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Multiple positive solutions of nonlinear singular m-point boundary value problem for second-order dynamic equations with sign changing coefficients on time scales

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Abstract: Let $\mathbb{T}$ be a time scale. In this paper, we study the existence of multiple positive solutions for the following nonlinear singular m-point boundary value problem dynamic equations with sign changing coefficients on time scales

$$\begin{cases} u^{\triangle\nabla}(t) + a(t)f(u(t)) = 0, \quad (0,T)_{\mathbb{T}}, \\ u^{\triangle}(0) = \sum_{i=1}^{m-2} a_i u^{\triangle}(\xi_i), \\ u(T) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u^{\triangle}(\xi_i), \end{cases}$$

where $1 \leq k \leq s \leq m-2$, a_i, $b_i \in (0,+\infty)$ with $0 < \sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i < 1$, $0 < \sum_{i=1}^{m-2} a_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \rho(T)$, $f \in C([0,+\infty),[0,+\infty))$, $a(t)$ may be singular at $t = 0$. We show that there exist two positive solutions by using two different fixed point theorems respectively. As an application, some examples are included to illustrate the main results. In particular, our criteria extend and improve some known results.

MSC: 34B15; 34B25

Keywords: M-point boundary value problem; Positive solutions; Fixed-point theorem; Time scales

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1 Introduction

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We make the blanket assumption that 0, T are points in $\mathbb{T}$. By an interval $(0,T)_{\mathbb{T}}$, we always mean the intersection of the real interval $(0,T)_{\mathbb{T}}$ with the given time scale, that is $(0,T) \cap \mathbb{T}$. The theory of dynamical systems on time scales is undergoing rapid development as it provides a unifying structure for the study of differential equations in the continuous case and the study of finite difference equations in the discrete case. Here, two-point boundary value problems have been extensively studied; for details, see [3,4,5,7] and the references therein. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [6,8,9,14,16,18-21] for some references along

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this line. Multi-point boundary value problems describe many phenomena in the applied mathematical sciences.

In [9], Anderson discussed the existence of positive solution of the following three-point boundary value problem on time scales

$$\begin{cases} u^{\triangle\nabla}(t) + q(t)f(u(t)) = 0, \ t \in (0, T), \\ u(0) = 0, \ \alpha u(\eta) = u(T). \end{cases}$$

The main tools are the Krasnoselskii's fixed point theorem and Leggett-Williams fixed point theorem.

In 2001, Ma [6] studied m-point boundary value problem (BVP)

$$\begin{cases} u''(t) + h(t)f(u) = 0, \qquad 0 \leq t \leq 1, \\ u(0) = 0, \qquad u(1) = \sum\limits_{i=1}^{m-2} \alpha_i u(\xi_i), \end{cases}$$

where $\alpha_i > 0$ $(i = 1, 2, \cdots, m-2)$, $\sum\limits_{i=1}^{m-2} \alpha_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $f \in C([0, +\infty), [0, +\infty))$, $h \in C([0, 1], [0, +\infty))$. Author established the existence of positive solutions theorems under the condition that f is either superlinear or sublinear.

In [8], Ma and Castaneda considered the following m-point boundary value problem (BVP)

$$\begin{cases} u''(t) + h(t)f(u) = 0, \qquad 0 \leq t \leq 1, \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \qquad u(1) = \sum\limits_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases}$$

where $\alpha_i > 0$, $\beta_i > 0$ $(i = 1, 2, \cdots, m-2)$, $\sum\limits_{i=1}^{m-2} \alpha_i < 1$, $\sum\limits_{i=1}^{m-2} \beta_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $f \in C([0, +\infty), [0, +\infty))$, $h \in C([0, 1], [0, +\infty))$. They showed the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem in cones.

Motivated by the results mentioned above, in this paper we study the existence of positive solutions for the following nonlinear singular m-point boundary value problem dynamic equation with sign changing coefficients on time scales

$$\begin{cases} u^{\triangle\nabla}(t) + a(t)f(u(t)) = 0, \quad (0, T)_{\mathbb{T}}, \\ u^{\triangle}(0) = \sum_{i=1}^{m-2} a_i u^{\triangle}(\xi_i), \\ u(T) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u^{\triangle}(\xi_i), \end{cases} \tag{1.1}$$

where $1 \leq k \leq s \leq m-2$, with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \rho(T)$ and a_i, b_i, f satisfy
(H_1) a_i, $b_i \in (0, +\infty)$, $0 < \sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i < 1$, $0 < \sum_{i=1}^{m-2} a_i < 1$;
(H_2) $f \in C([0, +\infty), [0, +\infty))$, $a(t) : [0, T]_{\mathbb{T}} \to [0, +\infty)$ and does not vanish identically on any closed subinterval of $[0, T]_{\mathbb{T}}$. Moreover

$$0 < \int_0^T a(s) \nabla s < \infty.$$

For convenience, we list here the following definitions which are needed later.

A time scale $[0,T]_{\mathbb{T}}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. The operators σ and ρ from $[0,T]_{\mathbb{T}}$ to $[0,T]_{\mathbb{T}}$ which defined by [1-3],

$$\sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\} \in \mathbb{T},\ \rho(t) = \sup\{\tau \in \mathbb{T} \mid \tau < t\} \in \mathbb{T},$$

are called the forward jump operator and the backward jump operator, respectively.

The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t,\ \rho(t) < t,\ \sigma(t) = t,\ \sigma(t) > t$, respectively. If $\mathbb{T}$ has a right scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^k = \mathbb{T}$.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$ (assume t is not left-scattered if $t = \sup \mathbb{T}$), then the delta derivative of f at the point t is defined by

$$f^{\Delta}(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

Similarly, for $t \in \mathbb{T}$ (assume t is not right-scattered if $t = \inf \mathbb{T}$), the nabla derivative of f at the point t is defined by

$$f^{\nabla}(t) := \lim_{s \to t} \frac{f(\rho(t)) - f(s)}{\rho(t) - s}.$$

A function f is left-dense continuous (i.e., ld-continuous), if f is continuous at each left-dense point in $\mathbb{T}$ and its right-sided limit exists at each right-dense point in $\mathbb{T}$. It is well-known that if f is ld-continuous.

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Throughout this article, $\mathbb{T}$ is closed subset of $\mathbb{R}$ with $0 \in \mathbb{T}_k, T \in \mathbb{T}^k$.

By a positive solution of BVP (1.1), we understand a function u which is positive on $(0,T)_{\mathbb{T}}$ and satisfies the differential equations as well as the boundary conditions in BVP (1.1).

2 Preliminaries and lemmas

In this section, we give some definitions and preliminaries that are important to our main results.

Definition 2.1. Let E be a real Banach space over R . A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $u \in P$, $a \geq 0$ implies $au \in P$; and

(ii) $u,\ -u \in P$ implies $u = 0$.

Definition 2.2. Given a cone P in a real Banach space E, a functional $\psi : P \to P$ is said to be increasing on P, provided $\psi(x) \leq \psi(y)$, for all $x, y \in P$ with $x \leq y$.

Definition 2.3. Given a nonnegative continuous functional γ on P of a real Banach space, we define for each $d > 0$ the set

$$P(\gamma, d) = \{x \in P | \gamma(x) < d\}.$$

Theorem 2.1[See11,12]. *Let E be a real Banach space, $P \subset E$ be a cone. Assume there exist positive numbers c and M, nonnegative increasing continuous functionals $\alpha,\ \gamma$ on P , and nonnegative continuous functional θ on P with $\theta(0) = 0$ such that*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \text{and} \quad ||x|| \leq M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose

$$A : \overline{P(\gamma, c)} \to P$$

is a completely continuous operator and there exist positive numbers $a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x) \quad \text{for } 0 \leq \lambda \leq 1 \text{ and } \ x \in \partial P(\theta, b),$$

and

(C_1) $\gamma(Ax) > c$ for all $x \in \partial P(\gamma, c)$;

(C_2) $\theta(Ax) < b$ for all $x \in \partial P(\theta, b)$;

(C_3) $P(\alpha, a) \neq \emptyset$ and $\alpha(Ax) > a$ for all $x \in \partial \underline{P(\alpha, a)}$.

Then A has at least two fixed points $x_1,\ x_2 \in \overline{P(\gamma,\ c)}$ satisfying

$$a < \alpha(x_1), \ \text{with } \theta(x_1) < b,$$

and

$$b < \theta(x_2), \ \text{with } \ \gamma(x_2) < c.$$

Theorem 2.2[See13]. *Let E be a real Banach space, $P \subset E$ be a cone. Assume there exist positive numbers c and M, nonnegative increasing continuous functionals $\alpha,\ \gamma$ on P , and nonnegative continuous functional θ on P with $\theta(0) = 0$ such that*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \text{and} \quad ||x|| \leq M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose

$$A : \overline{P(\gamma, c)} \to P$$

is a completely continuous operator and there exist positive numbers $a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x) \quad \text{for } 0 \leq \lambda \leq 1 \text{ and } \ x \in \partial P(\theta, b),$$

and

(C_1) $\gamma(Ax) < c$ for all $x \in \partial P(\gamma, c)$;

(C_2) $\theta(Ax) > b$ for all $x \in \partial P(\theta, b)$;

(C_3) $P(\alpha, a) \neq \emptyset$ and $\alpha(Ax) < a$ for all $x \in \partial P(\alpha, a)$.

Then A has at least two fixed points $x_1,\ x_2 \in \overline{P(\gamma,\ c)}$ satisfying

$$a < \alpha(x_1),\ \text{with}\ \theta(x_1) < b,$$

and

$$b < \theta(x_2),\ \text{with}\ \ \gamma(x_2) < c.$$

Lemma 2.1. *Let (H_1) and (H_2) hold. Then for $y \in C^+_{ld}[0,T]$, the boundary value problem*

$$\begin{cases} u^{\triangle\nabla}(t) + y(t) = 0, \quad 0 < t < T, \\ u^{\triangle}(0) = \sum_{i=1}^{m-2} a_i u^{\triangle}(\xi_i), \\ u(T) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u^{\triangle}(\xi_i), \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = -\int_0^t (t-s)y(s)\nabla s - \alpha t + \beta, \tag{2.2}$$

where

$$\alpha = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} y(s)\nabla s}{1 - \sum_{i=1}^{m-2} a_i},$$

$$\begin{aligned} \beta \ &= \frac{1}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big(\int_0^T (T-s)y(s)\nabla s - \sum_{i=1}^{k} b_i \int_0^{\xi_i} (\xi_i - s)y(s)\nabla s \\ &+ \textstyle\sum_{i=k+1}^{s} b_i \displaystyle\int_0^{\xi_i} (\xi_i - s)y(s)\nabla s + \sum_{i=s+1}^{m-2} b_i \int_0^{\xi_i} y(s)\nabla s \\ &+ \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} y(s)\nabla s}{1 - \sum_{i=1}^{m-2} a_i}\big(T - \sum_{i=1}^{k} b_i\xi_i + \sum_{i=k+1}^{s} b_i\xi_i + \sum_{i=s+1}^{m-2} b_i\big)\Big). \end{aligned}$$

Proof. Firstly, by integrating the equation of the problems (2.1) on $(0,t)$, we have

$$u^{\triangle}(t) = u^{\triangle}(0) - \int_0^t y(s)\nabla s. \tag{2.3}$$

Integrating (2.3) from 0 to t, we get

$$u(t) = u(0) + u^{\triangle}(0)t - \int_0^t (t-s)y(s)\nabla s. \tag{2.4}$$

Set $t = \xi_i (i = 1, 2, \cdots, m-2)$ in (2.3), we have

$$u^{\triangle}(\xi_i) = u^{\triangle}(0) - \int_0^{\xi_i} y(s)\nabla s, i = 1, 2, \cdots, m-2. \tag{2.5}$$

The boundary condition $u^{\triangle}(0) = \sum_{i=1}^{m-2} a_i u^{\triangle}(\xi_i)$ and (2.5) yield

$$u^{\triangle}(0) = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} y(s)\nabla s}{1 - \sum_{i=1}^{m-2} a_i}. \tag{2.6}$$

Set $t = T, \xi_i (i = 1, 2, \cdots, m-2)$ in (2.4), respectively, we have

$$u(T) = u(0) + u^{\triangle}(0)T - \int_0^T (T-s)y(s)\nabla s. \tag{2.7}$$

and

$$u(\xi_i) = u(0) + u^{\triangle}(0)\xi_i - \int_0^{\xi_i} (\xi_i - s)y(s)\nabla s, (i = 1, 2, \cdots, m-2). \tag{2.8}$$

Using the boundary condition $u(T) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u^{\triangle}(\xi_i)$, we have from (2.5)-(2.8),

$$\begin{aligned} u(0) &= \frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big(\int_0^T (T-s)y(s)\nabla s - \sum_{i=1}^{k} b_i \int_0^{\xi_i} (\xi_i - s)y(s)\nabla s \\ &+ \sum_{i=k+1}^{s} b_i \int_0^{\xi_i} (\xi_i - s)y(s)\nabla s + \sum_{i=s+1}^{m-2} b_i \int_0^{\xi_i} y(s)\nabla s \\ &+ \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} y(s)\nabla s}{1-\sum_{i=1}^{m-2} a_i}\big(T - \sum_{i=1}^{k} b_i\xi_i + \sum_{i=k+1}^{s} b_i\xi_i + \sum_{i=s+1}^{m-2} b_i\big)\Big). \end{aligned} \tag{2.9}$$

Substituting (2.6), (2.9) into (2.4), we have know that $u(t)$ satisfies (2.2). The proof of Lemma 2.2 is completed.

Let $E = C_{ld}[0,T]$, then E is Banach space, with respect to the norm $\|u\| = \sup_{t\in[0,T]} |u(t)|$. Now we define $P = \{u \in E | \ u$ is a concave, nonincreasing and nonnegative function$\}$. Obviously, P is a cone in E.

Define an operator $A : P \to E$ by setting

$$(Au)(t) = -\int_0^t (t-s)a(s)f(u(s))\nabla s - \widetilde{\alpha} t + \widetilde{\beta},$$

where

$$\widetilde{\alpha} = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(s)f(u(s))\nabla s}{1-\sum_{i=1}^{m-2} a_i},$$

$$\begin{aligned} \widetilde{\beta} &= \frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big(\int_0^T (T-s)a(s)f(u(s))\nabla s - \sum_{i=1}^{k} b_i \int_0^{\xi_i} (\xi_i - s)a(s)f(u(s))\nabla s \\ &+ \sum_{i=k+1}^{s} b_i \int_0^{\xi_i} (\xi_i - s)a(s)f(u(s))\nabla s + \sum_{i=s+1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))\nabla s \\ &+ \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(s)f(u(s))\nabla s}{1-\sum_{i=1}^{m-2} a_i}\big(T - \sum_{i=1}^{k} b_i\xi_i + \sum_{i=k+1}^{s} b_i\xi_i + \sum_{i=s+1}^{m-2} b_i\big)\Big). \end{aligned}$$

It is clear that the existence of a positive solution for the boundary value problems (1.1) is equivalent to the existence of a fixed point of the operator A .

Lemma 2.2. *Let (H_1) and (H_2) hold. If $x \in C_{ld}^{+}[0,T]$, the unique solution of the problem (2.1) satisfies $u(t) \geq 0$.*

Proof. According to $u^{\triangle\nabla}(t)=-y(t)\le 0$, we know that $u^{\triangle}(t)$ is nonincreasing on the interval $[0,T]$. From $u^{\triangle}(0)=\sum_{i=1}^{m-2}a_iu^{\triangle}(\xi_i)\le u^{\triangle}(0)\sum_{i=1}^{m-2}a_i$ and $\sum_{i=1}^{m-2}a_i<1$, we can get $u^{\triangle}(0)\le 0$, and

$$u^{\triangle}(t)\le u^{\triangle}(0)\le 0, \quad \text{for} \ \ t\in[0,T].$$

It tells us that $u(t)$ is nonincreasing on the interval $[0,T]$. $||u||=u(0), \inf_{t\in[0,T]}u(t)=u(T)$.

$$\begin{aligned}
u(T) &= -\int_0^T (T-s)y(s)\nabla s-\alpha T+\beta\\
&= \frac{1}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^s b_i}\Big(\sum_{i=1}^k b_i\int_0^T(T-s)y(s)\nabla s-\sum_{i=k+1}^s b_i\int_0^T(T-s)y(s)\nabla s\\
&-\textstyle\sum_{i=1}^k b_i\int_0^{\xi_i}(\xi_i-s)y(s)\nabla s+\sum_{i=k+1}^s b_i\int_0^{\xi_i}(\xi_i-s)y(s)\nabla s+\sum_{i=s+1}^{m-2}b_i\int_0^{\xi_i}y(s)\nabla s\\
&+\alpha\big((\textstyle\sum_{i=1}^k b_i-\sum_{i=k+1}^s b_i)T-\sum_{i=1}^k b_i\xi_i+\sum_{i=k+1}^s b_i\xi_i+\sum_{i=s+1}^{m-2}b_i\big)\Big)\\
&\ge \frac{1}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^s b_i}\Big((\sum_{i=1}^k b_i-\sum_{i=k+1}^s b_i)\int_0^{\xi_k}(T-\xi_k)y(s)\nabla s+\sum_{i=s+1}^{m-2}b_i\int_0^{\xi_i}y(s)\nabla s\\
&+\alpha\big((\textstyle\sum_{i=1}^k b_i-\sum_{i=k+1}^s b_i)(T-\xi_k)+\sum_{i=s+1}^{m-2}b_i\big)\Big)\\
&\ge 0.
\end{aligned}$$

So, $u(t)\ge 0$ for $t\in[0,T]$. The proof of Lemma 2.2 is completed.

Lemma 2.3. *$A:\overline{P(\gamma,d)}\to P$ is completely continuous.*

Proof. Let $u\in\overline{P(\gamma,d)}$, according to Lemma 2.2 we easily obtain $(Au)(t)\ge 0$. Clearly $(Au)^{\triangle\nabla}(t)=-a(s)f(u(s))\le 0$, we know that $(Au)(t)$ is concave and $(Au)^{\triangle}(t)$ nonincreasing on the interval $[0,T]_{\mathbb{T}}$. From $(Au)^{\triangle}(0)=\sum_{i=1}^{m-2}a_i(Au)^{\triangle}(\xi_i)\le (Au)^{\triangle}(0)\sum_{i=1}^{m-2}a_i$ and $\sum_{i=1}^{m-2}a_i<1$, we can get

$$(Au)^{\triangle}(0)\le 0,$$

and

$$(Au)^{\triangle}(t)\le (Au)^{\triangle}(0)\le 0, \quad \text{for} \ \ t\in[0,T]_{\mathbb{T}},$$

which implies that $(Au)(t)$ is nonincreasing on the interval $[0,T]_{\mathbb{T}}$. So $A(\overline{P(\gamma,d)})\subset P$. With standard argument one may show that A is a completely continuous operator by condition (H_2). The proof of Lemma 2.3 is completed.

Lemma 2.4. *If $u\in P$, then $u(t)\ge \frac{T-t}{T}||u||$, $t\in[0,T]_{\mathbb{T}}$.*

Proof. Since $u(t)$ is a concave, nonincreasing and nonnegative value on $[0,T]_{\mathbb{T}}$, we see that $u(0)\ge u(t)\ge u(T)\ge 0$ for $t\in[0,T]_{\mathbb{T}}$.

Let

$$x(t)=u(t)-\frac{T-t}{T}||u||, t\in[0,T]_{\mathbb{T}}.$$

Then

$$x^{\triangle\nabla}(t)\le t\in[0,T]_{\mathbb{T}}.$$

and

$$x(0)=0, x(T)\ge 0.$$

So, we get

$$x(t) = \frac{T-t}{T}x(0) + \frac{t}{T}x(T) - \int_0^T G(t,s)x^{\triangle\nabla}(s)\nabla s \geq 0 \text{ for } \quad t \in [0,T]_{\mathbb{T}},$$

where

$$G(t,s) = \frac{1}{T}\begin{cases} s(T-t), & 0 \leq s \leq t \leq T, \\ t(T-s), & 0 \leq t \leq s \leq T. \end{cases}$$

Hence, we obtain

$$u(t) \geq \frac{T-t}{T}||u||, \quad \text{for} \quad t \in [0,T]_{\mathbb{T}}.$$

The proof of Lemma 2.4 is completed.

3 Main results

In this section, let $h = \max\{t \in \mathbb{T}|\ 0 \leq t \leq \frac{T}{2}\}$ and fix $r \in \mathbb{T}$ such that $0 < r < h$, and define three nonnegative, increasing and continuous functionals γ, θ and α on P, by

$$\gamma(u) = \min_{r \leq t \leq h} u(t) = u(h),$$

$$\theta(u) = \max_{h \leq t \leq T} u(t) = u(h),$$

and

$$\alpha(u) = \max_{r \leq t \leq T} u(t) = u(r).$$

It is easy to see that $\gamma(u) = \theta(u) \leq \alpha(u)$ for $u \in P$. Moreover, by Lemma 2.4 we can know that $\gamma(u) = u(h) \geq \dfrac{T-h}{T}u(0) = \dfrac{T-h}{T}||u||$ for $u \in P$. Thus,

$$||u|| \leq \frac{T}{T-h}\gamma(u), \quad \text{for} \quad u \in P.$$

On the other hand, we have

$$\theta(\lambda u) = \lambda\theta(u), \lambda \in [0,1] \quad \text{and} \quad u \in \partial P(\theta, b).$$

For notational convenience, we define the following constants

$$\lambda = \frac{T-h}{T}\int_0^h (T-s)a(s)\nabla s,\ \lambda_r = \frac{T-r}{T}\int_r^T (T-s)a(s)\nabla s,$$

$$\eta = \frac{(1+\sum_{i=k+1}^{s} b_i)T + \sum_{i=s+1}^{m-2} b_i + \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}(T - \sum_{i=1}^{k} b_i\xi_i + \sum_{i=k+1}^{s} b_i\xi_i + \sum_{i=s+1}^{m-2} b_i)}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\int_0^T a(s)\nabla s.$$

Theorem 3.1. *Suppose conditions* (H_1) *and* (H_2) *hold, and assume that there are positive numbers* $a < b < c$ *such that*

$$0 < a < \frac{\lambda_r}{\eta}b < \frac{(T-h)\lambda_r}{T\eta}c,$$

and f satisfies the following conditions
(A_1): $f(u) > \frac{c}{\lambda}$ for $c \le u \le \frac{T}{T-h}c$;
(A_2): $f(u) < \frac{b}{\eta}$ for $0 \le u \le \frac{T}{T-h}b$;
(A_3): $f(u) > \frac{a}{\lambda_r}$ for $0 \le u \le a$.
Then, the boundary value problem (1.1) has at least two positive solutions u_1 and u_2 such that

$$a < \alpha(u_1), \quad \textit{with} \quad \theta(u_1) < b,$$

and

$$b < \theta(u_2) \quad \textit{with} \quad \gamma(u_2) < c.$$

Proof. By Lemma 2.3, we can know that $A : \overline{P(\gamma, c)} \to P$ is completely continuous.

We firstly show that if $u \in \partial P(\gamma, c)$, then $\gamma(Au) > c$.

Indeed, if $u \in \partial P(\gamma, c)$, then $\gamma(u) = \min_{r \le t \le h} u(t) = u(h) = c$. Since $u \in P, ||u|| \le \frac{T}{T-h}\gamma(u)$, we have

$$c \le u(t) \le \frac{T}{T-h}c, \quad t \in [0, h]_{\mathbb{T}}.$$

As a consequence of (A_1), $f(u) > \frac{c}{\lambda}$ for $0 \le t \le h$. Since $Au \in P$, we have from Lemma 2.4,

$$\begin{aligned}
\gamma(Au) \; &= (Au)(h) \ge \frac{T-h}{T}||Au|| = \frac{T-h}{T}(Au)(0) \\
&= \frac{T-h}{T}\frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big(\textstyle\int_0^T (T-s)a(s)f(u(s))\nabla s \\
&-\textstyle\sum_{i=1}^{k} b_i \int_0^{\xi_i}(\xi_i - s)a(s)f(u(s))\nabla s + \sum_{i=k+1}^{s} b_i \int_0^{\xi_i}(\xi_i - s)a(s)f(u(s))\nabla s \\
&+\textstyle\sum_{i=s+1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))\nabla s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(s)f(u(s))\nabla s\left(T - \sum_{i=1}^{k} b_i\xi_i + \sum_{i=k+1}^{s} b_i\xi_i + \sum_{i=s+1}^{m-2} b_i\right)}{1-\sum_{i=1}^{m-2} a_i}\Big) \\
&\ge \frac{T-h}{T}\frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big(\textstyle\int_0^T (T-s)a(s)f(u(s))\nabla s \\
&-\textstyle\sum_{i=1}^{k} b_i \int_0^{\xi_k}(\xi_k - s)a(s)f(u(s))\nabla s + \sum_{i=k+1}^{s} b_i \int_0^{\xi_k}(\xi_k - s)a(s)f(u(s))\nabla s\Big) \\
&\ge \frac{T-h}{T}\frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big(\textstyle\int_0^T (T-s)a(s)f(u(s))\nabla s \\
&-\textstyle(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i)\int_0^{\xi_k}(\xi_k - s)a(s)f(u(s))\nabla s\Big) \\
&\ge \frac{T-h}{T}\frac{(1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i)\int_0^T (T-s)a(s)f(u(s))\nabla s}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \\
&= \frac{T-h}{T}\int_0^T (T-s)a(s)f(u(s))\nabla s \\
&\ge \frac{T-h}{T}\frac{c}{\lambda}\int_0^h (T-s)a(s)\nabla s \\
&= c.
\end{aligned}$$

Then, condition (C_1) of Theorem 2.1 holds.

Let $u \in \partial P(\theta, b)$. Then $\theta(u) = \max_{h \le t \le T} u(t) = u(h) = b$. This implies $0 \le u(t) \le b$, for $h \le t \le T$, and since $u \in P$, we have $b \le u(t) \le ||u|| = u(0)$, for $h \le t \le T$. Note that

$$||u|| \le \frac{T}{T-h}\gamma(u) = \frac{T}{T-h}\theta(u) = \frac{T}{T-h}b \quad \text{for } u \in P.$$

So,

$$0 \le u(t) \le \frac{T}{T-h}b, \quad 0 \le t \le T.$$

From condition(A_2), $f(u) < \frac{b}{\eta}$ for $0 \le t \le T$, and so

$$\begin{aligned}
\theta(Au) &= (Au)(h) \le (Au)(0) \\
&\le \frac{1}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big(T\int_0^T a(s) f(u(s))\nabla s \\
&+ \textstyle\sum_{i=k+1}^{s} b_i T \int_0^T a(s) f(u(s))\nabla s + \sum_{i=s+1}^{m-2} b_i \int_0^T a(s) f(u(s))\nabla s \\
&+ \tfrac{\sum_{i=1}^{m-2} a_i \int_0^T a(s) f(u(s))\nabla s}{1 - \sum_{i=1}^{m-2} a_i}\big(T - \sum_{i=1}^{k} b_i\xi_i + \sum_{i=k+1}^{s} b_i\xi_i + \sum_{i=s+1}^{m-2} b_i\big)\Big) \\
&\le \frac{b}{\eta}\frac{1}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big((1 + \textstyle\sum_{i=k+1}^{s} b_i)T + \sum_{i=s+1}^{m-2} b_i \\
&+ \tfrac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i}\big(T - \sum_{i=1}^{k} b_i\xi_i + \sum_{i=k+1}^{s} b_i\xi_i + \sum_{i=s+1}^{m-2} b_i\big)\Big)\int_0^T a(s)\nabla s \\
&= b.
\end{aligned}$$

Then, condition (C_2) of Theorem 2.1 holds.

We note that $u(t) = \frac{a}{2}, t \in [0,T]_{\mathbb{T}}$, is a member of $P(\alpha, a)$ and $\alpha(u) = \frac{a}{2} < a$. So $P(\alpha, a) \neq \emptyset$.

Now, choose $u \in P(\alpha, a)$. Then $\alpha(u) = \max_{r \le t \le T} u(t) = u(r) = a$. This means that

$$0 \le u(t) \le a \quad \text{for} \quad r \le t \le T.$$

From condition(A_3), $f(u) > \frac{a}{\lambda_r}$ for $r \le t \le T$. Since $Au \in P$, we have from Lemma 2.4,

$$\begin{aligned}
\alpha(Au) &= (Au)(r) \ge \frac{T-r}{T}||Au|| = \frac{T-r}{T}(Au)(0) \\
&\ge \frac{T-r}{T}\frac{(1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i)\int_0^T (T-s)a(s)f(u(s))\nabla s}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \\
&\ge \frac{T-r}{T}\int_r^T (T-s)a(s)f(u(s))\nabla s \\
&\ge \frac{T-r}{T}\frac{a}{\lambda_r}\int_r^T (T-s)a(s)\nabla s \\
&= a.
\end{aligned}$$

Then, condition (C_3) of Theorem 2.1 holds.

Therefore, Theorem 2.1 implies that A has at least two fixed points which are positive solutions u_1 and u_2 such that

$$a < \alpha(u_1), \quad \text{with} \quad \theta(u_1) < b,$$

and

$$b < \theta(u_2) \quad \text{with} \quad \gamma(u_2) < c.$$

The proof of Theorem 3.1 is complete.
Theorem 3.2. *Suppose conditions (H_1) and (H_2) hold, and assume that there are positive numbers $a < b < c$ such that*

$$0 < a < \frac{T-r}{T}b < \frac{(T-r)\eta}{T\lambda}c,$$

and f satisfies the following conditions
(B_1): $f(u) < \frac{c}{\eta}$ for $0 \le u \le \frac{T}{T-h}c$;
(B_2): $f(u) > \frac{b}{\lambda}$ for $b \le u \le \frac{T}{T-h}b$;
(B_3): $f(u) < \frac{a}{\eta}$ for $0 \le u \le a$.
Then, the boundary value problem (1.1) has at least two positive solutions u_1 and u_2 such that

$$a < \alpha(u_1), \ \text{with } \theta(u_1) < b,$$

and

$$b < \theta(u_2), \ \text{with } \ \gamma(u_2) < c.$$

Proof. By using Theorem 2.2, the method of proof is similar to Theorem 3.1, so we omit it.

4 Examples

In the section, we present a simple example to explain our results.
Example 4.1. Let $\mathbb{T} = [0, \frac{1}{2}] \cup [\frac{3}{4}, 1]$. Consider the following five-point boundary value problem

$$\begin{cases} u^{\Delta\nabla}(t) + a(t)f(u) = 0, \quad t \in \mathbb{T}, \\ u^{\Delta}(0) = \frac{1}{8}u^{\Delta}(\frac{1}{4}) + \frac{1}{4}u^{\Delta}(\frac{1}{2}) + \frac{1}{2}u^{\Delta}(\frac{3}{4}), \\ u(1) = u(\frac{1}{4}) - \frac{1}{2}u(\frac{1}{2}) - 4u^{\Delta}(\frac{3}{4}), \end{cases} \tag{4.1}$$

where

$$f(u) = \begin{cases} 4, & 0 \le u \le 330, \\ 4 + \frac{1398}{585}(u - 330), & 330 \le u \le 1500, \\ u + 1300, & u \ge 1500. \end{cases}$$

Clearly, $a_1 = \frac{1}{8}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{2}$, $b_1 = 1$, $b_2 = \frac{1}{2}$, $b_3 = 4$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{2}$, $\xi_3 = \frac{3}{4}$, $a(t) = t^{-\frac{1}{2}}$. It is easy to see that $a(t)$ is singular at $t = 0$. Let $h = \frac{1}{2}$, $r = \frac{1}{4}$. By computing, we have

$$\lambda = \frac{5\sqrt{2}}{12}, \quad \eta = \frac{79(6\sqrt{2} - 5\sqrt{3} + 12)}{24}, \quad \lambda_r = \frac{20\sqrt{2} - 17\sqrt{3} + 10}{32}.$$

Choose $a = 1$, $b = 330$, $c = 1500$, then

$$f(u) = 4 > \frac{a}{\lambda_r} \approx 3.6, \quad 0 \leq u \leq 1;$$

$$f(u) = 4 < \frac{b}{\eta} \approx 8.5, \quad 0 \leq u \leq 330;$$

$$f(u) = u + 1300 > \frac{c}{\lambda} \approx 2547, \quad u \geq 1500.$$

By Theorem 3.1, we know the BVP (4.1) has at least two positive solutions u_1 and u_2 satisfying

$$1 < \max_{t\in[\frac{1}{4},1]} u_1(t), \quad \max_{t\in[\frac{1}{2},1]} u_1(t) < 330 < \max_{t\in[\frac{1}{2},1]} u_2(t), \quad \min_{t\in[\frac{1}{4},\frac{1}{2}]} u_2(t) < 1500.$$

Remark 4.1. In Example 4.1, $f_0 = \lim_{u\to 0} \frac{f(u)}{u} = \infty$, $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u} = 1$. It is said that f is not superlinear or sublinear, so the conclusion of [6,7,8] is invalid to the example.

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