$ are suitable meromorphic functions determined by

$$\begin{array}{c} B_1'(z) f_1(z) + B_2'(z) f_2(z) + ... + B_n'(z) f_n(z) = 0 \\ B_1'(z) f_1'(z) + B_2'(z) f_2'(z) + ... + B_n'(z) f_n'(z) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ B_1'(z) f_1^{(n-1)}(z) + B_2'(z) f_2^{(n-1)}(z) + ... + B_n'(z) f_n^{(n-1)}(z) = F(z). \end{array} \tag{5.2}$$

Since the Wronskian $W(f_1, f_2, ..., f_n)$ is a differential polynomial in $f_1, f_2, ..., f_n$ with constant coefficients, it is easy by using Theorem 1.2 to deduce that

$$\sigma_{p+1}(W) \leq \max\{\sigma_{p+1}(f_j) : j = 1, 2, ..., n\} = \sigma_p(A_0) = \sigma. \tag{5.3}$$

From (5.2), we get

$$B_j' = F.G_j(f_1, f_2, ..., f_n).W(f_1, f_2, ..., f_n)^{-1} \quad (j = 1, 2, ..., n), \tag{5.4}$$

where $G_j(f_1, f_2, ...f_n)$ are differential polynomials in $f_1, f_2, ..., f_n$ with constant coefficients. Thus

$$\sigma_{p+1}(G_j) \leq \max\{\sigma_{p+1}(f_j) : j = 1, 2, ..., n\}$$

$$= \sigma_p(A_0) = \sigma \quad (j = 1, 2, ..., n). \tag{5.5}$$

**(i)** If $q < p+1$ or $q = p+1$ and $\sigma_{p+1}(F) < \sigma_p(A_0) = \sigma$, then by Lemma 2.6, (5.3), (5.4) and (5.5) for $j = 1, 2, ..., n$, we have

$$\sigma_{p+1}(B_j) = \sigma_{p+1}\left(B_j'\right) \le \max\{\sigma_{p+1}(F), \sigma_p(A_0)\} = \sigma_p(A_0) = \sigma. \tag{5.6}$$

Then from (5.1) and (5.6), we get

$$\begin{aligned}\sigma_{p+1}(f) \le \max\{\sigma_{p+1}(f_j), \sigma_{p+1}(B_j) : j = 1, 2, ..., n\}\\ = \sigma_p(A_0) = \sigma < +\infty.\end{aligned} \tag{5.7}$$

From (1.17), it follows that

$$|A_0(z)| \le |A_n(z)|\left|\frac{f^{(n)}}{f}\right| + |A_{n-1}(z)|\left|\frac{f^{(n-1)}}{f}\right| + ... + |A_1(z)|\left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right|. \tag{5.8}$$

By Lemma 2.2, there exist a constant $B > 0$ and a set $E_2 \subset [0, +\infty)$ having finite linear measure such that for all $z$ satisfying $|z| = r \notin E_2$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le Br\,[T(2r, f)]^{n+1} \quad (j = 1, 2, ..., n). \tag{5.9}$$

Set

$$\max\{\sigma_p(A_j) : j = 1, 2, ..., n\} = \delta < \sigma_p(A_0) = \sigma. \tag{5.10}$$

We can rewrite (1.17) as

$$f^{(n)} + \frac{A_{n-1}(z)}{A_n(z)}f^{(n-1)} + ... + \frac{A_1(z)}{A_n(z)}f' + \frac{A_0(z)}{A_n(z)}f = \frac{F}{A_n(z)}. \tag{5.11}$$

Obviously, it follows that the poles of $f(z)$ can only occur at the zeros of $A_n(z)$. Note that the multiplicity of the poles of $f$ is uniformly bounded, and thus we have $i_\lambda\left(\frac{1}{f}\right) \le p$ and $\lambda_p(1/f) \le \delta < \sigma < +\infty$. By Hadamard factorization theorem, we can write $f$ as $f(z) = g(z)/d(z)$, where $g(z)$ and $d(z)$ are entire functions satisfying $i(f) = i(g) = t \ge p+1$, $\sigma_t(f) = \sigma_t(g)$ and $i(d) \le p$, $\sigma_p(d) = \lambda_p(1/f) < \sigma < +\infty$. Set

$$\max\{\sigma_{p+1}(F), \sigma_p(d)\} = \gamma < \sigma. \tag{5.12}$$

For any given $\varepsilon$ $(0 < 2\varepsilon < \sigma - \gamma)$ and a sufficiently large $r$, we have

$$|F(z)| \le \exp_p\left\{r^{\gamma+\varepsilon}\right\} \text{ and } |d(z)| \le \exp_{p-1}\left\{r^{\gamma+\varepsilon}\right\}. \tag{5.13}$$

Since $M(r,g) \geq 1$ for a sufficiently large $r$, we obtain from (5.13),

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)|\,|d(z)|}{|g(z)|} \leq \exp_p\left\{r^{\gamma+\varepsilon}\right\} \exp_{p-1}\left\{r^{\gamma+\varepsilon}\right\} \tag{5.14}$$

as $r \to +\infty$, $|z| = r$ and $|g(z)| = M(r,g)$. If $A_0(z), ..., A_{n-1}(z)$ and $A_n(z)$ satisfy the hypotheses of Theorem 1.2, then from (1.15), (1.16), (5.8), (5.9) and (5.14), it follows that

$$\begin{aligned} \exp_p\left\{\alpha\,|z_k|^{\sigma-\varepsilon}\right\} \leq{}& Bnr_k\left[T(2r_k,f)\right]^{n+1} \exp_p\left\{\beta\,|z_k|^{\sigma-\varepsilon}\right\} \\ &+ \exp_p\left\{|z_k|^{\gamma+\varepsilon}\right\} \exp_{p-1}\left\{|z_k|^{\gamma+\varepsilon}\right\} \end{aligned} \tag{5.15}$$

as $k \to +\infty$, $|z_k| = r_k \notin E_2$ and $|g(z_k)| = M(r_k,g)$. From (5.15) and Lemma 2.8, we get $\sigma_{p+1}(f) \geq \sigma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\sigma_{p+1}(f) \geq \sigma$. This and the fact that $\sigma_{p+1}(f) \leq \sigma$ yield $\sigma_{p+1}(f) = \sigma$. Thus by Lemma 2.7, we have $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p+1$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$.

**(ii)** If $q > p+1$ or $q = p+1$ and $\sigma_p(A_0) < \sigma_{p+1}(F) < +\infty$, then by Lemma 2.6, (5.3), (5.4) and (5.5) for $j = 1, 2, ..., n$, we have

$$\sigma_q(B_j) = \sigma_q\left(B_j'\right) \leq \max\left\{\sigma_q(F), \sigma_q(f_j) : j = 1, 2, ..., n\right\} = \sigma_q(F). \tag{5.16}$$

Then from (5.1) and (5.16), we get

$$\sigma_q(f) \leq \max\left\{\sigma_q(f_j), \sigma_q(B_j) : j = 1, 2, ..., n\right\} = \sigma_q(F). \tag{5.17}$$

On the other hand, if $q > p+1$ or $q = p+1$ and $\sigma_p(A_0) < \sigma_{p+1}(F) < +\infty$, it follows from (1.17) that a simple consideration of order implies $\sigma_q(f) \geq \sigma_q(F)$. By this inequality and (5.17) we obtain $\sigma_q(f) = \sigma_q(F)$.

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(Received December 15, 2009)