Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **29**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

EXISTENCE RESULTS FOR A PARTIAL NEUTRAL INTEGRO-DIFFERENTIAL EQUATION WITH STATE-DEPENDENT DELAY

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Abstract. In this paper we study the existence of mild solutions for a class of abstract partial neutral integro-differential equations with state-dependent delay.

Keywords: Integro-differential equations, neutral equation, resolvent of operators, state-dependent delay.
AMS-Subject Classification: 34K30, 35R10, 47D06.

1. Introduction

In this paper we study the existence of mild solutions for a class of abstract partial neutral integro-differential equations with state-dependent delay described in the form

$$\frac{d}{dt}[x(t)+\int_{-\infty}^{t} N(t-s)x(s)ds] = Ax(t)+\int_{-\infty}^{t} B(t-s)x(s)ds + f(t, x_{\rho(t,x_t)}), \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

where $t \in I = [0, b]$, $A, B(t)$ for $t \geq 0$ are closed linear operators defined on a common domain $D(A)$ which is dense in X, $N(t)$ $(t \geq 0)$ is bounded linear operators on X, the history $x_t : (-\infty, 0] \to X$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and $f : [0, b]\times\mathcal{B} \to X$ and $\rho : [0, b]\times\mathcal{B} \to (-\infty, b]$ are appropriate functions.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years. The literature devoted to this subject is concerned fundamentally with first order functional differential equations for which the state belong to some finite dimensional space, see among another works, [1, 3, 4, 5, 7, 9, 10, 11, 12, 19, 21, 22]. The problem of the existence of solutions for partial functional differential equations with state-dependent delay has been recently treated in the literature in [2, 14, 15, 16, 17]. Our purpose in this paper is to establish the existence of mild solutions for the partial neutral system without using many of the strong restrictions considered in the literature (see [6] for details).

† The work of this author was supported by FAPEMIG/Brazil, Grant CEX-APQ-00476-09.

2. PRELIMINARIES

In what follows we recall some definitions, notations and results that we need in the sequel. Throughout this paper, $(X, \|\cdot\|)$ is a Banach space and $A, B(t)$, $t \geq 0$, are closed linear operators defined on a common domain $\mathcal{D} = D(A)$ which is dense in X. The notation $[D(A)]$ represents the domain of A endowed with the graph norm. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this paper, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W endowed with the uniform operator topology and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. Furthermore, for appropriate functions $K : [0, \infty) \to Z$ the notation $\widehat{K}$ denotes the Laplace transform of K . The notation, $B_r(x, Z)$ stands for the closed ball with center at x and radius $r > 0$ in Z. On the other hand, for a bounded function $\gamma : [0, a] \to Z$ and $t \in [0, a]$, the notation $\| \gamma \|_{Z,t}$ is given by

$$\| \gamma \|_{Z,t} = \sup\{\| \gamma(s) \|_Z : s \in [0, t]\}, \tag{2.1}$$

and we simplify this notation to $\| \gamma \|_t$ when no confusion about the space Z arises.

To obtain our results, we assume that the integro-differential abstract Cauchy problem

$$\frac{d}{dt}\left[x(t) + \int_0^t N(t-s)x(s)\,ds\right] = Ax(t) + \int_0^t B(t-s)x(s)\,ds, \tag{2.2}$$

$$x(0) = z \in X. \tag{2.3}$$

has an associated resolvent operator of bounded linear operators $(\mathcal{R}(t))_{t\geq 0}$ on X.

Definition 2.1. *A one parameter family of bounded linear operators $(\mathcal{R}(t))_{t\geq 0}$ on X is called a resolvent operator of (2.2)-(2.3) if the following conditions are verified.*

(a) *The function $\mathcal{R}(\cdot) : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous, exponentially bounded and $\mathcal{R}(0)x = x$ for all $x \in X$.*

(b) *For $x \in D(A)$, $\mathcal{R}(\cdot)x \in C([0, \infty), [D(A)]) \bigcap C^1((0, \infty), X)$, and*

$$\frac{d}{dt}\left[\mathcal{R}(t)x + \int_0^t N(t-s)\mathcal{R}(s)x ds\right] = A\mathcal{R}(t)x + \int_0^t B(t-s)\mathcal{R}(s)x ds, \tag{2.4}$$

$$\frac{d}{dt}\left[\mathcal{R}(t)x + \int_0^t \mathcal{R}(t-s)N(s)x ds\right] = \mathcal{R}(t)Ax + \int_0^t \mathcal{R}(t-s)B(s)x ds, \tag{2.5}$$

for every $t \geq 0$.

The existence of a resolvent operator for problem (2.2)-(2.3) was studied in [6]. In this work we have considered the following conditions.

(P1) The operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X, and there are constants $M_0 > 0$ and $\vartheta \in (\pi/2, \pi)$ such that $\rho(A) \supseteq \Lambda_\vartheta = \{\lambda \in \mathbb{C} \setminus \{0\} : | arg(\lambda) | < \vartheta\}$ and $\| R(\lambda, A) \| \leq M_0 | \lambda |^{-1}$ for all $\lambda \in \Lambda_\vartheta$.

(P2) The function $N : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous and $\widehat{N}(\lambda)x$ is absolutely convergent for $x \in X$ and $Re(\lambda) > 0$. There exists $\alpha > 0$ and an analytical extension of $\widehat{N}(\lambda)$ (still denoted by $\widehat{N}(\lambda)$) to Λ_ϑ such that $\| \widehat{N}(\lambda) \| \leq N_0 \mid \lambda \mid^{-\alpha}$ for every $\lambda \in \Lambda_\vartheta$, and $\|\widehat{N}(\lambda)x\| \leq N_1|\lambda|^{-1}\|x\|_1$ for every $\lambda \in \Lambda_\vartheta$ and $x \in D(A)$.

(P3) For all $t \geq 0$, $B(t) : D(B(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0,\infty)$ for each $x \in D(A)$. There exists $b(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $Re(\lambda) > 0$ and $\| B(t)x \| \leq b(t) \| x \|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{B} : \Lambda_{\pi/2} \to \mathcal{L}([D(A)], X)$ has an analytical extension (still denoted by $\widehat{B}$) to Λ_ϑ such that $\|\widehat{B}(\lambda)x\| \leq \|\widehat{B}(\lambda)\| \, \|x\|_1$ for all $x \in D(A)$, and $\|\widehat{B}(\lambda)\| \to 0$ as $|\lambda| \to \infty$.

(P4) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and positive constants C_i, $i = 1,2$, such that $A(D) \subseteq D(A)$, $\widehat{B}(\lambda)(D) \subseteq D(A)$, $\widehat{N}(\lambda)(D) \subseteq D(A)$, $\|A\widehat{B}(\lambda)x\| \leq C_1\|x\|$ and $\|\widehat{N}(\lambda)x\|_1 \leq C_2|\lambda|^{-\alpha}\|x\|_1$ for every $x \in D$ and all $\lambda \in \Lambda_\vartheta$.

The following result has been established in [6, Theorem 2.1].

Theorem 2.1. *Assume that conditions* **(P1)**-**(P4)** *are fulfilled. Then there exists a unique resolvent operator for problem (2.2)-(2.3).*

In what follows, we always assume that the conditions **(P1)-(P4)** are verified.

We consider now the non-homogeneous problem

$$\frac{d}{dt}\left[x(t) + \int_0^t N(t-s)x(s)ds\right] = Ax(t) + \int_0^t B(t-s)x(s)\,ds + f(t), \ t \in I = [0,b], \tag{2.6}$$

with initial condition (2.3), where $f : [0,b] \to X$ is a continuous function.

Definition 2.2. *A function $x : [0,b] \to X$ is called a classical solution of problem (2.6) -(2.3) on $(0,b]$ if $x \in C([0,b],[D(A)]) \cap C^1((0,b],X)$, the condition (2.3) holds and the equation (2.6) is verified on $[0,b]$. If, in further, $x \in C([0,b],[D(A)]) \cap C^1([0,b],X)$ the function x is said a classical solution of problem (2.6)-(2.3) on $[0,b]$.*

It is clear from the preceding definition that $\mathcal{R}(\cdot)z$ is a solution of problem (2.6)-(2.3) on $(0,\infty)$ for $z \in D(A)$.

In [6, Theorem 2.4] we have established that the solutions of problem (2.6)-(2.3) are given by the variation of constants formula.

Theorem 2.2. *Let $z \in D(A)$. Assume that $f \in C([0,b],X)$ and $x(\cdot)$ is a classical solution of problem (2.6) -(2.3) on $(0,b]$. Then*

$$x(t) = \mathcal{R}(t)z + \int_0^t \mathcal{R}(t-s)f(s)\,ds, \quad t \in [0,b]. \tag{2.7}$$

Theorem 2.3. *(*[6, Lemma 3.1.1]*) If $R(\lambda_0, A)$ is compact for some $\lambda_0 \in \rho(A)$, then $\mathcal{R}(t)$ is compact for all $t > 0$.*

We will herein define the phase space $\mathcal{B}$ axiomatically, using ideas and notations developed in [18]. More precisely, $\mathcal{B}$ will denote the vector space of functions defined from $(-\infty,0]$ into X endowed with a seminorm denoted $\|\cdot\|_{\mathcal{B}}$ and such that the following axioms hold:

(**A**) If $x:(-\infty,\sigma+b)\to X$, $b>0,\sigma\in\mathbb{R}$, is continuous on $[\sigma,\sigma+b)$ and $x_\sigma\in\mathcal{B}$, then for every $t\in[\sigma,\sigma+b)$ the following conditions hold:
  (i) x_t is in $\mathcal{B}$.
  (ii) $\| x(t)\|\leq H\| x_t\|_{\mathcal{B}}$.
  (iii) $\| x_t\|_{\mathcal{B}}\leq K(t-\sigma)\sup\{\| x(s)\|: \sigma\leq s\leq t\}+M(t-\sigma)\| x_\sigma\|_{\mathcal{B}}$,
  where $H>0$ is a constant; $K,M:[0,\infty)\to[1,\infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and H,K,M are independent of $x(\cdot)$.

(**A1**) For the function $x(\cdot)$ in (**A**), the function $t\to x_t$ is continuous from $[\sigma,\sigma+b)$ into $\mathcal{B}$.

(**B**) The space $\mathcal{B}$ is complete.

Example 2.1. The phase space $\mathbf{C_r\times L^p(g,X)}$

Let $r\geq 0$, $1\leq p<\infty$ and let $g:(-\infty,-r]\to\mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [18]. Briefly, this means that g is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty,0]$ such that $g(\xi+\theta)\leq\gamma(\xi)g(\theta)$, for all $\xi\leq 0$ and $\theta\in(-\infty,-r)\setminus N_\xi$, where $N_\xi\subseteq(-\infty,-r)$ is a set with Lebesgue measure zero. The space $C_r\times L^p(g,X)$ consists of all classes of functions $\varphi:(-\infty,0]\to X$ such that φ is continuous on $[-r,0]$, Lebesgue-measurable, and $g\|\varphi\|^p$ is Lebesgue integrable on $(-\infty,-r)$. The seminorm in $C_r\times L^p(g,X)$ is defined by

$$\|\varphi\|_{\mathcal{B}}:=\sup\{\|\varphi(\theta)\|:-r\leq\theta\leq 0\}+\left(\int_{-\infty}^{-r}g(\theta)\|\varphi(\theta)\|^p\,d\theta\right)^{1/p}.$$

The space $\mathcal{B}=C_r\times L^p(g;X)$ satisfies axioms **(A), (A1), (B).** Moreover, when $r=0$ and $p=2$, we can take $H=1$, $M(t)=\gamma(-t)^{1/2}$ and $K(t)=1+\left(\int_{-t}^0 g(\theta)\,d\theta\right)^{1/2}$, for $t\geq 0$. (see [18, Theorem 1.3.8] for details).

For additional details concerning phase space we refer the reader to [18].

For completeness, we include the following well known result.

Theorem 2.4. (**Leray-Schauder Alternative**) [8, Theorem 6.5.4] *Let D be a closed convex subset of a Banach space Z with $0\in D$. Let $G:D\to D$ be a completely continuous map. Then, G has a fixed point in D or the set $\{z\in D: z=\lambda G(z),\ 0<\lambda<1\}$ is unbounded.*

3. Existence Results

In this section we study the existence of mild solutions for system (1.1)-(1.2). Throughout this section M is a positive constant such that $\| R(t)\|\leq M$ for every $t\in I$. In the rest of this work, φ is a fixed function in $\mathcal{B}$ and $f_i : [0,b] \to X$, $i = 1,2$, will be the functions defined by $f_1(t) = -\int_{-\infty}^{0} N(t-s)\varphi(s)ds$ and $f_2(t) = \int_{-\infty}^{0} B(t-s)\varphi(s)ds$. We adopt the notion of mild solutions for (1.1)-(1.2) from the one given in [6].

Definition 3.3. *A function $u : (-\infty, b] \to X$ is called a mild solution of the neutral system (1.1)-(1.2) on $[0,b]$ if $u_0 = \varphi, u_{\rho(t,u_t)} \in \mathcal{B}$, f_1 is differentiable on $[0,b]$, $f_1', f_2 \in L^1([0,b],X)$, $u_{|_{[0,a]}} \in C([0,b],X)$ and*

$$u(t) = \mathcal{R}(t)\varphi(0) + \int_0^t \mathcal{R}(t-s)f(s,u_{\rho(s,u_s)})ds + \int_0^t \mathcal{R}(t-s)(f_1'(s)+f_2(s))ds, \quad t \in [0,b].$$

To prove our results we always assume that $\rho : I \times \mathcal{B} \to (-\infty, b]$ is continuous and that $\varphi \in \mathcal{B}$. If $x \in C([0,b];X)$ we define $\overline{x} : (-\infty, b] \to X$ is the extension of x to $(-\infty, b]$ such that $\overline{x}_0 = \varphi$. We define $\widetilde{x} : (-\infty, b] \to X$ such that $\widetilde{x} = x + y$ where $y : (-\infty, b] \to X$ is the extension of $\varphi \in \mathcal{B}$ such that $y(t) = R(t)\varphi(0)$ for $t \in I$

In the sequel we introduce the following conditions:

($\mathbf{H_1}$) The function $f : [0,b] \times \mathcal{B} \to X$ verifies the following conditions.

(i) The function $f(t,\cdot) : \mathcal{B} \to X$ is continuous for every $t \in [0,b]$, and for every $\psi \in \mathcal{B}$, the function $f(\cdot,\psi) : [0,b] \to X$ is strongly measurable.

(ii) There exist $m_f \in C([0,b],[0,\infty))$ and a continuous non-decreasing function $\Omega_f : [0,\infty) \to (0,\infty)$ such that $\| f(t,\psi) \| \leq m_f(t)\Omega_f(\| \psi \|_{\mathcal{B}})$, for all $(t,\psi) \in [0,b] \times \mathcal{B}$.

($\mathbf{H}_\varphi$) The function $t \to \varphi_t$ is well defined and continuous from the set

$$\mathcal{R}(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in I \times \mathcal{B},\ \rho(s,\psi) \leq 0\}$$

into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho) \to (0,\infty)$ such that $\| \varphi_t \|_{\mathcal{B}} \leq J^\varphi(t) \| \varphi \|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho)$.

Remark 1. The condition ($\mathbf{H}_\varphi$) is frequently verified by continuous and bounded functions. In fact, if $\mathcal{B}$ verifies axiom C_2 in the nomenclature of [18], then there exists L > 0 such that $\| \varphi \|_{\mathcal{B}} \leq \mathrm{L} \sup_{\theta \leq 0} \| \varphi(\theta) \|$ for every $\varphi \in \mathcal{B}$ continuous and bounded, see [18, Proposition 7.1.1] for details. Consequently,

$$\| \varphi_t \|_{\mathcal{B}} \leq L \frac{\sup_{\theta \leq 0} \| \varphi(\theta) \|}{\| \varphi \|_{\mathcal{B}}} \| \varphi \|_{\mathcal{B}}$$

for every continuous and bounded function $\varphi \in \mathcal{B} \setminus \{0\}$ and every $t \leq 0$. We also observe that the space $C_r \times L^p(g;X)$ verifies axiom C_2, see [18, p.10] for details.

Remark 2. In the rest of this section, M_b and K_b are the constants $M_b = \sup_{s\in[0,b]} M(s)$ and $K_b = \sup_{s\in[0,b]} K(s)$.

Lemma 3.1 ([14, Lemma 2.1])**.** *Let $x : (-\infty, b] \to X$ is continuous on $[0,b]$ and $x_0 = \varphi$. If ($\mathbf{H}_\varphi$) be hold, then*

$$\| x_s \|_{\mathcal{B}} \leq (M_b + J^\varphi) \| \varphi \|_{\mathcal{B}} + K_b \sup\{\| x(\theta) \|; \theta \in [0, \max\{0,s\}] \},$$

$s \in \mathcal{R}(\rho^-) \cup I$, *where* $J^{\varphi} = \sup_{t \in \mathcal{R}(\rho^-)} J^{\varphi}(t)$.

Theorem 3.5. *Let conditions* $(\mathbf{H_1})$ *and* $(\mathbf{H}_{\varphi})$ *be hold and assume that* $\mathcal{R}(\cdot) \in C((0,b]; \mathcal{L}(X))$. *If* $M \liminf_{\xi \to \infty} \frac{\Omega_f(\xi)}{\xi} \int_0^b m_f(s) ds < 1$, *then there exists a mild solution of (1.1)-(1.2) on* $[0, b]$.

Proof: Let $\bar{\varphi} : (-\infty, b] \to X$ be the extension of φ to $(-\infty, b]$ such that $\bar{\varphi}(\theta) = \varphi(0)$ on $I = [0, b]$. Consider the space $S(b) = \{u \in C(I; X) : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology and define the operator $\Gamma : S(b) \to S(b)$ by

$$\Gamma x(t) \quad = \quad \mathcal{R}(t)\varphi(0) + \int_0^t \mathcal{R}(t-s)(f_1'(s) + f_2(s))ds + \int_0^t \mathcal{R}(t-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)})ds,$$

for $t \in [0, b]$. It is easy to see that $\Gamma S(b) \subset S(b)$. We prove that there exists $r > 0$ such that $\Gamma(B_r(\bar{\varphi}|_I, S(b))) \subseteq B_r(\bar{\varphi}|_I, S(b))$. If this property is false, then for every $r > 0$ there exist $x^r \in B_r(\bar{\varphi}|_I, S(b))$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - \varphi(0)\|$. Then, from Lemma 3.1 we find that

$$\begin{aligned}
\| \Gamma x^r(t^r) - \varphi(0) \| \quad &\leq \quad \| \mathcal{R}(t^r)\varphi(0) - \varphi(0) \| + M \| f_1' + f_2 \|_{L^1([0,b],X)} \\
&\quad + M \int_0^{t^r} m_f(t^r - s) \Omega_f(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}}) ds \\
&\leq \quad (M+1) H \| \varphi \|_{\mathcal{B}} + M \| f_1' + f_2 \|_{L^1([0,b],X)} \\
&\quad + \Omega_f \left((M_b + J^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b(r + \|\varphi(0)\|) \right) \int_0^b m_f(s) ds.
\end{aligned}$$

Therefore

$$1 \leq M \liminf_{\xi \to \infty} \frac{\Omega_f(\xi)}{\xi} \int_0^b m_f(s) ds,$$

which contradicts our assumption.

Let $r > 0$ be such that $\Gamma(B_r(\bar{\varphi}|_I, S(b))) \subseteq B_r(\bar{\varphi}|_I, S(b))$, in the sequel, r^* is the number defined by $r^* := (M_b + J^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b(r + \|\varphi(0)\|)$. To prove that Γ is a condensing operator, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\begin{aligned}
\Gamma_1 x(t) \quad &= \quad \mathcal{R}(t)\varphi(0) + \int_0^t \mathcal{R}(t-s)(f_1'(s) + f_2(s))ds, \\
\Gamma_2 x(t) \quad &= \quad \int_0^t \mathcal{R}(t-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)})ds,
\end{aligned}$$

for $t \in I$.

It is easy to see that $\Gamma_1(\cdot)$ is continuous and a contraction on $B_r(\bar{\varphi}|_I, S(b))$. Next we prove that $\Gamma_2(\cdot)$ is completely continuous from $B_r(\bar{\varphi}|_I, S(b))$ into $B_r(\bar{\varphi}|_I, S(b))$.

Step 1. The set $\Gamma_2(B_r(\bar{\varphi}|_I, S(b))(t)$ is relatively compact on X for every $t \in [0, b]$.

The case $t = 0$ is trivial. Let $0 < \epsilon < t < b$. From the assumptions, we can fix numbers $0 = t_0 < t_1 < \cdots < t_n = t - \epsilon$ such that $\| \mathcal{R}(t-s) - \mathcal{R}(t-s') \| \leq \epsilon$ if $s, s' \in [t_i, t_{i+1}]$, for some $i = 0, 1, 2, \cdots, n-1$. Let $x \in B_r(\bar{\varphi}|_I, S(b))$, under theses conditions, from the mean value theorem for the Bochner Integral (see [20, Lemma 2.1.3]) we see that

$$\begin{aligned}
\Gamma_2 x(t) = & \\
& \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \mathcal{R}(t-t_i) f(s, \overline{x}_{\rho(t,\overline{x}_s)}) ds \\
& + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (\mathcal{R}(t-s) - \mathcal{R}(t-t_i)) f(s, \overline{x}_{\rho(t,\overline{x}_s)}) ds \\
& + \int_{t_n}^{t} \mathcal{R}(t-s) f(s, \overline{x}_{\rho(t,\overline{x}_s)}) ds \\
\in \ & \sum_{i=1}^{n} (t_i - t_{i-1}) \overline{co(\{\mathcal{R}(t-t_i) f(s,\psi) : \psi \in B_{r^*}(0, \mathcal{B}), s \in [0,b]\})} \\
& + \epsilon\, r^{**} + M \Omega_f(r^*) \int_{t-\epsilon}^{t} m_f(s) ds \\
\in \ & \sum_{i=1}^{n} (t_i - t_{i-1}) \overline{co(\{\mathcal{R}(t-t_i) f(s,\psi) : \psi \in B_{r^*}(0, \mathcal{B}), s \in [0,b]\})} + \epsilon B_{r^{**}}(0, X) + C_\epsilon,
\end{aligned}$$

the first term of the left-hand side belong to a compact set in X and $diam(C_\epsilon) \to 0$ when $\epsilon \to 0$. This proves that $\Gamma_2(B_r(\bar{\varphi}|_I, S(b)))(t)$ is totally bounded and hence relatively compact in X for every $t \in [0, b]$.

Step 2. The set $\Gamma_2(B_r(\bar{\varphi}|_I, S(b)))$ is equicontinuous on $[0, b]$.

Let $0 < \epsilon < t < b$ and $0 < \delta < \epsilon$ such that $\| \mathcal{R}(s) - \mathcal{R}(s') \| \leq \epsilon$ for every $s, s' \in [\epsilon, b]$ with $| s - s' | \leq \delta$. Under these conditions, for $x \in B_r(\bar{\varphi}|_I, S(b))$ and $0 < h \leq \delta$ with $t + h \in [0, b]$, we get

$$\begin{aligned}
& \| \Gamma_2 x(t+h) - \Gamma_2 x(t) \| \\
& \quad \leq \int_0^{t-\epsilon} [\mathcal{R}(t+h-s) - \mathcal{R}(t-s)] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds \\
& \quad\quad + \int_{t-\epsilon}^{t} [\mathcal{R}(t+h-s) - \mathcal{R}(t-s)] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds \\
& \quad\quad + \int_{t}^{t+h} \mathcal{R}(t+h-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds \\
& \quad \leq \epsilon\, r^{**} + 2M\Omega_f(r^*) \int_{t-\epsilon}^{t} m_f(s) ds + M\Omega_f(r^*) \int_{t}^{t+h} m_f(s) ds
\end{aligned}$$

which shows that the set of functions $\Gamma_2(B_r(\bar{\varphi}|_I, S(b)))$ is right equicontinuous at $t \in (0, b)$. A similar procedure permit to prove the right equicontinuity at zero and the left equicontinuity at $t \in (0, b]$. Thus, $\Gamma_2(B_r(\bar{\varphi}|_I, S(b)))$ is equicontinuous. By using a procedure similar to the proof of [14, Theorem 2.3], we prove that that $\Gamma_2(\cdot)$ is continuous on $B_r(\bar{\varphi}|_I, S(b))$, which completes the proof that $\Gamma_2(\cdot)$ is completely continuous.

The existence of a mild solution for (1.1)-(1.2) is now a consequence of [20, Theorem 4.3.2]. This completes the proof. ■

Theorem 3.6. *Let conditions* $(\mathbf{H_1}), (\mathbf{H}_\varphi)$ *be hold,* $\rho(t,\psi) \leq t$ *for every* $(t,\psi) \in I \times \mathcal{B}$ *and assume that* $\mathcal{R}(\cdot) \in C((0,b]; \mathcal{L}(X))$*. If*

$$K_b M \int_0^b m_f(s)ds < \int_c^\infty \frac{1}{\Omega_f(s)} ds, \tag{3.1}$$

where $c = (M_b + K_b M H) \parallel \varphi \parallel_{\mathcal{B}} + K_b M \parallel f_1' + f_2 \parallel_{L^1([0,b],X)}$*, then there exists a mild solution of (1.1)-(1.2) on* $[0,b]$*.*

Proof: Let on the space $\mathcal{B}S(b) = \{u : (-\infty, b] \to X; u_0 = 0, u|_I \in C(I;X)\}$ endowed with the uniform convergence topology. We define the operator $\Gamma : \mathcal{B}S(b) \to \mathcal{B}S(b)$ by

$$\Gamma x(t) = \begin{cases} \displaystyle\int_0^t \mathcal{R}(t-s) f(s, \widetilde{x}_{\rho(s,\widetilde{x}_s)})ds + \int_0^t \mathcal{R}(t-s)(f_1'(s) + f_2(s))ds & t \in I = [0,b], \\ 0, & t \in (-\infty, 0], \end{cases}$$

In the sequel, we prove that Γ verifies the conditions of Theorem 2.4. We next establish an *a priori* estimate for the solutions of the integral equation $x = \lambda \Gamma x$ for $\lambda \in (0,1)$. Let x^λ be a solution of $x = \lambda \Gamma x$, $\lambda \in (0,1)$. By using Lemma 3.1, the notation

$$\alpha^\lambda(s) = (M_b + K_b M H) \parallel \varphi \parallel_{\mathcal{B}}) + K_b \|x^\lambda\|_s,$$

and the fact that $\rho(s, \widetilde{(x^\lambda)}_s) \leq s$, for each $s \in I$, we find that

$$\begin{aligned} \alpha^\lambda(t) \ &\leq \ (M_b + K_b M H) \parallel \varphi \parallel_{\mathcal{B}} + K_b M \parallel f_1' + f_2 \parallel_{L^1([0,b],X)} \\ &\quad + K_b M \int_0^t m_f(s) \Omega_f(\alpha^\lambda(s))ds. \end{aligned}$$

Denoting by $\beta_\lambda(t)$ the right hand side of the last inequality, we obtain that

$$\beta_\lambda'(t) \leq K_b M m_f(t) \Omega_f(\beta_\lambda(t))$$

and hence,

$$\int_c^{\beta_\lambda(t)} \frac{1}{\Omega_f(s)} ds \leq K_b M \int_0^b m_f(s) ds.$$

This inequality and (3.1) permit us to conclude that the set of functions $\{\beta_\lambda : \lambda \in (0,1)\}$ is bounded, which in turn shows that $\{x^\lambda : \lambda \in (0,1)\}$ is bounded in $\mathcal{B}S(b)$. A procedure similar to the proof of Theorem 3.5 allows us to show that Γ is completely continuous on $\mathcal{B}S(b)$. By the Theorem 2.4 the proof is ended. ■

4. Example

To finish this paper, we discuss the existence of solutions for the partial integro-differential system

$$\frac{\partial}{\partial t}\left[u(t,\xi)+\int_{-\infty}^{t}(t-s)^{\alpha}e^{-\omega(t-s)}u(s,\xi)ds\right]=\frac{\partial^2 u(t,\xi)}{\partial \xi^2}+\int_{-\infty}^{t}e^{-\gamma(t-s)}\quad\frac{\partial^2 u(s,\xi)}{\partial \xi^2}ds$$

$$+\int_{-\infty}^{t}a(s-t)u(s-\rho_1(t)\rho_2(\| u(t) \|),\xi)ds,\quad (t,\xi)\in I\times[0,\pi], \tag{4.1}$$

$$u(t,0) = u(t,\pi)=0, \qquad t\in[0,b], \tag{4.2}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \qquad \theta\leq 0,\ \xi\in[0,\pi]. \tag{4.3}$$

In this system, $\alpha\in(0,1)$, ω, γ are positive numbers, and $a:[0,\infty)\to\mathbb{R}$ is an appropriate function. Moreover, we have identified $\varphi(\theta)(\xi)=\varphi(\theta,\xi)$.

To represent this system in the abstract form (1.1)-(1.2), we choose the spaces $X=L^2([0,\pi])$ and $\mathcal{B}=C_0\times L^2(g,X)$, see Example 2.1 for details. We also consider the operators $A,B(t):D(A)\subseteq X\to X$, $t\geq 0$, given by $Ax=x''$, $B(t)x=e^{-\gamma t}Ax$ for $x\in D(A)=\{x\in X: x''\in X,\ x(0)=x(\pi)=0\}$ and $N(t)x=t^{\alpha}e^{-\omega t}x$ for $x\in X$.

The operator A is the infinitesimal generator of an analytic semigroup, $\rho(A)=\mathbb{C}\backslash\{-n^2: n\in\mathbb{N}\}$ and for all $\vartheta\in(\pi/2,\pi)$ there exists $M_{\vartheta}>0$ such that $\| R(\lambda,A)\|\leq M_{\vartheta}\mid\lambda\mid^{-1}$ for all $\lambda\in\Lambda_{\vartheta}$. Moreover, it is easy to see that conditions **(P2)-(P4)** in Section 2 are satisfied with $\widehat{N}(\lambda)=\frac{\Gamma(\alpha+1)}{(\lambda+\omega)^{\alpha+1}}$, $b(t)=e^{-\gamma t}$ and $D=C_0^{\infty}([0,\pi])$, where Γ is the gamma function and $C_0^{\infty}([0,\pi])$ is the space of infinitely differentiable functions that vanish at $\xi=0$ and $\xi=\pi$.

Under the above conditions we can represent the system

$$\frac{\partial}{\partial t}\left[u(t,\xi)+\int_{0}^{t}(t-s)^{\alpha}e^{-\omega(t-s)}u(s,\xi)ds\right]=\frac{\partial^2 u(t,\xi)}{\partial \xi^2}+\int_{0}^{t}e^{-\gamma(t-s)}\frac{\partial^2 u(s,\xi)}{\partial \xi^2}ds, \tag{4.4}$$

$$u(t,\pi)=u(t,0)=0, \tag{4.5}$$

in the abstract form (2.2)-(2.3).

The Proposition 4.1 below, is a consequence of [6, Theorem 2.1] and [6, Lemma 3.11] .

Proposition 4.1. *There exists a compact operator resolvent for (4.4)-(4.5).*

We next consider the problem of the existence of mild solutions for the system (4.1)-(4.3). To this end, we introduce the following conditions.

- **(a)** The function $a(\cdot)$ is continuous and $L_f=\left(\int_{-\infty}^{0}\frac{|a(-s)|^2}{g(s)}ds\right)^{\frac{1}{2}}<\infty$.
- **(b)** The functions $\varphi, A\varphi$ belong to $\mathcal{B}$ and the expressions $\sup\limits_{t\in[0,b]}\left[\int_{-\infty}^{0}\frac{(t-\tau)^{2\alpha}}{g(\tau)}e^{2\omega\tau}d\tau\right]^{\frac{1}{2}}$ and $\left(\int_{-\infty}^{0}\frac{e^{2\gamma\tau}}{g(\tau)}d\tau\right)^{\frac{1}{2}}$ are finite.

Under the conditions (**a**) and (**b**), the functions $f : [0,b]\times\mathcal{B} \to X$, $f_i : [0,b] \to X$, $i = 1,2$, given by

$$f(t,\varphi)(\xi) = \int_{-\infty}^{0} a(-s)\varphi(s,\xi)ds, \quad f_1(t)(\xi) = \int_{-\infty}^{0} (t-s)^{\alpha} e^{-\omega(t-s)}\varphi(s,\xi)ds,$$
$$f_2(t)(\xi) = \int_{-\infty}^{0} e^{-\gamma(t-s)} A\varphi(s,\xi)ds, \quad \rho(s,\psi) = \rho_1(s)\rho_2(\parallel \varphi(0) \parallel),$$

are well defined, which us to permit re-write the system (4.1)-(4.3) in the abstract form

$$\frac{d}{dt}[\, x(t) + \int_0^t N(t-s)x(s)ds + f_1(t)\,]$$
$$= Ax(t) + \int_0^t B(t-s)x(s)ds + f_2(t) + f(t, x_{\rho(t,x_t)}), \qquad t \in [0,b], \tag{4.6}$$
$$x_0 = \varphi \in \mathcal{B}. \tag{4.7}$$

We said that a function $u \in C([0,b];X)$ is a mild solution of (4.1)-(4.3) if $u(\cdot)$ is a mild solution of the associated abstract system (4.6)-(4.7).

Proposition 4.2. *Let $\varphi \in \mathcal{B}$ be such that condition $(\mathbf{H}_\varphi)$ holds, the functions ρ_1, ρ_2 are bounded and assume that the above conditions are fulfilled. If $L_f < 1$ and*

$$\sup_{t\in[0,a]} \left[\int_{-\infty}^{0} \frac{1}{\rho(\tau)} [\frac{e^{-\omega(t-\tau)}}{(t-\tau)^{1-\alpha}}]^2 d\tau \right]^{\frac{1}{2}} < \infty, \tag{4.8}$$

then there exists a mild solution of (4.1)-(4.3) on $[0,b]$.

Proof: From the condition (**a**) it is easy to see that f is a bounded linear operator with $\parallel f \parallel_{\mathcal{L}(\mathcal{B},X)} \leq L_f$, and from condition (**b**) it follows that f_1 and f_2 are continuous. If the condition (4.8) is valid, then f_1 is differentiable and

$$f_1'(t)(\xi) = \int_{-\infty}^{0} [(t-s)^{\alpha-1} + \omega(t-s)^{\alpha}]e^{-\omega(t-s)}\varphi(s,\xi)ds, \;\; \forall\,(t,\xi) \in [0,b]\times[0,\pi]. \tag{4.9}$$

Moreover, using this expression we can prove that $f_1 \in C^1([0,b],X)$. Finally, from Theorem 3.5 we can assert that there exists a unique mild solution for the system (4.1)-(4.3) on $[0,b]$. ■

Acknowledgment. The author wishes to thank the referee for comments and suggestions.

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(Received February 6, 2010)

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