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HIGHER ORDER MULTI-POINT BOUNDARY VALUE PROBLEMS WITH SIGN-CHANGING NONLINEARITIES AND NONHOMOGENEOUS BOUNDARY CONDITIONS

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ABSTRACT. We study classes of nth order boundary value problems consisting of an equation having a sign-changing nonlinearity $f(t,x)$ together with several different sets of nonhomogeneous multi-point boundary conditions. Criteria are established for the existence of nontrivial solutions, positive solutions, and negative solutions of the problems under consideration. Conditions are determined by the behavior of $f(t,x)/x$ near 0 and $\pm\infty$ when compared to the smallest positive characteristic values of some associated linear integral operators. This work improves and extends a number of recent results in the literature on this topic. The results are illustrated with examples.

1. INTRODUCTION

Throughout this paper, let $m \geq 1$ be an integer, and for any $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, we write $\overline{x} = \sum_{i=1}^m |x_i|$ and $\langle x, y\rangle = \sum_{i=1}^m x_i y_i$. Let

$$\alpha = (\alpha_1, \ldots, \alpha_m),\ \beta = (\beta_1, \ldots, \beta_m),\ \gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}_+^m,$$

and

$$\xi = (\xi_1, \ldots, \xi_m) \in (0,1)^m$$

be fixed, where $\mathbb{R}_+ = [0,\infty)$ and ξ_i, $i = 1, \ldots, m$, satisfy $0 < \xi_1 < \xi_2 < \ldots < \xi_m < 1$. To clarify our notation, we wish to point out that while $u(t)$ is a scalar valued function, $u(\xi) = (u(\xi_1), \ldots, u(\xi_m))$ is a vector. In this paper, we are concerned with the existence of nontrivial solutions of boundary value problems (BVPs) consisting of the scalar nth order differential equation

$$u^{(n)} + g(t)f(t,u) = 0,\ t \in (0,1), \tag{1.1}$$

and one of the three nonhomogeneous multi–point boundary conditions (BCs)

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi)\rangle + \lambda_i,\ i = 0, \ldots, n-3, \\ u^{(n-1)}(0) = \langle \beta, u^{(n-1)}(\xi)\rangle - \lambda_{n-2}, \\ u^{(n-2)}(1) = \langle \gamma, u^{(n-2)}(\xi)\rangle + \lambda_{n-1}, \end{cases} \tag{1.2}$$

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$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi) \rangle + \lambda_i, \ i = 0, \ldots, n-3, \\ u^{(n-2)}(0) = \langle \beta, u^{(n-2)}(\xi) \rangle + \lambda_{n-2}, \\ u^{(n-1)}(1) = \langle \gamma, u^{(n-1)}(\xi) \rangle + \lambda_{n-1}, \end{cases} \tag{1.3}$$

and

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi) \rangle + \lambda_i, \ i = 0, \ldots, n-3, \\ u^{(n-2)}(0) = \langle \beta, u^{(n-2)}(\xi) \rangle + \lambda_{n-2}, \\ u^{(n-2)}(1) = \langle \gamma, u^{(n-2)}(\xi) \rangle + \lambda_{n-1}, \end{cases} \tag{1.4}$$

where $n \geq 2$ is an integer, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : (0,1) \to \mathbb{R}_+$ are continuous, $g \not\equiv 0$ on any subinterval of $(0,1)$, $\lambda_i \in \mathbb{R}_+$, and $u^{(i)}(\xi) = \big(u^{(i)}(\xi_1), \ldots, u^{(i)}(\xi_m)\big)$ for $i = 0, \ldots, n-1$. By a *nontrivial solution* of BVP (1.1), (1.2), we mean a function $u \in C^{n-1}[0,1] \cap C^n(0,1)$ such that $u(t) \not\equiv 0$ on $(0,1)$, $u(t)$ satisfies Eq. (1.1) and BC (1.2). If $u(t) > 0$ on $(0,1)$, then $u(t)$ is a *positive solution*. Similar definitions also apply for BVPs (1.1), (1.3) and (1.1), (1.4) as well as for *negative solutions* of these problems.

We remark that in case $n = 2$, the first equations in BCs (1.2), (1.3), and (1.4) vanish, and BVPs (1.1), (1.2) and (1.1), (1.3) and (1.1), (1.4) now reduce to the second order BVPs consisting of the equation

$$u'' + g(t)f(t,u) = 0, \ t \in (0,1), \tag{1.5}$$

one of the BCs

$$u'(0) = \langle \beta, u'(\xi) \rangle - \lambda_0, \ u(1) = \langle \gamma, u(\xi) \rangle + \lambda_1, \tag{1.6}$$

$$u(0) = \langle \beta, u(\xi) \rangle + \lambda_0, \ u'(1) = \langle \gamma, u'(\xi) \rangle + \lambda_1, \tag{1.7}$$

and

$$u(0) = \langle \beta, u(\xi) \rangle + \lambda_0, \ u(1) = \langle \gamma, u(\xi) \rangle + \lambda_1. \tag{1.8}$$

When f is positone (i.e., $f \geq 0$), existence of solutions of the above second order BVPs, or some of their variations, has been extensively investigated in recent years. For instance, papers [10, 21, 22, 23, 25, 26, 29, 41, 42] studied BVPs with one–parameter BCs and [13, 14, 15, 16, 17] studied BVPs with two–parameter BCs. In particular, for one–parameter problems, Ma [25] studied BVP (1.5), (1.8) with $m = 1$ and $\beta = \lambda_0 = 0$. Under certain assumptions, he showed that there exists $\lambda_1^* > 0$ such that BVP (1.5), (1.8) has at least one positive solution for $0 < \lambda_1 < \lambda_1^*$ and has no positive solution for $\lambda_1 > \lambda_1^*$; later, Sun et al. [29] proved similar results for BVP (1.5), (1.6) with $\beta = (0, \ldots, 0)$ and $\lambda_0 = 0$; Kwong and Wong [21] further significantly improved the results in [29] and also constructed a counterexample to point out that one of the main results in [29] is actually false; Zhang and Sun [41] recently obtained results, similar to those in [25], for BVP (1.5), (1.7) with $\lambda_0 = 0$. Paper [21] does contain some optimal existence criteria. As for the second order two–parameter problems, Kong and Kong [13, 14, 15, 16] studied BVPs (1.5), (1.6) and (1.5), (1.8) with λ_0, $\lambda_1 \in \mathbb{R}$ and established many existence, nonexistence, and multiplicity results for positive solutions of the problems. Moreover, under some conditions, they proved that there exists a continuous curve Γ separating the (λ_0, λ_1)–plane into two disjoint connected regions Λ^E and Λ^N with $\Gamma \subseteq \Lambda^E$ such that BVPs (1.5), (1.6) and (1.5), (1.8) have at least two solutions for $(\lambda_0, \lambda_1) \in \Lambda^E \setminus \Gamma$, have at least one solution for $(\lambda_0, \lambda_1) \in \Gamma$, and have no solution for $(\lambda_0, \lambda_1) \in \Lambda^N$. The uniqueness of positive solutions and the dependence of positive solutions on the parameters λ_0 and λ_1 are investigated in [17] for BVP (1.5), (1.8). Recently, higher order positone BVPs with nonhomogeneous BCs have also been studied in the literature, for example, in [7, 8, 18, 19, 20, 28, 31, 32, 37]. In particular, paper [20] studied BVPs (1.1), (1.2) and (1.1), (1.3) and proved several optimal existence criteria for positive solutions of these problems under the assumption that f is nonnegative. In the present paper, we allow f to change sign.

However, very little has been done in the literature on BVPs with nonhomogeneous BCs when the nonlinearities are sign–changing functions. As far as we know, the only work to tackle this situation is the recent paper [6], where BVP (1.5), (1.8) is considered with $m = 2$, $\beta = (\beta_1, 0)$, and $\gamma = (0, \gamma_2)$, and where sufficient conditions for the existence of nontrivial solutions are obtained. Motivated partially by the recent papers [6, 12, 18, 20, 24], here we will derive several new criteria for the existence of nontrivial solutions, positive solutions, and negative solutions of BVPs (1.1), (1.2) and (1.1), (1.3) and (1.1), (1.4) when the nonlinear term f is a sign-changing function and not necessarily bounded from below. The proof uses topological degree theory together with a comparison between the behavior of the quotient $f(t, x)/x$ for x near 0 and $\pm\infty$ and the smallest positive characteristic values (given by (3.1) below) of some related linear operators L, $\tilde{L}$, and $\hat{L}$ (defined by (2.31)–(2.33) in Section 2). These characteristic values are known to exist by the Krein–Rutman theorem. Our results extend and improve many recent works on BVPs with nonhomogeneous BCs, especially those in papers [6, 10, 13, 14, 15, 16, 18, 19, 20, 25, 29, 41, 42]. We believe that our results are new even for homogeneous problems, i.e., when $\lambda_i = 0$, $i = 0, \ldots, n-1$, in BCs (1.2), (1.3), and (1.4). For other studies on optimal existence criteria on BVPs with homogeneous BCs, we refer the reader to [2, 3, 11, 27, 30, 33, 34, 35, 36, 38, 40] and the references therein. In particular, Webb and Infante [35] studied some higher order problems and obtained sharp results for the existence of one positive solution. They gave some non-existence results as well. The nonlocal boundary conditions in [35] are different from the ones studied here. Webb and Infante were mainly concerned with homogeneous boundary conditions but nonhomogeneous boundary conditions were also treated.

We assume the following condition holds throughout without further mention:

(H) $0 \le \overline{\alpha} < 1,\ \ 0 \le \overline{\beta} < 1,\ \ 0 \le \overline{\gamma} < 1,$ and $\int_0^1 g(s)ds < \infty$.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas, Sections 3 contains the main results of this paper and several examples, and the proofs of the main results are presented in Section 4.

2. Preliminary results

In this section, we present some preliminary results that will be used in the statements and the proofs of the main results. In the rest of this paper, the bold $\mathbf{0}$ stands for the zero element in any given Banach space. We refer the reader to [9, Lemma 2.5.1] for the proof of the following well known lemma.

Lemma 2.1. *Let Ω be a bounded open set in a real Banach space X with $\mathbf{0} \in \Omega$ and $T : \overline{\Omega} \to X$ be compact. If*

$$Tu \neq \tau u \quad \text{for all } u \in \partial\Omega \text{ and } \tau \ge 1,$$

then the Leray-Schauder degree

$$\deg(I - T, \Omega, \mathbf{0}) = 1.$$

Let $(X,\ ||\cdot||)$ be a real Banach space and $L : X \to X$ be a linear operator. We recall that λ is an *eigenvalue* of L with a corresponding *eigenfunction* φ if φ is nontrivial and $L\varphi = \lambda\varphi$. The reciprocals of eigenvalues are called the *characteristic values* of L. Recall also that a cone P in X is called a *total cone* if $X = \overline{P - P}$.

The following Krein-Rutman theorem can be found in either [1, Theorem 19.2] or [39, Proposition 7.26].

Lemma 2.2. *Assume that P is a total cone in a real Banach space X. Let $L : X \to X$ be a compact linear operator with $L(P) \subseteq P$ and the spectral radius, r_L, of L satisfy $r_L > 0$. Then r_L is an eigenvalue of L with an eigenfunction in P.*

Let X^* be the dual space of X, P be a total cone in X, and P^* be the dual cone of P, i.e.,

$$P^* = \{l \in X^* \ : \ l(u) \geq 0 \text{ for all } u \in P\}.$$

Let $L, M : X \to X$ be two linear compact operators such that $L(P) \subseteq P$ and $M(P) \subseteq P$. If their spectral radii r_L and r_M are positive, then by Lemma 2.2, there exist $\varphi_L, \varphi_M \in P \setminus \{\mathbf{0}\}$ such that

$$L\varphi_L = r_L\varphi_L \quad \text{and} \quad M\varphi_M = r_M\varphi_M. \tag{2.1}$$

Assume there exists $h \in P^* \setminus \{\mathbf{0}\}$ such that

$$L^*h = r_M h, \tag{2.2}$$

where L^* is the dual operator of L. Choose $\delta > 0$ and define

$$P(h, \delta) = \{u \in P \ : \ h(u) \geq \delta||u||\}. \tag{2.3}$$

Then, $P(h, \delta)$ is a cone in X.

In the following, Lemma 2.3 is a generalization of [12, Theorem 2.1] and it is proved in [24, Lemma 2.5] for the case when L and M are two specific linear operators, but the proof there also works for any general linear operators L and M satisfying (2.1) and (2.2). Lemma 2.4 generalizes [4, Lemma 3.5] and it is proved in [5, Lemma 2.5]. From here on, for any $R > 0$, let $B(\mathbf{0}, R) = \{u \in X \ : \ ||u|| < R\}$ be the open ball of X centered at $\mathbf{0}$ with radius R.

Lemma 2.3. *Assume that the following conditions hold:*

(A1) *There exist φ_L, $\varphi_M \in P \setminus \{\mathbf{0}\}$ and $h \in P^* \setminus \{\mathbf{0}\}$ such that* (2.1) *and* (2.2) *hold and $L(P) \subseteq P(h, \delta)$;*

(A2) $H : X \to P$ *is a continuous operator and satisfies*

$$\lim_{||u||\to\infty} \frac{||Hu||}{||u||} = 0;$$

(A3) $F : X \to X$ *is a bounded continuous operator and there exists* $u_0 \in X$ *such that* $Fu + Hu + u_0 \in P$ *for all* $u \in X$*;*

(A4) *There exist* $v_0 \in X$ *and* $\epsilon > 0$ *such that*

$$LFu \geq r_M^{-1}(1+\epsilon)Lu - LHu - v_0 \quad \textit{for all } u \in X.$$

Let $T = LF$*. Then there exists* $R > 0$ *such that the Leray-Schauder degree*

$$\deg(I - T, B(\mathbf{0}, R), \mathbf{0}) = 0.$$

Lemma 2.4. *Assume that (A1) and the following conditions hold:*

(A2)* $H : X \to P$ *is a continuous operator and satisfies*

$$\lim_{||u||\to 0} \frac{||Hu||}{||u||} = 0;$$

(A3)* $F : X \to X$ *is a bounded continuous operator and there exists* $r_1 > 0$ *such that*

$$Fu + Hu \in P \quad \textit{for all } u \in X \textit{ with } ||u|| < r_1;$$

(A4)* *There exist* $\epsilon > 0$ *and* $r_2 > 0$ *such that*

$$LFu \geq r_M^{-1}(1+\epsilon)Lu \quad \textit{for all } u \in X \textit{ with } ||u|| < r_2.$$

Let $T = LF$*. Then there exists* $0 < R < \min\{r_1, r_2\}$ *such that the Leray-Schauder degree*

$$\deg(I - T, B(\mathbf{0}, R), \mathbf{0}) = 0.$$

Now let

$$G(t,s) = \begin{cases} 1-s, & 0 \leq t \leq s \leq 1, \\ 1-t, & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.4}$$

$$\tilde{G}(t,s) = \begin{cases} t, & 0 \leq t \leq s \leq 1, \\ s, & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.5}$$

and

$$\hat{G}(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.6}$$

Then, it is well known that $G(t,s)$ is the Green's function of the BVP

$$-u'' = 0 \quad \text{on } (0,1),\ u'(0) = u(1) = 0,$$

$\tilde{G}(t,s)$ is the Green's function of the BVP

$$-u'' = 0 \quad \text{on } (0,1),\ u(0) = u'(1) = 0,$$

and $\hat{G}(t,s)$ is the Green's function of the BVP

$$-u'' = 0 \quad \text{on } (0,1),\ u(0) = u(1) = 0.$$

Recall that the characteristic function χ on an interval I is given by

$$\chi_I(t) = \begin{cases} 1, & t \in I, \\ 0, & t \notin I. \end{cases}$$

In the sequel, we write

$$G(\xi,s) = \big(G(\xi_1,s),\ldots,G(\xi_m,s)\big),$$

$$\chi_{[0,\xi]}(s) = \big(\chi_{[0,\xi_1]}(s),\ldots,\chi_{[0,\xi_m]}(s)\big),$$

$$\xi(1-\xi) = \big(\xi_1(1-\xi_1),\ldots,\xi_m(1-\xi_m)\big),$$

and for any $v \in C[0,1]$, we let

$$v(\xi) = (v(\xi_1),\ldots,v(\xi_m)).$$

We also use some other similar notations that will be clear from the context and will not be listed here.

Define

$$H_1(t,s) = \frac{\langle \alpha, \chi_{[0,\xi]}(s)\rangle}{1-\overline{\alpha}} + \chi_{[0,t]}(s), \tag{2.7}$$

$$H_0(t,s) = G(t,s) + \frac{\langle \gamma, G(\xi,s)\rangle}{1-\overline{\gamma}} + \frac{[1-\langle\gamma,\xi\rangle-(1-\overline{\gamma})t]\langle\beta,\chi_{[0,\xi]}(s)\rangle}{(1-\overline{\beta})(1-\overline{\gamma})}, \tag{2.8}$$

$$\tilde{H}_0(t,s) = \tilde{G}(t,s) + \frac{\langle \beta, \tilde{G}(\xi,s)\rangle}{1-\overline{\beta}} + \frac{[\langle\beta,\xi\rangle+(1-\overline{\beta})t]\langle\gamma,\chi_{[\xi,1]}(s)\rangle}{(1-\overline{\beta})(1-\overline{\gamma})}, \tag{2.9}$$

$$\begin{aligned} \hat{H}_0(t,s) \quad = \quad & \hat{G}(t,s) + \frac{t}{\rho}[(1-\overline{\beta})\langle\gamma,\hat{G}(\xi,s)\rangle - (1-\overline{\gamma})\langle\beta,\hat{G}(\xi,s)\rangle] \\ & + \frac{1}{\rho}[(1-\langle\gamma,\xi\rangle)\langle\beta,\hat{G}(\xi,s)\rangle + \langle\beta,\xi\rangle\langle\gamma,\hat{G}(\xi,s)\rangle], \end{aligned} \tag{2.10}$$

where

$$\rho = (1-\overline{\beta})(1-\langle\gamma,\xi\rangle) + (1-\overline{\gamma})\langle\beta,\xi\rangle > 0. \tag{2.11}$$

For $i = 1, \dots, n-1$, define $K_i(t,s)$, $\tilde{K}_i(t,s)$, and $\hat{K}_i(t,s)$ recursively as follows

$$K_1(t,s) = H_0(t,s),\ K_i(t,s) = \int_0^1 H_1(t,\tau)K_{i-1}(\tau,s)d\tau,\ i = 2,\dots,n-1, \tag{2.12}$$

$$\tilde{K}_1(t,s) = \tilde{H}_0(t,s),\ \tilde{K}_i(t,s) = \int_0^1 H_1(t,\tau)\tilde{K}_{i-1}(\tau,s)d\tau,\ i = 2,\dots,n-1,$$

and

$$\hat{K}_1(t,s) = \hat{H}_0(t,s),\ \hat{K}_i(t,s) = \int_0^1 H_1(t,\tau)\hat{K}_{i-1}(\tau,s)d\tau,\ i = 2,\dots,n-1.$$

Remark 2.1. It is easy to see that, for $i = 1, \dots, n-1$, $K_i(t,s) \geq 0$, $\tilde{K}_i(t,s) \geq 0$, $\hat{K}_i(t,s) \geq 0$ for $t,s \in [0,1]$, and $K_i(t,s) > 0$, $\tilde{K}_i(t,s) > 0$, $\hat{K}_i(t,s) > 0$ for $t,s \in (0,1)$.

The following lemma provides the equivalent integral forms for some BVPs.

Lemma 2.5. *Let $k \in L(0,1) \cap C(0,1)$. Then we have the following:*

(a) *The function $u(t)$ is a solution of the BVP consisting of the equation*

$$u^{(n)} + k(t) = 0,\ t \in (0,1), \tag{2.13}$$

and the BC

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi)\rangle,\ i = 0,\dots,n-3, \\ u^{(n-1)}(0) = \langle \beta, u^{(n-1)}(\xi)\rangle, \\ u^{(n-2)}(1) = \langle \gamma, u^{(n-2)}(\xi)\rangle, \end{cases}$$

if and only if

$$u(t) = \int_0^1 K_{n-1}(t,s)k(s)ds.$$

(b) *The function $u(t)$ is a solution of the BVP consisting of Eq.* (2.13) *and the BC*

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi)\rangle,\ i = 0,\dots,n-3, \\ u^{(n-2)}(0) = \langle \beta, u^{(n-2)}(\xi)\rangle, \\ u^{(n-1)}(1) = \langle \gamma, u^{(n-1)}(\xi)\rangle, \end{cases}$$

if and only if

$$u(t) = \int_0^1 \tilde{K}_{n-1}(t,s)k(s)ds.$$

(c) *The function $u(t)$ is a solution of the BVP consisting of Eq.* (2.13) *and the BC*

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi)\rangle,\ i = 0,\dots,n-3, \\ u^{(n-2)}(0) = \langle \beta, u^{(n-2)}(\xi)\rangle, \\ u^{(n-2)}(1) = \langle \gamma, u^{(n-2)}(\xi)\rangle, \end{cases}$$

if and only if

$$u(t) = \int_0^1 \hat{K}_{n-1}(t,s)k(s)ds.$$

Parts (a) and (b) of Lemma 2.5 were proved in [20, Lemma 2.2], and part (c) can be proved similarly. We omit the proof of part (c) of the lemma.

Lemma 2.6 below obtains some useful estimates for $K_{n-1}(t,s)$, $\tilde{K}_{n-1}(t,s)$, and $\hat{K}_{n-1}(t,s)$.

Lemma 2.6. *We have the following:*

(a) *The function $K_{n-1}(t,s)$ satisfies*

$$\mu(t)(1-s) \le K_{n-1}(t,s) \le \nu(1-s) \quad \text{for } t,s \in [0,1], \tag{2.14}$$

where

$$\mu(t) = \begin{cases} 1 - t + \dfrac{\overline{\gamma} - \langle \gamma, \xi \rangle}{1 - \overline{\gamma}}, & n = 2, \\ \displaystyle\int_0^t \frac{(t-\tau)^{n-3}}{(n-3)!} \left(1 - \tau + \frac{\overline{\gamma} - \langle \gamma, \xi \rangle}{1 - \overline{\gamma}}\right) d\tau, & n \ge 3, \end{cases} \tag{2.15}$$

and

$$\nu = \left(\frac{1}{1-\overline{\gamma}} + \frac{1 - \langle \gamma, \xi \rangle}{(1-\overline{\beta})(1-\overline{\gamma})}\right) \frac{1 - \xi_m + \overline{\beta}}{1 - \xi_m} \frac{1}{(1-\overline{\alpha})^{n-2}}, \tag{2.16}$$

(b) *The function $\tilde{K}_{n-1}(t,s)$ satisfies*

$$\tilde{\mu}(t)s \le \tilde{K}_{n-1}(t,s) \le \tilde{\nu}s \quad \text{for } t,s \in [0,1],$$

where

$$\tilde{\mu}(t) = \begin{cases} t + \dfrac{\langle \beta, \xi \rangle}{1 - \overline{\beta}}, & n = 2, \\ \displaystyle\int_0^t \frac{(t-\tau)^{n-3}}{(n-3)!} \left(\tau + \frac{\langle \beta, \xi \rangle}{1 - \overline{\beta}}\right) d\tau, & n \ge 3, \end{cases} \tag{2.17}$$

and

$$\tilde{\nu} = \left(\frac{1}{1-\overline{\beta}} + \frac{\langle \beta, \xi \rangle + 1 - \overline{\beta}}{(1-\overline{\beta})(1-\overline{\gamma})}\right) \frac{\xi_1 + \overline{\gamma}}{\xi_1} \frac{1}{(1-\overline{\alpha})^{n-2}}, \tag{2.18}$$

(c) *The function $\hat{K}_{n-1}(t,s)$ satisfies*

$$\hat{\mu}(t)s(1-s) \le \hat{K}_{n-1}(t,s) \le \hat{\nu}s(1-s) \quad \text{for } t,s \in [0,1],$$

where

$$\hat{\mu}(t) = \begin{cases} t(1-t) + \min\left\{\hat{a}, \hat{b}\right\}, & n = 2, \\ \displaystyle\int_0^t \frac{(t-\tau)^{n-3}}{(n-3)!} \left(\tau(1-\tau) + \min\left\{\hat{a}, \hat{b}\right\}\right) d\tau, & n \geq 3, \end{cases} \tag{2.19}$$

and

$$\hat{\nu} = \left(1 + \max\left\{\hat{c}, \hat{d}\right\}\right) \frac{1}{(1-\overline{\alpha})^{n-2}}, \tag{2.20}$$

with

$$\hat{a} = \frac{1}{\rho}[(1 - \langle \gamma, \xi \rangle)\langle \beta, \xi(1-\xi)\rangle + \langle \beta, \xi\rangle\langle \gamma, \xi(1-\xi)\rangle], \tag{2.21}$$

$$\hat{b} = \frac{1}{\rho}[(1 - \overline{\beta} + \langle \beta, \xi\rangle)\langle \gamma, \xi(1-\xi)\rangle + (\overline{\gamma} - \langle \gamma, \xi\rangle)\langle \beta, \xi(1-\xi)\rangle], \tag{2.22}$$

$$\hat{c} = \frac{1}{\rho}[(1 - \langle \gamma, \xi\rangle)\overline{\beta} + \langle \beta, \xi\rangle\overline{\gamma}], \tag{2.23}$$

$$\hat{d} = \frac{1}{\rho}[(1 - \overline{\beta} + \langle \beta, \xi\rangle)\overline{\gamma} + (\overline{\gamma} - \langle \gamma, \xi\rangle)\overline{\beta}], \tag{2.24}$$

and ρ *is defined by* (2.11).

Proof. We first prove part (a). From (2.4), it is clear that

$$(1-t)(1-s) \leq G(t,s) \leq (1-s) \quad \text{for } t, s \in [0,1].$$

Then, from (2.8), it is easy to see that

$$H_0(t,s) \geq (1-t)(1-s) + \frac{\langle \gamma, G(\xi, s)\rangle}{1-\overline{\gamma}} \geq \left(1 - t + \frac{\overline{\gamma} - \langle \gamma, \xi\rangle}{1-\overline{\gamma}}\right)(1-s) \tag{2.25}$$

and

$$\begin{aligned} H_0(t,s) &\leq 1 - s + \frac{\overline{\gamma}}{1-\overline{\gamma}}(1-s) + \frac{1 - \langle \gamma, \xi\rangle}{(1-\overline{\beta})(1-\overline{\gamma})}\overline{\beta}\chi_{[0,\xi_m]}(s) \\ &= \frac{1}{1-\overline{\gamma}}(1-s) + \frac{1 - \langle \gamma, \xi\rangle}{(1-\overline{\beta})(1-\overline{\gamma})}\overline{\beta}\chi_{[0,\xi_m]}(s) \\ &\leq \left(\frac{1}{1-\overline{\gamma}} + \frac{1 - \langle \gamma, \xi\rangle}{(1-\overline{\beta})(1-\overline{\gamma})}\right)(1 - s + \overline{\beta}\chi_{[0,\xi_m]}(s)) \end{aligned} \tag{2.26}$$

for $t, s \in [0,1]$. For $s \in [0, \xi_m]$, since

$$\left(\frac{1 - s + \overline{\beta}}{1-s}\right)' = \frac{\overline{\beta}}{(1-s)^2} \geq 0,$$

we see that $(1 - s + \overline{\beta})/(1-s)$ is nondecreasing on $[0, \xi_m]$, and so

$$\frac{1 - s + \overline{\beta}}{1-s} \leq \frac{1 - \xi_m + \overline{\beta}}{1-\xi_m} \quad \text{for } s \in [0, \xi_m].$$

This in turn implies

$$1-s+\overline{\beta}\chi_{[0,\xi_m]}(s)=1-s+\overline{\beta}\le\frac{1-\xi_m+\overline{\beta}}{1-\xi_m}(1-s)\quad\text{for } s\in[0,\xi_m].$$

Note that

$$1-s+\overline{\beta}\chi_{[0,\xi_m]}(s)=1-s\le\frac{1-\xi_m+\overline{\beta}}{1-\xi_m}(1-s)\quad\text{for } s\in(\xi_m,1].$$

Then,

$$1-s+\overline{\beta}\chi_{[0,\xi_m]}(s)\le\frac{1-\xi_m+\overline{\beta}}{1-\xi_m}(1-s)\quad\text{for } s\in[0,1].$$

Combing the above inequality with (2.26) yields

$$H_0(t,s)\le\left(\frac{1}{1-\overline{\gamma}}+\frac{1-\langle\gamma,\xi\rangle}{(1-\overline{\beta})(1-\overline{\gamma})}\right)\frac{1-\xi_m+\overline{\beta}}{1-\xi_m}(1-s).\tag{2.27}$$

When $n=2$, note from (2.12) that $K_{n-1}(t,s)=H_0(t,s)$, and by (2.15) and (2.16),

$$\mu(t)=1-t+\frac{\overline{\gamma}-\langle\gamma,\xi\rangle}{1-\overline{\gamma}}\quad\text{and}\quad\nu=\left(\frac{1}{1-\overline{\gamma}}+\frac{1-\langle\gamma,\xi\rangle}{(1-\overline{\beta})(1-\overline{\gamma})}\right)\frac{1-\xi_m+\overline{\beta}}{1-\xi_m}.$$

Then, (2.14) follows from (2.25) and (2.27).

Now we assume that $n\ge3$. For $t,s\in[0,1]$, from (2.7),

$$\chi_{[0,t]}(s)\le H_1(t,s)\le\frac{\overline{\alpha}}{1-\overline{\alpha}}+1=\frac{1}{1-\overline{\alpha}}.\tag{2.28}$$

Then, from (2.12), (2.25), (2.27), (2.28), it follows that

$$K_2(t,s)\ge\int_0^t\left(1-\tau+\frac{\overline{\gamma}-\langle\gamma,\xi\rangle}{1-\overline{\gamma}}\right)d\tau\ (1-s)$$

and

$$K_2(t,s)\le\left(\frac{1}{1-\overline{\gamma}}+\frac{1-\langle\gamma,\xi\rangle}{(1-\overline{\beta})(1-\overline{\gamma})}\right)\frac{1-\xi_m+\overline{\beta}}{1-\xi_m}\frac{1}{1-\overline{\alpha}}\ (1-s).$$

Combining the above inequalities with (2.12) and (2.28), we see that

$$\begin{aligned}K_3(t,s)\ &\ge\ \int_0^t\int_0^v\left(1-\tau+\frac{\overline{\gamma}-\langle\gamma,\xi\rangle}{1-\overline{\gamma}}\right)d\tau dv\ (1-s)\\ &=\ \int_0^t(t-\tau)\left(1-\tau+\frac{\overline{\gamma}-\langle\gamma,\xi\rangle}{1-\overline{\gamma}}\right)d\tau\ (1-s)\end{aligned}$$

and

$$K_3(t,s)\le\left(\frac{1}{1-\overline{\gamma}}+\frac{1-\langle\gamma,\xi\rangle}{(1-\overline{\beta})(1-\overline{\gamma})}\right)\frac{1-\xi_m+\overline{\beta}}{1-\xi_m}\frac{1}{(1-\overline{\alpha})^2}\ (1-s).$$

An induction argument easily shows that

$$K_{n-1}(t,s)\ge\int_0^t\frac{(t-\tau)^{n-3}}{(n-3)!}\left(1-\tau+\frac{\overline{\gamma}-\langle\gamma,\xi\rangle}{1-\overline{\gamma}}\right)d\tau\ (1-s)=\mu(t)(1-s)$$

and

$$\begin{aligned} K_{n-1}(t,s) &\leq \left(\frac{1}{1-\overline{\gamma}} + \frac{1-\langle \gamma, \xi \rangle}{(1-\overline{\beta})(1-\overline{\gamma})}\right) \frac{1-\xi_m+\overline{\beta}}{1-\xi_m} \frac{1}{(1-\overline{\alpha})^{n-2}} (1-s) \\ &= \nu(1-s) \end{aligned}$$

for $t, s \in [0,1]$. Thus, (2.14) holds. This prove part (a).

Next, we show part (b). From (2.5), we have

$$ts \leq \tilde{G}(t,s) \leq s \quad \text{for } t,s \in [0,1].$$

Then, from (2.9), it is easy to see that

$$\tilde{H}_0(t,s) \geq ts + \frac{\langle \beta, \tilde{G}(\xi,s) \rangle}{1-\overline{\beta}} \geq \left(t + \frac{\langle \beta, \xi \rangle}{1-\overline{\beta}}\right) s$$

and

$$\begin{aligned} \tilde{H}_0(t,s) &\leq s + \frac{\overline{\beta}}{1-\overline{\beta}} s + \frac{\langle \beta, \xi \rangle + 1 - \overline{\beta}}{(1-\overline{\beta})(1-\overline{\gamma})} \overline{\gamma} \chi_{[\xi_1,1]}(s) \\ &= \frac{1}{1-\overline{\beta}} s + \frac{\langle \beta, \xi \rangle + 1 - \overline{\beta}}{(1-\overline{\beta})(1-\overline{\gamma})} \overline{\gamma} \chi_{[\xi_1,1]}(s) \\ &\leq \left(\frac{1}{1-\overline{\beta}} + \frac{\langle \beta, \xi \rangle + 1 - \overline{\beta}}{(1-\overline{\beta})(1-\overline{\gamma})}\right) (s + \overline{\gamma} \chi_{[\xi_1,1]}(s)) \end{aligned} \tag{2.29}$$

for $t, s \in [0,1]$. Note that $(s+\overline{\gamma})/s$ is nonincreasing on $[\xi_1, 1]$. Then,

$$\frac{s+\overline{\gamma}}{s} \leq \frac{\xi_1+\overline{\gamma}}{\xi_1} \quad \text{for } s \in [\xi_1, 1].$$

Thus,

$$s + \overline{\gamma} \chi_{[\xi_1,1]}(s) = s + \overline{\gamma} \leq \frac{\xi_1+\overline{\gamma}}{\xi_1} s \quad \text{for } s \in [\xi_1, 1].$$

Since

$$s + \overline{\gamma} \chi_{[\xi_1,1]}(s) = s \leq \frac{\xi_1+\overline{\gamma}}{\xi_1} s \quad \text{for } s \in [0, \xi_1),$$

we have

$$s + \overline{\gamma} \chi_{[\xi_1,1]}(s) \leq \frac{\xi_1+\overline{\gamma}}{\xi_1} s \quad \text{for } s \in [0, 1].$$

Then, from (2.29),

$$\tilde{H}_0(t,s) \leq \left(\frac{1}{1-\overline{\beta}} + \frac{\langle \beta, \xi \rangle + 1 - \overline{\beta}}{(1-\overline{\beta})(1-\overline{\gamma})}\right) \frac{\xi_1+\overline{\gamma}}{\xi_1} s.$$

The rest of the proof is similar to the latter part of the one used in showing (2.14), and hence is omitted.

Finally, we prove part (c). From (2.6), we have

$$t(1-t)s(1-s) \le \hat{G}(t,s) \le s(1-s) \quad \text{for } t,s \in [0,1].$$

Let $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ be defined by (2.21)–(2.24), and define

$$\begin{aligned} p(t) &= \frac{t}{\rho}[(1-\overline{\beta})\langle \gamma, \hat{G}(\xi,s)\rangle - (1-\overline{\gamma})\langle \beta, \hat{G}(\xi,s)\rangle] \\ &\quad + \frac{1}{\rho}[(1-\langle \gamma,\xi\rangle)\langle \beta, \hat{G}(\xi,s)\rangle + \langle \beta,\xi\rangle\langle \gamma, \hat{G}(\xi,s)\rangle]. \end{aligned}$$

Then,

$$\begin{aligned} \hat{a}s(1-s) &= \frac{1}{\rho}[(1-\langle \gamma,\xi\rangle)\langle \beta, \xi(1-\xi)\rangle + \langle \beta,\xi\rangle\langle \gamma, \xi(1-\xi)\rangle]s(1-s) \\ &\le p(0) = \frac{1}{\rho}[(1-\langle \gamma,\xi\rangle)\langle \beta, \hat{G}(\xi,s)\rangle + \langle \beta,\xi\rangle\langle \gamma, \hat{G}(\xi,s)\rangle] \\ &\le \frac{1}{\rho}[(1-\langle \gamma,\xi\rangle)\overline{\beta} + \langle \beta,\xi\rangle\overline{\gamma}]s(1-s) \\ &= \hat{c}s(1-s) \end{aligned}$$

and

$$\begin{aligned} \hat{b}s(1-s) &= \frac{1}{\rho}[(1-\overline{\beta}+\langle \beta,\xi\rangle)\langle \gamma, \xi(1-\xi)\rangle + (\overline{\gamma}-\langle \gamma,\xi\rangle)\langle \beta, \xi(1-\xi)\rangle]s(1-s) \\ &\le p(1) = \frac{1}{\rho}[(1-\overline{\beta}+\langle \beta,\xi\rangle)\langle \gamma, \hat{G}(\xi,s)\rangle + (\overline{\gamma}-\langle \gamma,\xi\rangle)\langle \beta, \hat{G}(\xi,s)\rangle] \\ &\le \frac{1}{\rho}[(1-\overline{\beta}+\langle \beta,\xi\rangle)\overline{\gamma} + (\overline{\gamma}-\langle \gamma,\xi\rangle)\overline{\beta}]s(1-s) \\ &= \hat{d}s(1-s), \end{aligned}$$

i.e.,

$$\hat{a}s(1-s) \le p(0) \le \hat{c}s(1-s) \quad \text{and} \quad \hat{b}s(1-s) \le p(1) \le \hat{d}s(1-s).$$

Moreover, from (2.10), we see that $\hat{H}_0(t,s) = \hat{G}(t,s) + p(t)$. Thus,

$$\begin{aligned} \hat{H}_0(t,s) &\ge t(1-t)s(1-s) + \min\{p(0), p(1)\} \\ &\ge t(1-t)s(1-s) + \min\{\hat{a}, \hat{b}\}s(1-s) \\ &= \left(t(1-t) + \min\{\hat{a}, \hat{b}\}\right) s(1-s) \end{aligned}$$

and

$$
\begin{aligned}
\hat{H}_0(t,s) &\le s(1-s) + \max\{p(0), p(1)\} \\
&\le s(1-s) + \max\{\hat{c}, \hat{d}\} s(1-s) \\
&= \left(1 + \max\{\hat{c}, \hat{d}\}\right) s(1-s).
\end{aligned}
$$

The rest of the proof is similar to the latter part of the one used in showing (2.14), and hence is omitted. This completes the proof of the lemma. □

In the remainder of the paper, let $X = C[0,1]$ be the Banach space of continuous functions equipped with the norm $||u|| = \max_{t\in[0,1]} |u(t)|$. Define a cone P in X by

$$
P = \{u \in X \ : \ u(t) \ge 0 \text{ on } [0,1]\}. \tag{2.30}
$$

Let the linear operators $L, M, \tilde{L}, \tilde{M}, \hat{L}, \hat{M} : X \to X$ be defined by

$$
Lu(t) = \int_0^1 K_{n-1}(t,s)g(s)u(s)ds, \quad Mu(t) = \int_0^1 K_{n-1}(s,t)g(s)u(s)ds, \tag{2.31}
$$

$$
\tilde{L}u(t) = \int_0^1 \tilde{K}_{n-1}(t,s)g(s)u(s)ds, \quad \tilde{M}u(t) = \int_0^1 \tilde{K}_{n-1}(s,t)g(s)u(s)ds, \tag{2.32}
$$

and

$$
\hat{L}u(t) = \int_0^1 \hat{K}_{n-1}(t,s)g(s)u(s)ds, \quad \hat{M}u(t) = \int_0^1 \hat{K}_{n-1}(s,t)g(s)u(s)ds. \tag{2.33}
$$

Remark 2.2. When $g \equiv 1$, and the pair L and M are considered as operators in the space $L^2(0,1)$, then L and M are adjoints of each other, so r_L and r_M would be equal. Since the function g is present, they are not adjoints of each other. However, for g as in this paper, as can be seen from Proposition 5.1 in the Appendix, the proof of which is provided by J. R. L. Webb, we do have that r_L and r_M are equal. Similar statements hold for the other pairs of operators as well.

From here on, let r_L, r_M, $r_{\tilde{L}}$, $r_{\tilde{M}}$, $r_{\hat{L}}$, and $r_{\hat{M}}$ be the spectral radii of L, M, $\tilde{L}$, $\tilde{M}$, $\hat{L}$, and $\hat{M}$, respectively. The next lemma provides some information about the above operators.

Lemma 2.7. *The operators $L, M, \tilde{L}, \tilde{M}, \hat{L}, \hat{M}$ map P into P and are compact. Moreover, we have*

(a) $r_L > 0$ *and* r_L *is an eigenvalue of* L *with an eigenfunction* $\varphi_L \in P$*;*

(b) $r_M > 0$ *and* r_M *is an eigenvalue of* M *with an eigenfunction* $\varphi_M \in P$;

(c) $r_{\tilde{L}} > 0$ *and* $r_{\tilde{L}}$ *is an eigenvalue of* $\tilde{L}$ *with an eigenfunction* $\varphi_{\tilde{L}} \in P$;

(d) $r_{\tilde{M}} > 0$ *and* $r_{\tilde{M}}$ *is an eigenvalue of* $\tilde{M}$ *with an eigenfunction* $\varphi_{\tilde{M}} \in P$;

(e) $r_{\hat{L}} > 0$ *and* $r_{\hat{L}}$ *is an eigenvalue of* $\hat{L}$ *with an eigenfunction* $\varphi_{\hat{L}} \in P$;

(f) $r_{\hat{M}} > 0$ *and* $r_{\hat{M}}$ *is an eigenvalue of* $\hat{M}$ *with an eigenfunction* $\varphi_{\hat{M}} \in P$.

The proof of the compactness and cone invariance for these operators is standard. By virtue of Lemma 2.2, part (a) was proved in [20, Lemma 2.6] and parts (b)–(f) can be proved essentially by the same way. We omit the proof of the lemma.

3. Main results

For convenience, we will use the following notations.

$$f_0 = \liminf_{x\to 0^+} \min_{t\in[0,1]} \frac{f(t,x)}{x}, \quad f_0^* = \liminf_{x\to 0} \min_{t\in[0,1]} \frac{f(t,x)}{x},$$

$$f_\infty = \liminf_{x\to \infty} \min_{t\in[0,1]} \frac{f(t,x)}{x}, \quad f_\infty^* = \liminf_{|x|\to \infty} \min_{t\in[0,1]} \frac{f(t,x)}{x},$$

$$F_0 = \limsup_{x\to 0} \max_{t\in[0,1]} \left|\frac{f(t,x)}{x}\right|, \quad F_\infty = \limsup_{|x|\to \infty} \max_{t\in[0,1]} \left|\frac{f(t,x)}{x}\right|.$$

Let r_M, $r_{\tilde{M}}$, $r_{\hat{M}}$, φ_M, $\varphi_{\tilde{M}}$, and $\varphi_{\hat{M}}$ be given as in Lemma 2.7. Define

$$\mu_M = \frac{1}{r_M}, \quad \mu_{\tilde{M}} = \frac{1}{r_{\tilde{M}}}, \quad \text{and} \quad \mu_{\hat{M}} = \frac{1}{r_{\hat{M}}}. \tag{3.1}$$

Clearly, μ_M is the smallest positive characteristic value of M and satisfies $\varphi_M = \mu_M M\varphi_M$. Similar statements hold for $\mu_{\tilde{M}}$ and $\mu_{\hat{M}}$.

We need the following assumptions.

(H1) There exist three nonnegative functions $a, b \in C[0,1]$ and $c \in C(\mathbb{R})$ such that $c(x)$ is even and nondecreasing on $\mathbb{R}_+$,

$$f(t,x) \geq -a(t) - b(t)c(x) \quad \text{for all} \quad (t,x) \in [0,1]\times\mathbb{R}, \tag{3.2}$$

and

$$\lim_{x\to\infty} \frac{c(x)}{x} = 0. \tag{3.3}$$

(H2) There exist a constant $r \in (0,1)$ and two nonnegative functions $d \in C[0,1]$ and $e \in C(\mathbb{R})$ such that e is even and nondecreasing on $\mathbb{R}^+$,

$$f(t,x) \geq -d(t)e(x) \quad \text{for all} \quad (t,x) \in [0,1] \times [-r,0], \tag{3.4}$$

and

$$\lim_{x \to 0} \frac{e(x)}{x} = 0. \tag{3.5}$$

(H3) $xf(t,x) \geq 0$ for $(t,x) \in [0,1] \times \mathbb{R}$.

Remark 3.1. Here, we want to emphasize that, in (H1), we assume that $f(t,x)$ is bounded from below by $-a(t) - b(t)c(x)$ for all $(t,x) \in [0,1] \times \mathbb{R}$; however in (H2), we only require that $f(t,x)$ is bounded from below by $-d(t)e(x)$ for $t \in [0,1]$ and x in a small left-neighborhood of 0.

We first state our existence results for BVP (1.1), (1.2).

Theorem 3.1. *Assume that (H1) holds and $F_0 < \mu_M < f_\infty$. Then, for $(\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}^n_+$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, BVP* (1.1), (1.2) *has at least one nontrivial solution.*

Theorem 3.2. *Assume that (H2) holds and $F_\infty < \mu_M < f_0$. Then, for $(\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}^n_+$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, BVP* (1.1), (1.2) *has at least one nontrivial solution.*

Theorem 3.3. *Assume that (H3) holds, $F_0 < \mu_M < f^*_\infty$, and $\lambda_i = 0$ for $i = 0, \ldots, n-1$. Then BVP* (1.1), (1.2) *has at least one positive solution and one negative solution.*

Theorem 3.4. *Assume that (H3) holds, $F_\infty < \mu_M < f^*_0$, and $\lambda_i = 0$ for $i = 0, \ldots, n-1$. Then BVP* (1.1), (1.2) *has at least one positive solution and one negative solution.*

Remark 3.2. If the nonlinear term $f(t,x)$ is separable, say $f(t,x) = f_1(t)f_2(x)$, then conditions such as $\mu_M < f_\infty$ and $\mu_M < f_0$ imply that $f_1(t) > 0$ on $[0,1]$. However, the function $g(t)$ in Eq. (1.1) may have zeros on $(0,1)$.

We now present some applications of the above theorems. To this end, let

$$A = \frac{1}{\nu \int_0^1 (1-s)g(s)ds} \quad \text{and} \quad B = \frac{\nu}{\underline{\mu} \int_0^1 (1-s)g(s)\mu(s)ds}, \tag{3.6}$$

where $\mu(t)$ and ν are defined by (2.15) and (2.16), respectively, and $\underline{\mu} = \min_{t\in[\theta_1,\theta_2]} \mu(t)$ with $0 < \theta_1 < \theta_2 < 1$ being fixed constants.

The following corollaries are immediate consequences of Theorems 3.1–3.4.

Corollary 3.1. *Assume that (H1) holds and $F_0/A < 1 < f_\infty/B$. Then the conclusion of Theorem 3.1 holds.*

Corollary 3.2. *Assume that (H2) holds and $F_\infty/A < 1 < f_0/B$. Then the conclusion of Theorem 3.2 holds.*

Corollary 3.3. *Assume that (H3) holds, $F_0/A < 1 < f^*_\infty/B$, and $\lambda_i = 0$ for $i = 0, \ldots, n-1$. Then the conclusion of Theorem 3.3 holds.*

Corollary 3.4. *Assume that (H3) holds, $F_\infty/A < 1 < f^*_0/B$, and $\lambda_i = 0$ for $i = 0, \ldots, n-1$. Then the conclusion of Theorem 3.4 holds.*

Replacing μ_M by $\mu_{\tilde{M}}$ and $\mu_{\hat{M}}$ gives the following results for BVPs (1.1), (1.3) and (1.1), (1.4), respectively, that are analogous to Theorems 3.1–3.4 and Corollary 3.1–3.4 for BVP (1.1), (1.2).

Theorem 3.5. *Assume that (H1) holds and $F_0 < \mu_{\tilde{M}} < f_\infty$. Then, for $(\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}^n_+$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, BVP* (1.1), (1.3) *has at least one nontrivial solution.*

Theorem 3.6. *Assume that (H2) holds and $F_\infty < \mu_{\tilde{M}} < f_0$. Then, for $(\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}^n_+$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, BVP* (1.1), (1.3) *has at least one nontrivial solution.*

Theorem 3.7. *Assume that (H3) holds, $F_0 < \mu_{\tilde{M}} < f^*_\infty$, and $\lambda_i = 0$ for $i = 0, \ldots, n-1$. Then BVP* (1.1), (1.3) *has at least one positive solution and one negative solution.*

Theorem 3.8. *Assume that (H3) holds, $F_\infty < \mu_{\tilde{M}} < f^*_0$, and $\lambda_i = 0$ for $i = 0, \ldots, n-1$. Then BVP* (1.1), (1.3) *has at least one positive solution and one negative solution.*

Let

$$\tilde{A} = \frac{1}{\tilde{\nu} \int_0^1 sg(s)ds} \quad \text{and} \quad \tilde{B} = \frac{\tilde{\nu}}{\underline{\tilde{\mu}} \int_0^1 sg(s)\tilde{\mu}(s)ds}, \tag{3.7}$$

where $\tilde{\mu}(t)$ and $\tilde{\nu}$ are defined by (2.17) and (2.18), respectively, and $\underline{\tilde{\mu}} = \min_{t\in[\theta_1,\theta_2]} \tilde{\mu}(t)$ with $0 < \theta_1 < \theta_2 < 1$ being fixed constants.

Corollary 3.5. *Assume that (H1) holds and $F_0/\tilde{A} < 1 < f_\infty/\tilde{B}$. Then the conclusion of Theorem 3.5 holds.*

Corollary 3.6. *Assume that (H2) holds and $F_\infty/\tilde{A} < 1 < f_0/\tilde{B}$. Then the conclusion of Theorem 3.6 holds.*

Corollary 3.7. *Assume that (H3) holds, $F_0/\tilde{A} < 1 < f_\infty^*/\tilde{B}$, and $\lambda_i = 0$ for $i = 0, \dots, n-1$. Then the conclusion of Theorem 3.7 holds.*

Corollary 3.8. *Assume that (H3) holds, $F_\infty/\tilde{A} < 1 < f_0^*/\tilde{B}$, and $\lambda_i = 0$ for $i = 0, \dots, n-1$. Then the conclusion of Theorem 3.8 holds.*

Theorem 3.9. *Assume that (H1) holds and $F_0 < \mu_{\hat{M}} < f_\infty$. Then, for $(\lambda_0, \dots, \lambda_{n-1}) \in \mathbb{R}_+^n$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, BVP (1.1), (1.4) has at least one nontrivial solution.*

Theorem 3.10. *Assume that (H2) holds and $F_\infty < \mu_{\hat{M}} < f_0$. Then, for $(\lambda_0, \dots, \lambda_{n-1}) \in \mathbb{R}_+^n$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, BVP (1.1), (1.4) has at least one nontrivial solution.*

Theorem 3.11. *Assume that (H3) holds, $F_0 < \mu_{\hat{M}} < f_\infty^*$, and $\lambda_i = 0$ for $i = 0, \dots, n-1$. Then BVP (1.1), (1.4) has at least one positive solution and one negative solution.*

Theorem 3.12. *Assume that (H3) holds, $F_\infty < \mu_{\hat{M}} < f_0^*$, and $\lambda_i = 0$ for $i = 0, \dots, n-1$. Then BVP (1.1), (1.4) has at least one positive solution and one negative solution.*

Let

$$\hat{A} = \frac{1}{\hat{\nu} \int_0^1 s(1-s)g(s)ds} \quad \text{and} \quad \hat{B} = \frac{\hat{\nu}}{\underline{\hat{\mu}} \int_0^1 s(1-s)g(s)\hat{\mu}(s)ds}, \tag{3.8}$$

where $\hat{\mu}(t)$ and $\hat{\nu}$ are defined by (2.19) and (2.20), respectively, and $\underline{\hat{\mu}} = \min_{t\in[\theta_1,\theta_2]} \hat{\mu}(t)$ with $0 < \theta_1 < \theta_2 < 1$ being fixed constants.

Corollary 3.9. *Assume that (H1) holds and $F_0/\hat{A} < 1 < f_\infty/\hat{B}$. Then the conclusion of Theorem 3.9 holds.*

Corollary 3.10. *Assume that (H2) holds and $F_\infty/\hat{A} < 1 < f_0/\hat{B}$. Then the conclusion of Theorem 3.10 holds.*

Corollary 3.11. *Assume that (H3) holds, $F_0/\hat{A} < 1 < f^*_\infty/\hat{B}$, and $\lambda_i = 0$ for $i = 0, \ldots, n-1$. Then the conclusion of Theorem 3.11 holds.*

Corollary 3.12. *Assume that (H3) holds, $F_\infty/\hat{A} < 1 < f^*_0/\hat{B}$, and $\lambda_i = 0$ for $i = 0, \ldots, n-1$. Then the conclusion of Theorem 3.12 holds.*

Although not explicitly discussed in this paper since we ask that $m \geq 1$, it is interesting to examine our results in the case of two-point boundary conditions, that is, if $\alpha = \beta = \gamma = 0$. All theorems and corollaries stated in this section remain valid in this case. Notice that the quantities ξ_1 and ξ_m do not affect the values in (2.16) and (2.18).

Remark 3.3. We wish to point out that the optimal existence results given in this section, Theorems 3.1–3.12, are all related to the smallest positive characteristic values of the operators M, $\tilde{M}$, and $\hat{M}$ as given by (3.1). In [20], the authors proved optimal existence results for positive solutions of BVPs (1.1), (1.2) and (1.1), (1.3), in terms of the smallest positive characteristic values of the operators L and $\tilde{L}$ as defined by (2.31) and (2.32). In that paper it was assumed that f was nonnegative. Other optimal type results for existence of positive solutions under different types of boundary conditions from the ones used in this paper can be found in [2, 35, 38].

Remark 3.4. In this paper, we do not study the multiplicity and nonexistence of solutions of the problems under consideration. Since this paper is somewhat long, we will leave such investigations to future work. Other papers investigating these kinds of questions for different boundary conditions include [34, 38].

Remark 3.5. In Theorems 3.1 and 3.2 (as well as Theorems 3.5, 3.6, 3.9, 3.10), we have the requirement that $\sum_{i=0}^{n-1} \lambda_i$ be sufficiently small. It is worth mentioning that this "smallness" can be estimated. For example, for $n = 2$, $m = 3$, $\beta = (\beta_1, 0)$, and $\gamma = (0, \gamma_2)$, this was described in the paper [6] (see [6, Remark 3.2 and Example 3.2]).

We conclude this section with several examples.

Example 3.1. In equations (1.1) and (1.2), let $m = 1$, $n = 3$, $\alpha = \beta = \gamma = \xi = 1/2$, $g(t) \equiv 1$ on $[0, 1]$,

$$f(t,x) = \begin{cases} \sum_{i=1}^{k} a_i(t)x^i, & x \in [-1, \infty), \\ \sum_{i=1}^{k} (-1)^i a_i(t) - \bar{b}(t)|x|^\kappa + \bar{b}(t), & x \in (-\infty, -1), \end{cases} \tag{3.9}$$

where $k > 1$ is an integer, $a_i, \bar{b} \in C[0, 1]$ with $0 \le ||a_1|| < 1/10$ and $a_k(t) > 0$ on $[0, 1]$, and $0 \le \kappa < 1$. Then, for $(\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}_+^n$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, BVP (1.1), (1.2) has at least one nontrivial solution.

To see this, we first note that $f \in C([0, 1] \times \mathbb{R})$ and assumption (H) is satisfied. Let

$$a(t) = \sum_{i=1}^{k} |a_i(t)| + |\bar{b}(t)|, \quad b(t) = |\bar{b}(t)|, \quad \text{and} \quad c(x) = |x|^\kappa.$$

Then, it is easy to see that (H1) holds.

From (3.6) with $\theta_1 = 1/4$ and $\theta_2 = 3/4$, and by a simple calculation, we have

$$A = 1/10 \quad \text{and} \quad B = 3072/11.$$

Moreover, (3.9) implies that

$$F_0 = \limsup_{x\to 0} \max_{t\in[0,1]} \left| \frac{f(t,x)}{x} \right| = ||a_1|| < \frac{1}{10} \quad \text{and} \quad f_\infty = \liminf_{x\to\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x} = \infty.$$

Hence, $F_0/A < 1 < f_\infty/B$. The conclusion then follows from Corollary 3.1.

Example 3.2. In equations (1.1)–(1.4), let $m \ge 1$ and $n \ge 2$ be any integers, $\xi = (\xi_1, \ldots, \xi_m) \in (0, 1)^m$ with $0 < \xi_1 < \ldots < \xi_m < 1$, α, β, $\gamma \in \mathbb{R}_+^m$ with $0 \le \overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma} < 1$. Also let $g : (0, 1) \to \mathbb{R}_+$ be continuous, $g \not\equiv 0$ on any subinterval of $(0, 1)$, $\int_0^1 g(s)ds < \infty$, and

$$f(t,x) = \begin{cases} -16t^2 + 13 + (|x|^{1/2} - 2)x^{1/3}, & x < -4, \\ -t^2x^2 + 3|x| + 1, & -4 \le x \le 0, \\ 1 - tx^{1/2}, & x > 0. \end{cases} \tag{3.10}$$

Then, for $(\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}^n_+$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, we have

(i) BVP (1.1), (1.2) has at least one nontrivial solution.

(ii) BVP (1.1), (1.3) has at least one nontrivial solution.

(iii) BVP (1.1), (1.4) has at least one nontrivial solution.

To see this, we first note that $f \in C([0,1] \times \mathbb{R})$ and assumption (H) is satisfied. Now with $d(t) = t^2$ and $e(x) = x^2$, from (3.10), we see that (3.4) and (3.5) hold for any $r \in (0,1)$, and so (H2) holds. Moreover, from (3.10), we have $f_0 = \infty$ and $F_\infty = 0$. Thus,

$$F_\infty < \mu_M < f_0, \quad F_\infty < \mu_{\tilde{M}} < f_0, \quad \text{and} \quad F_\infty < \mu_{\hat{M}} < f_0,$$

where μ_M, $\mu_{\tilde{M}}$, and $\mu_{\hat{M}}$ are defined in (3.1). The conclusions (i), (ii), and (iii) then follow from Theorems 3.2, 3.6, and 3.10, respectively.

Example 3.3. In equations (1.1)–(1.4), let $m \geq 1$ and $n \geq 2$ be any integers, $\lambda_i = 0$ for $i = 0, \ldots, n-1$, $\xi = (\xi_1, \ldots, \xi_m) \in (0,1)^m$ with $0 < \xi_1 < \ldots < \xi_m < 1$, α, β, $\gamma \in \mathbb{R}^m_+$ with $0 \leq \overline{\alpha}, \overline{\beta}, \overline{\gamma} < 1$. Also let $f(t,x) = x^3$ and $g(t) = t^{-1/2}$. Then, we have

(i) BVP (1.1), (1.2) has at least one positive solution and one negative solution.

(ii) BVP (1.1), (1.3) has at least one positive solution and one negative solution.

(iii) BVP (1.1), (1.4) has at least one positive solution and one negative solution.

To see this, we first note that assumptions (H) and (H3) are satisfied. Moreover, we have $F_0 = 0$ and $f^*_\infty = \infty$. Thus,

$$F_0 < \mu_M < f^*_\infty, \quad F_0 < \mu_{\tilde{M}} < f^*_\infty, \quad \text{and} \quad F_0 < \mu_{\hat{M}} < f^*_\infty,$$

where μ_M, $\mu_{\tilde{M}}$, and $\mu_{\hat{M}}$ are defined in (3.1). The conclusions (i), (ii), and (iii) then follow from Theorems 3.3, 3.7, and 3.11, respectively.

Additional examples may also be readily given to illustrate the other results. We leave the details to the interested reader.

4. Proofs of the main results

Let $\phi(t)$ be the unique solution of the BVP

$$u'' = 0, \quad t \in (0,1),$$

$$u'(0) = \langle \beta, u'(\xi) \rangle - 1, \; u(1) = \langle \gamma, u(\xi) \rangle,$$

and let $\psi(t)$ be the unique solution of the BVP

$$u'' = 0, \quad t \in (0,1),$$

$$u'(0) = \langle \beta, u'(\xi) \rangle, \; u(1) = \langle \gamma, u(\xi) \rangle + 1.$$

Then,

$$\phi(t) = -\frac{t}{1-\overline{\beta}} + \frac{1 - \langle \gamma, \xi \rangle}{(1-\overline{\beta})(1-\overline{\gamma})} \quad \text{and} \quad \psi(t) = \frac{1}{1-\overline{\gamma}} \quad \text{on } [0,1].$$

Let $\tilde{\phi}(t)$ be the unique solution of the BVP

$$u'' = 0, \quad t \in (0,1),$$

$$u(0) = \langle \beta, u(\xi) \rangle, \; u'(1) = \langle \gamma, u'(\xi) \rangle + 1,$$

and let $\tilde{\psi}(t)$ be the unique solution of the BVP

$$u'' = 0, \quad t \in (0,1),$$

$$u(0) = \langle \beta, u(\xi) \rangle + 1, \; u'(1) = \langle \gamma, u'(\xi) \rangle.$$

Then, we have

$$\tilde{\phi}(t) = \frac{t}{1-\overline{\gamma}} + \frac{\langle \beta, \xi \rangle}{(1-\overline{\beta})(1-\overline{\gamma})} \quad \text{and} \quad \tilde{\psi}(t) = \frac{1}{1-\overline{\beta}} \quad \text{on } [0,1].$$

Similarly, let $\hat{\phi}(t)$ be the unique solution of the BVP

$$u'' = 0, \quad t \in (0,1),$$

$$u(0) = \langle \beta, u(\xi) \rangle, \; u(1) = \langle \gamma, u(\xi) \rangle + 1,$$

and let $\hat{\psi}(t)$ be the unique solution of the BVP

$$u'' = 0, \quad t \in (0,1),$$

$$u(0) = \langle \beta, u(\xi) \rangle + 1, \; u(1) = \langle \gamma, u(\xi) \rangle.$$

Then, we have

$$\hat{\phi}(t) = \frac{1}{\rho}[(1-\overline{\beta})t + \langle \beta, \xi \rangle] \quad \text{and} \quad \hat{\psi}(t) = \frac{1}{\rho}[(\overline{\gamma}-1)t + (1 - \langle \gamma, \xi \rangle)] \quad \text{on } [0,1].$$

Clearly, $\phi(t)$, $\psi(t)$, $\tilde{\phi}(t)$, $\tilde{\psi}(t)$, $\hat{\phi}(t)$, and $\hat{\psi}(t)$ are nonnegative on $[0,1]$.

Let $J_1(t,s) = H_1(t,s)$, where $H_1(t,s)$ is defined by (2.7). If $n \geq 4$, then we recursively define

$$J_k(t,s) = \int_0^1 H_1(t,\tau)J_{k-1}(\tau,s)d\tau,\ k = 2,\dots,n-2.$$

We now define several functions $y_i(t)$, $\tilde{y}_i(t)$, and $\hat{y}_i(t)$, $i = 0,\dots,n-1$, as follows: when $n = 2$, let

$$\begin{aligned} y_0(t) &= \phi(t) \quad \text{and} \quad y_1(t) = \psi(t), \\ \tilde{y}_0(t) &= \tilde{\phi}(t) \quad \text{and} \quad \tilde{y}_1(t) = \tilde{\psi}(t), \\ \hat{y}_0(t) &= \hat{\phi}(t) \quad \text{and} \quad \hat{y}_1(t) = \hat{\psi}(t); \end{aligned} \tag{4.1}$$

and when $n \geq 3$, let

$$y_0(t) = \frac{1}{1-\overline{\alpha}},\ y_k(t) = \frac{1}{1-\overline{\alpha}}\int_0^1 J_k(t,s)ds,\ k = 1,\dots,n-3, \tag{4.2}$$

$$y_{n-2}(t) = \int_0^1 J_{n-2}(t,s)\phi(s)ds, \tag{4.3}$$

$$y_{n-1}(t) = \int_0^1 J_{n-2}(t,s)\psi(s)ds, \tag{4.4}$$

and

$$\begin{aligned} \tilde{y}_k(t) &= y_k(t),\ k = 0,\dots,n-3, \\ \tilde{y}_{n-2}(t) &= \int_0^1 J_{n-2}(t,s)\tilde{\phi}(s)ds, \\ \tilde{y}_{n-1}(t) &= \int_0^1 J_{n-2}(t,s)\tilde{\psi}(s)ds, \end{aligned}$$

and

$$\begin{aligned} \hat{y}_k(t) &= y_k(t),\ k = 0,\dots,n-3, \\ \hat{y}_{n-2}(t) &= \int_0^1 J_{n-2}(t,s)\hat{\phi}(s)ds, \\ \hat{y}_{n-1}(t) &= \int_0^1 J_{n-2}(t,s)\hat{\psi}(s)ds. \end{aligned}$$

Clearly, $y_i(t) \geq 0$, $\tilde{y}_i(t) \geq 0$, and $\hat{y}_i(t) \geq 0$ for $t \in [0,1]$ and $i = 0,\dots,n-1$.

The following lemma gives some properties of $y_i(t)$, $\tilde{y}_i(t)$, and $\hat{y}_i(t)$ when $n \geq 3$.

Lemma 4.1. *Assume that $n \geq 3$. Then, we have the following:*

(a) *For any* $k \in \{0, \ldots, n-3\}$, $y_k(t)$ *is the unique solution of the BVP consisting of the equation*

$$u^{(n)} = 0,\ t \in (0,1), \tag{4.5}$$

and the BC

$$\begin{cases} u^{(k)}(0) = \langle \alpha, u^{(k)}(\xi)\rangle + 1, \\ u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi)\rangle,\ i = 0, \ldots, n-3,\ i \neq k, \\ u^{(n-1)}(0) = \langle \beta, u^{(n-1)}(\xi)\rangle, \\ u^{(n-2)}(1) = \langle \gamma, u^{(n-2)}(\xi)\rangle, \end{cases}$$

and $y_{n-2}(t)$ *is the unique solution of the BVP consisting of Eq.* (4.5) *and the BC*

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi)\rangle,\ i = 0, \ldots, n-3, \\ u^{(n-1)}(0) = \langle \beta, u^{(n-1)}(\xi)\rangle - 1, \\ u^{(n-2)}(1) = \langle \gamma, u^{(n-2)}(\xi)\rangle, \end{cases}$$

and $y_{n-1}(t)$ *is the unique solution of the BVP consisting of Eq.* (4.5) *and the BC*

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi)\rangle,\ i = 0, \ldots, n-3, \\ u^{(n-1)}(0) = \langle \beta, u^{(n-1)}(\xi)\rangle, \\ u^{(n-2)}(1) = \langle \gamma, u^{(n-2)}(\xi)\rangle + 1. \end{cases}$$

(b) *For any* $k \in \{0, \ldots, n-3\}$, $\tilde{y}_k(t)$ *is the unique solution of the BVP consisting of Eq.* (4.5) *and the BC*

$$\begin{cases} u^{(k)}(0) = \langle \alpha, u^{(k)}(\xi)\rangle + 1, \\ u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi)\rangle,\ i = 0, \ldots, n-3,\ i \neq k, \\ u^{(n-2)}(0) = \langle \beta, u^{(n-2)}(\xi)\rangle, \\ u^{(n-1)}(1) = \langle \gamma, u^{(n-1)}(\xi)\rangle, \end{cases}$$

and $\tilde{y}_{n-2}(t)$ *is the unique solution of the BVP consisting of Eq.* (4.5) *and the BC*

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi)\rangle,\ i = 0, \ldots, n-3, \\ u^{(n-2)}(0) = \langle \beta, u^{(n-2)}(\xi)\rangle + 1, \\ u^{(n-1)}(1) = \langle \gamma, u^{(n-1)}(\xi)\rangle, \end{cases}$$

and $\tilde{y}_{n-1}(t)$ *is the unique solution of the BVP consisting of Eq.* (4.5) *and the BC*

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi)\rangle,\ i = 0, \ldots, n-3, \\ u^{(n-2)}(0) = \langle \beta, u^{(n-2)}(\xi)\rangle, \\ u^{(n-1)}(1) = \langle \gamma, u^{(n-1)}(\xi)\rangle + 1. \end{cases}$$

(c) *For any* $k \in \{0, \ldots, n-3\}$, $\hat{y}_k(t)$ *is the unique solution of the BVP consisting of Eq.* (4.5) *and the BC*

$$\begin{cases} u^{(k)}(0) = \langle \alpha, u^{(k)}(\xi) \rangle + 1, \\ u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi) \rangle, \ i = 0, \ldots, n-3, \ i \neq k, \\ u^{(n-2)}(0) = \langle \beta, u^{(n-2)}(\xi) \rangle, \\ u^{(n-2)}(1) = \langle \gamma, u^{(n-2)}(\xi) \rangle, \end{cases}$$

and $\hat{y}_{n-2}(t)$ *is the unique solution of the BVP consisting of Eq.* (4.5) *and the BC*

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi) \rangle, \ i = 0, \ldots, n-3, \\ u^{(n-2)}(0) = \langle \beta, u^{(n-2)}(\xi) \rangle + 1, \\ u^{(n-2)}(1) = \langle \gamma, u^{(n-2)}(\xi) \rangle, \end{cases}$$

and $\hat{y}_{n-1}(t)$ *is the unique solution of the BVP consisting of Eq.* (4.5) *and the BC*

$$\begin{cases} u^{(i)}(0) = \langle \alpha, u^{(i)}(\xi) \rangle, \ i = 0, \ldots, n-3, \\ u^{(n-2)}(0) = \langle \beta, u^{(n-2)}(\xi) \rangle, \\ u^{(n-2)}(1) = \langle \gamma, u^{(n-2)}(\xi) \rangle + 1. \end{cases}$$

Parts (a) and (b) of Lemma 4.1 were proved in [20, Lemma 2.4], and part (c) can be proved similarly. We omit the proof of part (c) of the lemma.

For any $\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}^n_+$, let

$$u(t) = v(t) + \sum_{i=0}^{n-1} \lambda_i y_i(t), \ t \in [0, 1]. \tag{4.6}$$

Here, the reader is reminded that $y_i(t)$, $i = 0, \ldots, n-1$, are defined by (4.1) if $n = 2$, and are given by (4.2)–(4.4) if $n \geq 3$. Then, by Lemma 4.1 (a), BVP (1.1), (1.2) is equivalent to the BVP consisting of the equation

$$v^{(n)} + g(t) f\left(t, v + \sum_{i=0}^{n-1} \lambda_i y_i(t)\right), \ t \in (0, 1), \tag{4.7}$$

and the homogeneous BC

$$\begin{cases} v^{(i)}(0) = \langle \alpha, v^{(i)}(\xi) \rangle, \ i = 0, \ldots, n-3, \\ v^{(n-1)}(0) = \langle \beta, v^{(n-1)}(\xi) \rangle, \\ v^{(n-2)}(1) = \langle \gamma, v^{(n-2)}(\xi) \rangle. \end{cases} \tag{4.8}$$

Moreover, if $v(t)$ is a solution of BVP (4.7), (4.8), then $u(t)$ given by (4.6) is a solution of BVP (1.1), (1.2).

Let P, L, and M be defined as in (2.30) and (2.31). By Lemma 2.7, L and M map P into P and are compact. Define operators $F_\lambda, T : X \to X$ by

$$F_\lambda v(t) = f\left(t, v + \sum_{i=0}^{n-1} \lambda_i y_i(t)\right) \tag{4.9}$$

and

$$Tv(t) = LF_\lambda v(t) = \int_0^1 K_{n-1}(t,s)g(s)F_\lambda v(s)ds, \tag{4.10}$$

where K_{n-1} is defined by (2.12) with $i = n-1$. Then, F_λ is bounded, and a standard argument shows that T is compact. Moreover, by Lemma 2.5 (a), a solution of BVP (4.7), (4.8) is equivalent to a fixed point of T in X.

Proof of Theorem 3.1. We first verify that conditions (A1)–(A4) of Lemma 2.3 are satisfied. By Lemma 2.7 (a) and (b), there exist $\varphi_L, \varphi_M \in P \setminus \{\mathbf{0}\}$ such that (2.1) holds. To show (2.2), we let

$$h(v) = \int_0^1 \varphi_M(t)g(t)v(t)dt, \ v \in X. \tag{4.11}$$

Then $h \in P^* \setminus \{\mathbf{0}\}$, and from (2.1) and (2.31),

$$\begin{aligned} (L^*h)(v) = h(Lv) &= \int_0^1 \varphi_M(t)g(t)Lv(t)dt \\ &= \int_0^1 \varphi_M(t)g(t)\left(\int_0^1 K_{n-1}(t,s)g(s)v(s)ds\right)dt \\ &= \int_0^1 g(s)v(s)\left(\int_0^1 K_{n-1}(t,s)g(t)\varphi_M(t)dt\right)ds \\ &= \int_0^1 g(s)v(s)M\varphi_M(s)ds \\ &= r_M \int_0^1 g(s)v(s)\varphi_M(s)ds = r_M h(v), \end{aligned}$$

i.e., h satisfies (2.2).

From the fact that $\varphi_M = \mu_M M\varphi_M$, (2.31), and (3.1),

$$r_M\varphi_M(s) = \int_0^1 K_{n-1}(t,s)g(t)\varphi_M(t)dt. \tag{4.12}$$

Then, by Lemma 2.6 (a) (see (2.14)), we have

$$\begin{aligned} r_M\varphi_M(s) &\geq (1-s)\int_0^1 \mu(t)g(t)\varphi_M(t)dt \\ &= \left(\frac{1}{\nu}\int_0^1 \mu(t)g(t)\varphi_M(t)dt\right)\nu(1-s) \\ &\geq \left(\frac{1}{\nu}\int_0^1 \mu(t)g(t)\varphi_M(t)dt\right)K_{n-1}(t,s) \\ &= \delta K_{n-1}(t,s) \quad \text{for } t,s\in[0,1], \end{aligned} \tag{4.13}$$

where $\mu(t)$ and ν are defined by (2.15) and (2.16), and

$$\delta = \frac{1}{\nu}\int_0^1 \mu(t)g(t)\varphi_M(t)dt.$$

To see that $\delta > 0$, first note that $\mu(t) > 0$ on $(0,1)$ and $\nu > 0$. If $\delta = 0$, then $g(t)\varphi_M(t) \equiv 0$ on $(0,1)$. Since $r_M > 0$, this implies $\varphi_M(t) \equiv 0$ by (4.12), which is a contradiction.

Let $P(h,\delta)$ be defined by (2.3). For any $v \in P$ and $t \in [0,1]$, from (2.31), (4.11), and (4.13), it follows that

$$\begin{aligned} h(Lv) = r_M h(v) &= r_M\int_0^1 \varphi_M(s)g(s)v(s)ds \\ &\geq \delta\int_0^1 K_{n-1}(t,s)g(s)v(s)ds \\ &= \delta Lv(t). \end{aligned}$$

Hence, $h(Lv) \geq \delta||Lv||$, i.e., $L(P) \subseteq P(h,\delta)$. Therefore, (A1) of Lemma 2.3 holds.

Let $\lambda = (\lambda_0,\ldots,\lambda_{n-1}) \in \mathbb{R}^n_+$ and

$$H_\lambda v(t) = \bar{b}c\left(|v(t)| + \sum_{i=0}^{n-1}\lambda_i||y_i||\right)$$

for $v \in X$, where $\bar{b} = \max_{t\in[0,1]} b(t)$. Since c is nondecreasing on $\mathbb{R}^+$, we have

$$H_\lambda v(t) \leq \bar{b}c\left(||v|| + \sum_{i=0}^{n-1}\lambda_i||y_i||\right) \quad \text{for all } v\in P \text{ and } t\in[0,1].$$

Then, from the fact that c is even, it follows that

$$H_\lambda v(t) \leq \bar{b}c\left(||v|| + \sum_{i=0}^{n-1}\lambda_i||y_i||\right) \quad \text{for all } v\in X \text{ and } t\in[0,1].$$

Thus,

$$||H_\lambda v|| \le \bar{b}c\left(||v|| + \sum_{i=0}^{n-1}\lambda_i||y_i||\right) \quad \text{for all } v \in X.$$

From (3.3), we see that

$$\lim_{||v||\to\infty} \frac{||H_\lambda v||}{||v||} = 0 \quad \text{for any } v \in X.$$

Thus, (A2) of Lemma 2.3 holds with $H = H_\lambda$.

Let F_λ be defined by (4.9), and $u_0(t) = a(t)$. Then, from (H1), we have $F_\lambda v + H_\lambda v + u_0 \in P$ for all $v \in X$. Hence, (A3) of Lemma 2.3 holds with $F = F_\lambda$ and $H = H_\lambda$.

Since $f_\infty > \mu_M$, there exist $\epsilon > 0$ and $N > 0$ such that

$$f(t,x) \ge \mu_M(1+\epsilon)x \quad \text{for } (t,x) \in [0,1]\times[N,\infty).$$

In view of (3.2), we see that there exists $\zeta > 0$ large enough so that

$$f(t,x) \ge \mu_M(1+\epsilon)x - \bar{b}c(x) - \zeta \quad \text{for } (t,x) \in [0,1]\times\mathbb{R}.$$

From (3.1) and (4.9), we have

$$\begin{aligned} F_\lambda v(t) &\ge \mu_M(1+\epsilon)\left(v(t) + \sum_{i=0}^{n-1}\lambda_i y_i(t)\right) - \bar{b}c\left(v(t) + \sum_{i=0}^{n-1}\lambda_i y_i(t)\right) - \zeta \\ &\ge \mu_M(1+\epsilon)v(t) - \bar{b}c\left(|v(t)| + \sum_{i=0}^{n-1}\lambda_i||y_i||\right) - \zeta \\ &= r_M^{-1}(1+\epsilon)v(t) - H_\lambda v(t) - \zeta \quad \text{for all } v \in X. \end{aligned}$$

Thus,

$$LF_\lambda v(t) \ge r_M^{-1}(1+\epsilon)Lv(t) - LH_\lambda v(t) - L\zeta \quad \text{for all } v \in X.$$

Therefor, (A4) of Lemma 2.3 holds with $F = F_\lambda$, $H = H_\lambda$, and $v_0 = L\zeta$.

We have verified that all the conditions of Lemma 2.3 hold, so there exists $R_1 > 0$ such that

$$\deg(I - T, B(\mathbf{0}, R_1), \mathbf{0}) = 0. \tag{4.14}$$

Next, since $F_0 < \mu_M$, there exist $0 < q < 1$ and $0 < R_2 < R_1$ such that

$$|f(t,x)| \le \mu_M(1-q)|x| \quad \text{for } (t,x) \in [0,1]\times[-2R_2, 2R_2]. \tag{4.15}$$

In what follows, let $\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}^n_+$ be small enough so that

$$\sum_{i=0}^{n-1} \lambda_i ||y_i|| < R_2 \tag{4.16}$$

and

$$C_1 := \mu_M(1-q)\left(\sum_{i=0}^{n-1} \lambda_i ||y_i||\right) \max_{t\in[0,1]} \int_0^1 K_{n-1}(t,s)g(s)ds < qR_2. \tag{4.17}$$

We claim that

$$Tv \neq \tau v \quad \text{for all } v \in \partial B(\mathbf{0}, R_2) \text{ and } \tau \geq 1. \tag{4.18}$$

If this is not the case, then there exist $\bar{v} \in \partial B(\mathbf{0}, R_2)$ and $\bar{\tau} \geq 1$ such that $T\bar{v} = \bar{\tau}\bar{v}$. It follows that $\bar{v} = \bar{s}T\bar{v}$, where $\bar{s} = 1/\bar{\tau}$. Clearly, $\bar{s} \in (0, 1]$. From (4.9), (4.15), and (4.16), we have

$$\begin{aligned} |F_\lambda \bar{v}(t)| &\leq \mu_M(1-q)\left|\bar{v}(t) + \sum_{i=0}^{n-1} \lambda_i y_i(t)\right| \\ &\leq \mu_M(1-q)\left(|\bar{v}(t)| + \sum_{i=0}^{n-1} \lambda_i ||y_i||\right). \end{aligned} \tag{4.19}$$

Assume $R_2 = ||\bar{v}|| = |\bar{v}(\bar{t})|$ for some $\bar{t} \in [0, 1]$. Then, from (2.31), (3.1), (4.10), (4.17), and (4.19), we obtain

$$\begin{aligned} R_2 &= |\bar{v}(\bar{t})| = \bar{s}|T\bar{v}(\bar{t})| \leq \int_0^1 K_{n-1}(\bar{t}, s))g(s)|F_\lambda \bar{v}(s)|ds \\ &\leq \mu_M(1-q)\int_0^1 K_{n-1}(\bar{t}, s)g(s)\left(v(s) + \sum_{i=0}^{n-1} \lambda_i ||y_i||\right)ds \\ &= \mu_M(1-q)\int_0^1 K_{n-1}(\bar{t}, s)g(s)|v(s)|ds \\ &\quad +\mu_M(1-q)\left(\sum_{i=0}^{n-1} \lambda_i ||y_i||\right)\int_0^1 K_{n-1}(\bar{t}, s)g(s)ds \\ &\leq \mu_M(1-q)L|v(\bar{t})| + C_1 = r_M^{-1}(1-q)LR_2 + C_1. \end{aligned}$$

Consequently,

$$\begin{aligned} h(R_2) &\leq r_M^{-1}(1-q)h(LR_2) + h(C_1) \\ &= r_M^{-1}(1-q)(L^*h)(R_2) + h(C_1) \\ &= r_M^{-1}(1-q)r_M h(R_2) + h(C_1) \\ &= (1-q)h(R_2) + h(C_1). \end{aligned}$$

Thus,

$$(C_1 - qR_2)h(1) \geq 0.$$

Since $h(1) > 0$, we have $C_1 \geq qR_2$. But this contradicts (4.17). Thus, (4.18) holds.

Now, Lemma 2.1 implies

$$\deg(I - T, B(\mathbf{0}, R_2), \mathbf{0}) = 1. \tag{4.20}$$

By the additivity property of the Leray-Schauder degree, (4.14), and (4.20), we have

$$\deg(I - T, B(\mathbf{0}, R_1) \setminus \overline{B(\mathbf{0}, R_2)}) = -1.$$

Then, from the solution property of the Leray-Schauder degree, T has at least one fixed point v in $B(\mathbf{0}, R_1)\setminus\overline{B(\mathbf{0}, R_2)}$, which is a solution of BVP (4.7), (4.8). Therefore, we have shown that, for $\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}_+^n$ satisfying (4.16) and (4.17), BVP (4.7), (4.8) has at least one solution $v(t)$ satisfying $||v|| \geq R_2$. Thus, for each $\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}_+^n$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, BVP (1.1), (1.2) has at least one solution $u(t) = v(t) + \sum_{i=0}^{n-1} \lambda_i y_i(t)$ satisfying

$$||u|| \geq ||v|| - \sum_{i=0}^{n-1} \lambda_i ||y_i|| \geq R_2 - \sum_{i=0}^{n-1} \lambda_i ||y_i|| > 0.$$

This completes the proof of the theorem. □

Proof of Theorem 3.2. We first verify that conditions (A1) and (A2)*–(A4)* of Lemma 2.4 are satisfied. As in the proof of Theorem 3.1, there exist $\varphi_L, \varphi_M \in P \setminus \{\mathbf{0}\}$ and $h \in P^* \setminus \{\mathbf{0}\}$ defined by (4.11) such that (A1) holds.

From the fact that e is even and nondecreasing on $\mathbb{R}^+$, it is easy to see that

$$e(v(t)) \leq e(||v||) \quad \text{for all } v \in X \text{ and } t \in [0,1].$$

Thus,

$$||e(v)|| \leq e(||v||) \quad \text{for all } v \in X.$$

This, together with (3.5), implies that

$$\lim_{||v||\to 0} \frac{||e(v)||}{||v||} = 0 \quad \text{for any } v \in X.$$

Let $Hv(t) = \bar{d}e(v(t))$ for $v \in X$, where $\bar{d} = \max_{t\in[0,1]} d(t)$. Then, (A2)* of Lemma 2.4 holds.

Since $f_0 > \mu_M$, there exist $\epsilon > 0$ and $0 < \zeta_1 < 1$ such that

$$f(t,x) \geq \mu_M(1+\epsilon)x = r_M^{-1}(1+\epsilon)x \geq 0 \quad \text{for } (t,x) \in [0,1]\times[0,2\zeta_1]. \tag{4.21}$$

Let $\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}_+^n$ be small enough so that

$$\sum_{i=0}^{n-1} \lambda_i ||y_i|| \leq \zeta_1 \tag{4.22}$$

and F_λ be defined by (4.9). Then, from (4.21), we have

$$\begin{aligned} F_\lambda v(t) &\geq \mu_M(1+\epsilon)\left(v(t) + \sum_{i=0}^{n-1}\lambda_i y_i(t)\right) \\ &\geq \mu_M(1+\epsilon)v(t) = r_M^{-1}(1+\epsilon)v(t) \quad \text{for all } v \in P \text{ with } ||v|| \leq \zeta_1. \end{aligned} \tag{4.23}$$

Let r be given in (H2). Now, in view of (3.4) and (4.23), we see that (A3)* of Lemma 2.4 holds with $F = F_\lambda$ and $r_1 = \min\{r, \zeta_1\}$.

From (3.5), there exists $0 < \zeta_2 < \min\{r, \zeta_1\}$ such that

$$-e(x) \geq \bar{d}^{-1} r_M^{-1}(1+\epsilon)x \quad \text{for } x \in [-\zeta_2, 0].$$

Then, from (3.4),

$$f(t,x) \geq d(t)\bar{d}^{-1} r_M^{-1}(1+\epsilon)x \geq r_M^{-1}(1+\epsilon)x \quad \text{for } (t,x) \in [0,1]\times[-\zeta_2,0]. \tag{4.24}$$

From (4.21) and (4.24), it is easy to see that

$$\begin{aligned} F_\lambda v(t) &\geq \mu_M(1+\epsilon)\left(v(t) + \sum_{i=0}^{n-1}\lambda_i y_i(t)\right) \\ &\geq \mu_M(1+\epsilon)v(t) = r_M^{-1}(1+\epsilon)v(t) \quad \text{for all } v \in X \text{ with } ||v|| \leq \zeta_2, \end{aligned}$$

which clearly implies that

$$LF_\lambda v(t) \geq r_M^{-1}(1+\epsilon)Lv(t) \quad \text{for all } v \in X \text{ with } ||u|| < \zeta_2.$$

Hence, (A4)* of Lemma 2.4 holds with $F = F_\lambda$ and $r_2 = \zeta_2$.

We have verified that all the conditions of Lemma 2.4 hold, so there exists $R_3 > 0$ such that

$$\deg(I - T, B(\mathbf{0}, R_3), \mathbf{0}) = 0. \tag{4.25}$$

Next, since $F_\infty < \mu_M$, there exist $0 < z < 1$ and $R > R_3$ such that

$$|f(t,x)| \le \mu_M(1-z)|x| = r_M^{-1}(1-z)|x| \quad \text{for } (t,|x|) \in [0,1] \times (R, \infty). \tag{4.26}$$

Let

$$\begin{aligned} C_2 &= r_M^{-1}(1-z)\left(\sum_{i=0}^{n-1} \lambda_i ||y_i||\right) \max_{t\in[0,1]} \int_0^1 K_{n-1}(t,s)g(s)ds \\ &\quad + \max_{t\in[0,1],|x|\le R} |f(t,x)| \max_{t\in[0,1]} \int_0^1 K_{n-1}(t,s)g(s)ds. \end{aligned} \tag{4.27}$$

Then $0 < C_2 < \infty$. Choose R_4 large enough so that

$$R_4 > \max\{R,\ z^{-1}C_2\}. \tag{4.28}$$

We claim that

$$Tv \ne \tau v \quad \text{for all } v \in \partial B(\mathbf{0}, R_4) \text{ and } \tau \ge 1. \tag{4.29}$$

If this is not the case, then there exist $\bar{v} \in \partial B(\mathbf{0}, R_4)$ and $\bar{\tau} \ge 1$ such that $T\bar{v} = \bar{\tau}\bar{v}$. It follows that $\bar{v} = \bar{s}T\bar{v}$, where $\bar{s} = 1/\bar{\tau}$. Clearly, $\bar{s} \in (0,1]$. Assume $R_4 = ||\bar{v}|| = |\bar{v}(\bar{t})|$ for some $\bar{t} \in [0,1]$. Let

$$J_1(\bar{v}) = \left\{ t \in [0,1] \ : \ \left| \bar{v}(t) + \sum_{i=0}^{n-1} \lambda_i y_i(t) \right| > R \right\},$$

$$J_2(\bar{v}) = [0,1] \setminus J_1(\bar{v}),$$

and

$$p(\bar{v}(t)) = \min\left\{ \left| \bar{v}(t) + \sum_{i=0}^{n-1} \lambda_i y_i(t) \right|,\ R \right\} \quad \text{for } t \in [0,1].$$

Then, from (2.31), (4.10), (4.26), and (4.27), it follows that

$$\begin{aligned}
R_4 &= |\bar{v}(\bar{t})| = \bar{s}|T\bar{v}(\bar{t})| \\
&\leq \int_0^1 K_{n-1}(\bar{t},s)g(s)|F_\lambda \bar{v}(s)|ds \\
&= \int_{J_1(\bar{v})} K_{n-1}(\bar{t},s)g(s)|F_\lambda \bar{v}(s)|ds + \int_{J_2(\bar{v})} K_{n-1}(\bar{t},s)g(s)|F_\lambda \bar{v}(s)|ds \\
&\leq r_M^{-1}(1-z)\int_{J_1(\bar{v})} K_{n-1}(\bar{t},s)g(s)\left|\bar{v}(s) + \sum_{i=0}^{n-1}\lambda_i y_i(s)\right|ds \\
&\quad + \int_{J_2(\bar{v})} K_{n-1}(\bar{t},s)g(s)|F_\lambda p(\bar{v}(s))|ds \\
&\leq r_M^{-1}(1-z)\int_0^1 K_{n-1}(\bar{t},s)g(s)|\bar{v}(s)|ds \\
&\quad + r_M^{-1}(1-z)\left(\sum_{i=0}^{n-1}\lambda_i ||y_i||\right)\int_0^1 K_{n-1}(\bar{t},s)g(s)ds \\
&\quad + \int_0^1 K_{n-1}(\bar{t},s)g(s)|F_\lambda p(\bar{v}(s))|ds \\
&\leq r_M^{-1}(1-z)L|v(\bar{t})| + C_2 = r_M^{-1}(1-z)LR_4 + C_2.
\end{aligned}$$

Hence, for h defined by (4.11), we have

$$\begin{aligned}
h(R_4) &\leq r_M^{-1}(1-z)h(LR_4) + h(C_2) \\
&= r_M^{-1}(1-z)(L^*h)(R_4) + h(C_2) \\
&= r_M^{-1}(1-z)r_M h(R_4) + h(C_2) \\
&= (1-z)h(R_4) + h(C_2),
\end{aligned}$$

which implies

$$(zR_4 - C_2)h(1) \leq 0.$$

In view of the fact that $h(1) > 0$, it follows that $R_4 \leq z^{-1}C_2$. This contradicts (4.28) and so (4.29) holds. By Lemma 2.1, we have

$$\deg(I - T, B(\mathbf{0}, R_4), \mathbf{0}) = 1. \tag{4.30}$$

By the additivity property of the Leray-Schauder degree, (4.25), and (4.30), we obtain

$$\deg(I - T, B(\mathbf{0}, R_4) \setminus \overline{B(\mathbf{0}, R_3)}) = 1.$$

Thus, from the solution property of the Leray-Schauder degree, T has at least one fixed point v in $B(\mathbf{0}, R_4)\backslash\overline{B(\mathbf{0}, R_3)}$, which is a solution of BVP (4.7), (4.8). Therefore, we have shown that, for $\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}^n_+$ satisfying (4.22), BVP (4.7), (4.8) has at least one solution $v(t)$ satisfying $||v|| \geq R_3$. Thus, for each $\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{R}^n_+$ with $\sum_{i=0}^{n-1} \lambda_i$ sufficiently small, BVP (1.1), (1.2) has at least one solution $u(t) = v(t) + \sum_{i=0}^{n-1} \lambda_i y_i(t)$ satisfying

$$||u|| \geq ||v|| - \sum_{i=0}^{n-1} \lambda_i ||y_i|| \geq R_3 - \sum_{i=0}^{n-1} \lambda_i ||y_i|| > 0.$$

This completes the proof of the theorem. □

Proof of Theorem 3.3. For $(t, x) \in [0, 1] \times \mathbb{R}$, let

$$f_1(t,x) = \begin{cases} f(t,x), & x \geq 0, \\ -f(t,x), & x < 0. \end{cases} \tag{4.31}$$

In virtue of (H3), we see that $f_1 : [0, 1]\times\mathbb{R} \to \mathbb{R}$ is continuous and nonnegative. Then, (H1) with $f = f_1$ is trivially satisfied. Moreover, from $F_0 < \mu_M < f^*_\infty$, it follows that $F_{1,0} < \mu_M < f_{1,\infty}$, where

$$F_{1,0} = \limsup_{x\to 0} \max_{t\in[0,1]} \left|\frac{f_1(t,x)}{x}\right| \quad \text{and} \quad f_{1,\infty} = \liminf_{x\to\infty} \min_{t\in[0,1]} \frac{f_1(t,x)}{x}.$$

Thus, by Theorem 3.1, we know that the BVP consisting of the equation

$$u^{(n)} + g(t)f_1(t,u) = 0, \ t \in (0,1),$$

and BC (1.2) has at least one nontrivial solution $u_1(t)$. By Lemma 2.5 (a), we have

$$u_1(t) = \int_0^1 K_{n-1}(t,s)g(s)f_1(s,u_1(s))ds.$$

Then, by Remark 2.1, $u_1(t) > 0$ on $(0, 1)$. Therefore, from (4.31), $f_1(t, u(t)) = f(t, u(t))$, and so $u_1(t)$ is a positive solution of BVP (1.1), (1.2).

For $(t, x) \in [0, 1] \times \mathbb{R}$, let

$$f_2(t,x) = \begin{cases} -f(t,-x), & x \geq 0, \\ f(t,-x), & x < 0. \end{cases} \tag{4.32}$$

In virtue of (H3), we see that $f_2 : [0, 1]\times\mathbb{R} \to \mathbb{R}$ is continuous and nonnegative. Then, (H1) with $f = f_2$ is trivially satisfied. Moreover, from $F_0 < \mu_M < f^*_\infty$, it follows that

$F_{2,0} < \mu_M < f_{2,\infty}$, where

$$F_{2,0} = \limsup_{x\to 0} \max_{t\in[0,1]} \left|\frac{f_2(t,x)}{x}\right| \quad \text{and} \quad f_{2,\infty} = \liminf_{x\to\infty} \min_{t\in[0,1]} \frac{f_2(t,x)}{x}.$$

Thus, as above, we know that the BVP consisting of the equation

$$u^{(n)} + g(t)f_2(t,u) = 0, \ t \in (0,1),$$

and BC (1.2) has at least one solution $v(t)$ satisfying $v(t) > 0$ on $(0,1)$ and

$$v(t) = \int_0^1 K_{n-1}(t,s)g(s)f_2(s,v(s))ds.$$

Then, from (4.32),

$$-v(t) = \int_0^1 K_{n-1}(t,s)g(s)f(s,-v(s))ds.$$

Therefore, $u_2(t) := -v(t)$ is a negative solution of BVP (1.1), (1.2), and the theorem is proved. □

Using Theorem 3.2, Theorem 3.4 can be proved by similar ideas as those given in the proof of Theorem 3.3. We omit the details here.

Lemma 4.2. *Let μ_M, $\mu_{\tilde{M}}$, and $\mu_{\hat{M}}$ be given in* (3.1). *Then we have the following:*

(a) $A \le \mu_M \le B$, *where A and B are defined in* (3.6).

(b) $\tilde{A} \le \mu_{\tilde{M}} \le \tilde{B}$, *where $\tilde{A}$ and $\tilde{B}$ are defined in* (3.7).

(c) $\hat{A} \le \mu_{\hat{M}} \le \hat{B}$, *where $\hat{A}$ and $\hat{B}$ are defined in* (3.8).

Proof. We first prove part (a). Let φ_M be given as in Lemma 2.7 (b). Then,

$$\varphi_M(t) = \mu_M \int_0^1 K_{n-1}(t,s)g(s)\varphi_M(s)ds \quad \text{for } t \in [0,1].$$

By Lemma 2.6 (a), we have

$$\varphi_M(t) \le \mu_M \nu \int_0^1 (1-s)g(s)\varphi_M(s)ds \quad \text{on } [0,1] \tag{4.33}$$

and

$$\varphi_M(t) \ge \mu_M \mu(t) \int_0^1 (1-s)g(s)\varphi_M(s)ds \quad \text{on } [0,1]. \tag{4.34}$$

Thus,

$$\varphi_M(t) \ge \frac{1}{\nu}\mu(t)||\varphi_M|| \quad \text{on } [0,1]. \tag{4.35}$$

From (4.33), we have

$$\varphi_M(t) \leq \mu_M \nu ||\varphi_M|| \int_0^1 (1-s)g(s)ds \quad \text{on } [0,1].$$

Hence,

$$\mu_M \geq \frac{1}{\nu \int_0^1 (1-s)g(s)ds} = A.$$

From (4.34) and (4.35), we see that

$$\begin{aligned} \varphi_M(t) &\geq \frac{1}{\nu}\mu_M \mu(t) ||\varphi_M|| \int_0^1 (1-s)g(s)\mu(s)ds \\ &\geq \frac{1}{\nu}\mu_M \underline{\mu}\, ||\varphi_M|| \int_0^1 (1-s)g(s)\mu(s)ds \quad \text{for } t \in [\theta_1, \theta_2]. \end{aligned}$$

Hence,

$$\mu_M \leq \frac{\nu}{\underline{\mu} \int_0^1 (1-s)g(s)\mu(s)ds} = B.$$

This proves part (a).

By Lemma 2.6 (b) and (c), and a similar argument as above, parts (b) and (c) can be proved. This completes the proof of the lemma. □

Proof of Corollary 3.1. The conclusion follows readily from Theorem 3.1 and Lemma 4.1 (a). □

Proof of Corollary 3.2. The conclusion follows readily from Theorem 3.2 and Lemma 4.1 (a). □

Proof of Corollary 3.3. The conclusion follows readily from Theorem 3.3 and Lemma 4.1 (a). □

Proof of Corollary 3.4. The conclusion follows readily from Theorem 3.4 and Lemma 4.1 (a). □

Finally, we remark that, by virtue of Lemma 2.5 (b) and (c), Lemma 4.1 (b) and (c), and Lemma 4.2 (b) and (c), Theorems 3.5–3.12 and Corollaries 3.5–3.12 can be proved by techniques similar to those in the proofs of Theorems 3.1–3.4 and Corollaries 3.1–3.4. To avoid redundancy and save space, we omit the details.

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Appendix: Equality of two eigenvalues

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Let P be the cone of non-negative functions in $C[0,1]$,

$$P = \{u \in C[0,1] : u(t) \geq 0, t \in [0,1]\}.$$

We consider linear operators defined by

$$Lu(t) = \int_0^1 C(t,s)g(s)u(s)\,ds, \quad Mu(t) = \int_0^1 C(s,t)g(s)u(s)\,ds,$$

where $C(t,s) \geq 0$ for almost all $t,s \in [0,1]$ together with some implicit conditions that ensure that L, M are well defined bounded linear operators on $C[0,1]$. For example, we might ask that C be continuous and g belong to L^1.

Proposition 5.1. *Let $g \in L^1(0,1)$ be such that $g(s) \geq 0$ for almost all $s \in [0,1]$. Suppose that L and M are well defined bounded linear operators on $C[0,1]$ and that λ_L and λ_M are positive eigenvalues of L, and M with respective eigenfunctions φ, ψ in $P \setminus \{0\}$. Suppose that $\int_0^1 g(s)\varphi(s)\psi(s)\,ds > 0$. Then $\lambda_L = \lambda_M$.*

Proof. We have

$$\lambda_L \varphi(t) = \int_0^1 C(t,s)g(s)\varphi(s)\,ds \quad \text{and} \quad \lambda_M \psi(t) = \int_0^1 C(s,t)g(s)\psi(s)\,ds.$$

Then, for almost every $t \in (0,1)$,

$$\lambda_L g(t)\varphi(t)\psi(t) = \int_0^1 g(t)\psi(t)C(t,s)g(s)\varphi(s)\,ds.$$

Integrating over $[0,1]$ gives

$$\begin{aligned}
\lambda_L \int_0^1 g(t)\varphi(t)\psi(t)\,dt &= \int_0^1 \Big(\int_0^1 C(t,s)g(s)\varphi(s)\,ds\Big)g(t)\psi(t)\,dt \\
&= \int_0^1 \Big(\int_0^1 C(t,s)g(t)\psi(t)\,dt\Big)g(s)\varphi(s)\,ds \\
&= \lambda_M \int_0^1 g(s)\varphi(s)\psi(s)\,ds,
\end{aligned}$$

where changing the order of integration is justified by Tonelli's theorem since

$$\int_0^1 g(t)\varphi(t)\psi(t)\,dt \leq \|\varphi\|\,\|\psi\| \int_0^1 g(t)\,dt < \infty.$$

As $\int_0^1 g(t)\varphi(t)\psi(t)\,dt > 0$, this proves $\lambda_L = \lambda_M$. □

Remark 5.1. The argument here is similar to one used in [38]. In the proof, L and M do not need to be compact operators, but they are compact if G is continuous and $g \in L^1$. Compactness is useful so that the Krein-Rutman theorem can be applied to assert that r_L and r_M are positive eigenvalues with eigenfunctions in $P \setminus \{0\}$.

Remark 5.2. The positivity condition $\int_0^1 g(t)\varphi(t)\psi(t)\,dt > 0$ is satisfied if there is a subinterval $[t_0, t_1]$ such that $\varphi(t)\psi(t) > 0$ for $t \in [t_0, t_1]$ and g is positive almost everywhere on $[t_0, t_1]$, say g is continuous on $(0,1)$ and $g \not\equiv 0$ on any subinterval of

$(0, 1)$. Under quite general conditions, it is known that $\varphi(t) > 0$ for $t \in [t_0, t_1]$ for an arbitrarily chosen $[t_0, t_1]$ in $(0, 1)$ (see, for example, [36]), and then $[t_0, t_1]$ can be chosen so that $\varphi(t)\psi(t) > 0$ for $t \in [t_0, t_1]$.

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