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Dynamic analysis of an impulsively controlled predator-prey system

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Abstract

In this paper, we study an impulsively controlled predator-prey model with Monod-Haldane functional response. By using the Floquet theory, we prove that there exists a stable prey-free solution when the impulsive period is less than some critical value, and give the condition for the permanence of the system. In addition, we show the existence and stability of a positive periodic solution by using bifurcation theory.

Key words: Predator-prey model, Monod-Haldane type functional response, impulsive differential equation, Floquet theory.

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1 Introduction

In population dynamics, one of central goals is to understand the dynamical relationship between predator and prey. One important component of the predator-prey relationship is the predator's rate of feeding on prey, i.e., the so-called predator's functional response. Functional response refers to the change in the density of prey attached per unit time per predator as the prey density changes. Holling [7] gave three different kinds of functional response for different kinds of species to model the phenomena of predation, which made the standard Lotka-Volterra system more realistic. These functional responses are monotonic in the first quadrant. But, some experiments and observations indicate that a non-monotonic response occurs at a level: when the nutrient concentration reaches a high level an inhibitory effect on the specific growth rate may occur. To model such an inhibitory effect, the authors in [1, 14] suggested a function

$$p(x) = \frac{\alpha x}{b + x^2}$$

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called the Monod-Haldane function, or Holling type-IV function. We can write a predator-prey model with Monod-Haldane type functional response as follows.

$$\begin{cases} x'(t) = ax(t)(1 - \dfrac{x(t)}{K}) - \dfrac{cx(t)y(t)}{b + x^2(t)}, \\ y'(t) = -Dy(t) + \dfrac{mx(t)y(t)}{b + x^2(t)}, \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ represent population densities of prey and predator at time t. All parameters are positive constants. Usually, a is the intrinsic growth rate of the prey, K is the carrying capacity of the prey, the constant D is the death rate of the predator, m is the rate of conversion of a consumed prey to a predator and b measures the level of prey interference with predation.

As Cushing [5] pointed out that it is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed (for example, those due to seasonal effects of weather, food supply, mating habits or harvesting seasons and so on). Such perturbations were often treated continually. But, there are still some other perturbations such as fire, flood, etc, that are not suitable to be considered continually. These impulsive perturbations bring sudden changes to the system.

In this paper, with the idea of impulsive perturbations, we consider the following predator-prey model with periodic constant impulsive immigration of the predator and periodic harvesting on the prey.

$$\begin{cases} \left.\begin{aligned} x'(t) &= ax(t)(1 - \dfrac{x(t)}{K}) - \dfrac{cx(t)y(t)}{b + x^2(t)}, \\ y'(t) &= -Dy(t) + \dfrac{mx(t)y(t)}{b + x^2(t)}, \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} x(t^+) &= (1 - p_1)x(t), \\ y(t^+) &= (1 - p_2)y(t) + q, \end{aligned}\right\} t = nT, \\ (x(0^+), y(0^+)) = \mathbf{x}_0, \end{cases} \tag{2}$$

where T is the period of the impulsive immigration or stock of the predator, $p_i(0 \leq p_i < 1, = 1, 2)$ are the harvesting control parameters, q is the size of immigration or stock of the predator. This model is an example of impulsive differential equations whose theories and applications were greatly developed by the efforts of Bainov and Lakshmikantham et al. [4, 8].

Recently, many researchers have intensively investigated systems with impulsive perturbations (cf, [2, 3, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21]). Most of such systems have dealt with impulsive harvesting and immigration of predators at different fixed times. On the contrary, here we consider the impulsive harvesting and immigration at the same time in our model which has not been studied well until now.

The main purpose of this paper is to study the dynamics of the system (2).

The organization of this paper is as follows. In the next section, we introduce some notations and lemmas related to impulsive differential equations which are used in this paper. In Section 3, we show the stability of prey-free periodic solutions and give a sufficient condition for the permanence of system (2) by applying the Floquet theory and the comparison theorem. In Section 4, we show the existence of nontrivial periodic solutions via the bifurcation theorem. Finally, in conclusion, we give a bifurcation diagram that shows the system has various dynamic aspects including chaos.

2 Preliminaries

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^2_+ = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 : x, y \geq 0\}$. Denote $\mathbb{N}$ the set of all of nonnegative integers and $f = (f_1, f_2)^T$ the right hand of (2). Let $V : \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{R}_+$, then V is said to be in a class V_0 if

(1)V is continuous on$(nT, (n+1)T] \times \mathbb{R}^2_+$, and $\lim\limits_{\substack{(t,\mathbf{y})\to(nT,\mathbf{x}) \\ t>nT}} V(t, \mathbf{y}) = V(nT^+, \mathbf{x})$ exists.

(2)V is locally Lipschitz in $\mathbf{x}$.

Definition 2.1. *Let $V \in V_0$, $(t, \mathbf{x}) \in (nT, (n+1)T] \times \mathbb{R}^2_+$. The upper right derivatives of $V(t, \mathbf{x})$ with respect to the impulsive differential system (2) is defined as*

$$D^+V(t, \mathbf{x}) = \limsup_{h\to 0^+} \frac{1}{h}[V(t + h, \mathbf{x} + hf(t, \mathbf{x})) - V(t, \mathbf{x})].$$

Remarks 2.1. *(1) The solution of the system (2) is a piecewise continuous function $\mathbf{x} : \mathbb{R}_+ \to \mathbb{R}^2_+$, $\mathbf{x}(t)$ is continuous on $(nT, (n+1)T], n \in \mathbb{N}$ and $\mathbf{x}(nT^+) = \lim_{t\to nT^+} \mathbf{x}(t)$ exists.*
(2) The smoothness properties of f guarantee the global existence and uniqueness of solution of the system (2). (See [8] for the details).

We will use the following important comparison theorem on an impulsive differential equation [8].

Lemma 2.2. *(Comparison theorem) Suppose $V \in V_0$ and*

$$\begin{cases} D^+V(t, \mathbf{x}) \leq g(t, V(t, \mathbf{x})),\ t \neq n\tau \\ V(t, \mathbf{x}(t^+)) \leq \psi_n(V(t, \mathbf{x})),\ t = n\tau, \end{cases} \tag{3}$$

$g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous on $(n\tau, (n+1)\tau] \times \mathbb{R}_+$ and for $u(t) \in \mathbb{R}_+, n \in \mathbb{N}$, $\lim_{(t,y)\to(n\tau^+,u)} g(t, y) = g(n\tau^+, u)$ exists, $\psi_n : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing. Let $r(t)$ be

the maximal solution of the scalar impulsive differential equation

$$\begin{cases} u'(t) = g(t, u(t)),\ t \neq n\tau, \\ u(t^+) = \psi_n(u(t)),\ t = n\tau, \\ u(0^+) = u_0, \end{cases} \tag{4}$$

existing on $[0, \infty)$. *Then* $V(0^+, \mathbf{x}_0) \leq u_0$ *implies that* $V(t, \mathbf{x}(t)) \leq r(t), t \geq 0$, *where* $\mathbf{x}(t)$ *is any solution of (3).*

Note that $\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0$ whenever $x(t) = y(t) = 0, t \neq nT$. So, we can easily show the following lemma.

Lemma 2.3. *Let* $\mathbf{x}(t) = (x(t), y(t))$ *be a solution of (2).*
(1) If $\mathbf{x}(0^+) \geq 0$ *then* $\mathbf{x}(t) \geq 0$ *for all* $t \geq 0$.
(2) If $\mathbf{x}(0^+) > 0$ *then* $\mathbf{x}(t) > 0$ *for all* $t \geq 0$.

Now, we show that all solutions of (2) are uniformly ultimately bounded.

Lemma 2.4. *There is an* $M > 0$ *such that* $x(t), y(t) \leq M$ *for all* t *large enough, where* $(x(t), y(t))$ *is a solution of (2).*

Proof. Let $\mathbf{x}(t) = (x(t), y(t))$ be a solution of (2) and let $V(t, \mathbf{x}) = mx(t) + cy(t)$. Then $V \in V_0$, and if $t \neq nT$

$$D^+V + \beta V = -\frac{ma}{K}x^2(t) + m(a+\beta)x(t) + c(\beta - D)y(t). \tag{5}$$

Clearly, the right hand of (5), is bounded when $0 < \beta < D$. When $t = nT$, $V(nT^+) = mx(nT^+) + cy(nT^+) = (1-p_1)mx(nT) + (1-p_2)cy(nT) + cq \leq V(nT) + cq$. So we can choose $0 < \beta_0 < D$ and $M_0 > 0$ such that

$$\begin{cases} D^+V \leq -\beta_0 V + M_0, t \neq nT, \\ V(nT^+) \leq V(nT) + cq. \end{cases} \tag{6}$$

From Lemma 2.2, we can obtain that

$$V(t) \leq (V(0^+) - \frac{M_0}{\beta_0})\exp(-\beta_0 t) + \frac{cq(1-\exp(-(n+1)\beta_0 T))}{1-\exp(-\beta_0 T)}\exp(-\beta_0(t - nT)) + \frac{M_0}{\beta_0}$$

for $t \in (nT, (n+1)T]$. Therefore, $V(t)$ is bounded by a constant for sufficiently large t. Hence there is an $M > 0$ such that $x(t) \leq M, y(t) \leq M$ for a solution $(x(t), y(t))$ with all t large enough. □

Now, we consider the following impulsive differential equation.

$$\begin{cases} y'(t) = -Dy(t),\ t \neq nT, \\ y(t^+) = (1-p_2)y(t) + q,\ t = nT, \\ y(0^+) = y_0. \end{cases} \tag{7}$$

We can easily obtain the following results.

Lemma 2.5. *(1)* $y^*(t) = \dfrac{q\exp(-D(t-nT))}{1-(1-p_2)\exp(-DT)}$*,* $t \in (nT,(n+1)T]$*,* $n \in \mathbb{N}$ *is a positive periodic solution of (7) with the initial value* $y^*(0^+) = \dfrac{q}{1-(1-p_2)\exp(-DT)}$.
(2) $y(t) = (1-p_2)^{n+1}\Big(y(0^+) - \dfrac{q\exp(-DT)}{1-(1-p_2)\exp(-DT)}\Big)\exp(-Dt) + y^*(t)$ *is the solution of (7) with* $y_0 \geq 0$*,* $t \in (nT,(n+1)T]$ *and* $n \in \mathbb{N}$*.*
(3) All solutions $y(t)$ *of (2) with* $y_0 \geq 0$ *tend to* $y^*(t)$*. i.e.,* $|y(t) - y^*(t)| \to 0$ *as* $t \to \infty$*.*

It is from Lemma 2.5 that the general solution $y(t)$ of (7) can be identified with the positive periodic solution $y^*(t)$ of (7) for sufficiently large t and we can obtain the complete expression for the prey-free periodic solution of (2)

$$(0, y^*(t)) = \left(0, \frac{q\exp(-D(t-nT))}{1-(1-p_2)\exp(-DT)}\right) \text{ for } t \in (nT,(n+1)T].$$

3 Extinction and permanence

Now, we present a condition which guarantees local stability of the prey-free periodic solution $(0, y^*(t))$.

Theorem 3.1. *The prey-free solution* $(0, y^*(t))$ *is locally asymptotically stable if*

$$aT + \ln(1-p_2) < \frac{cq(1-\exp(-DT))}{bD(1-(1-p_2)\exp(-DT))}.$$

Proof. The local stability of the periodic solution $(0, y^*(t))$ of (2) may be determined by considering the behavior of small amplitude perturbations of the solution. Let $(x(t), y(t))$ be any solution of (2). Define $x(t) = u(t), y(t) = y^*(t) + v(t)$. Then they may be written as

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t)\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, 0 \leq t \leq T, \tag{8}$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi}{dt} = \begin{pmatrix} a - \frac{c}{b}y^*(t) & 0 \\ \frac{m}{b}y^*(t) & -D \end{pmatrix}\Phi(t) \tag{9}$$

and $\Phi(0) = I$, the identity matrix. The linearization of the third and fourth equation of (2) becomes

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \end{pmatrix} = \begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix}\begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}. \tag{10}$$

Note that all eigenvalues of $S = \begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix} \Phi(T)$ are $\mu_1 = \exp(-DT) < 1$ and $\mu_2 = (1-p_2)\exp(\int_0^T a - \frac{c}{b}y^*(t)dt)$. Since

$$\int_0^T y^*(t)dt = \frac{q(1-\exp(-DT))}{D(1-(1-p_2)\exp(-DT))},$$

we have

$$\mu_2 = (1-p_2)\exp\left(aT - \frac{cq(1-\exp(-DT))}{bD(1-(1-p_2)\exp(-DT))}\right).$$

By Floquet Theory in [4], $(0, y^*(t))$ is locally asymptotically stable if $|\mu_2| < 1$.i.e.,

$$aT < \frac{cq(1-\exp(-DT))}{bD(1-(1-p_2)\exp(-DT))} + \ln\frac{1}{1-p_2}.$$

□

Definition 3.1. *The system (2) is permanent if there exist* $M \geq m > 0$ *such that, for any solution* $(x(t), y(t))$ *of (2) with* $\mathbf{x}_0 > 0$,

$$m \leq \lim_{t\to\infty}\inf x(t) \leq \lim_{t\to\infty}\sup x(t) \leq M \text{ and } m \leq \lim_{t\to\infty}\inf y(t) \leq \lim_{t\to\infty}\sup y(t) \leq M.$$

Theorem 3.2. *The system (2) is permanent if*

$$aT + \ln(1-p_2) > \frac{cq(1-\exp(-DT))}{bD(1-(1-p_2)\exp(-DT))}.$$

Proof. Let $(x(t), y(t))$ be any solution of (2)with $\mathbf{x}_0 > 0$. From Lemma 2.4, we may assume that $x(t) \leq M$, $y(t) \leq M$, $t \geq 0$ and $M > \dfrac{ab}{c}$. Let $m_2 = \dfrac{q\exp(-DT)}{1-(1-p_2)\exp(-DT)} - \epsilon_2$, $\epsilon_2 > 0$. From Lemma 2.5, clearly we have $y(t) \geq m_2$ for all t large enough. Now we shall find an $m_1 > 0$ such that $x(t) \geq m_1$ for all t large enough. We will do this in the following two steps.

(Step 1) Since

$$aT > \frac{cq(1-\exp(-DT))}{bD(1-(1-p_2)\exp(-DT))} + \ln\frac{1}{1-p_2},$$

we can choose $m_3 > 0$, $\epsilon_1 > 0$ small enough such that $\delta = \dfrac{mm_3}{b+m_3} < D$ and $R = (1-p_2)\exp\left(aT - \dfrac{a}{K}Tm_3 - \dfrac{cq(1-\exp(-DT))}{bD(1-(1-p_2)\exp(-DT))} - \dfrac{c\epsilon_1}{b}T\right) > 1$. Suppose that $x(t) < m_3$ for all t. Then we get $y'(t) \leq y(t)(-D+\delta)$ from above assumptions.

By Lemma 2.2, we have $y(t) \le u(t)$ and $u(t) \to u^*(t)$, $t \to \infty$, where $u(t)$ is the solution of

$$\begin{cases} u'(t) = (-D+\delta)u(t),\ t \neq nT, \\ u(t^+) = (1-p_2)u(t) + q,\ t = nT, \\ u(0^+) = y_0, \end{cases} \tag{11}$$

and $u^*(t) = \dfrac{q\exp((-D+\delta)(t-nT))}{1-(1-p_2)\exp((-D+\delta)T)}$, $t \in (nT, (n+1)T]$. Then there exists $T_1 > 0$ such that $y(t) \le u(t) \le u^*(t) + \epsilon_1$ and

$$\begin{aligned} x'(t) &= x(t)\left(a - \frac{a}{K}x(t) - \frac{cy(t)}{b + x^2(t)}\right) \\ &\ge x(t)\big(a - \frac{a}{K}m_3 - \frac{c}{b}y(t)\big) \\ &\ge x(t)\big(a - \frac{a}{K}m_3 - \frac{c}{b}(u^*(t) + \epsilon_1)\big) \text{ for } t \ge T_1. \end{aligned}$$

Let $N_1 \in \mathbb{N}$ and $N_1T \ge T_1$. We have, for $n \ge N_1$

$$\begin{cases} x'(t) & \ge x(t)\left(a - \frac{a}{K}m_3 - \frac{c}{b}(u^*(t)+\epsilon_1)\right), t \neq nT, \\ x(t^+) & = (1-p)x(t), t = nT. \end{cases} \tag{12}$$

Integrating (12) on $(nT, (n+1)T](n \ge N_1)$, we obtain

$$x((n+1)T) \ge x(nT^+)\exp\left(\int_{nT}^{(n+1)T} a - \frac{a}{K}m_3 - \frac{c}{b}(u^*(t) + \epsilon_1)dt\right) = x(nT)R.$$

Then $x((N_1+k)T) \ge x(N_1T)R^k \to \infty$ as $k \to \infty$ which is a contradiction. Hence there exists a $t_1 > 0$ such that $x(t_1) \ge m_3$.

(Step 2) If $x(t) \ge m_3$ for all $t \ge t_1$, then we are done. If not, we may let $t^* = \inf_{t>t_1}\{x(t) < m_3\}$.Then $x(t) \ge m_3$ for $t \in [t_1, t^*]$ and, by the continuity of $x(t)$, we have $x(t^*) = m_3$. In this step, we have only to consider two possible cases.

Case (1) $t^* = n_1T$ for some $n_1 \in \mathbb{N}$. Then $(1-p_1)m_3 \le x(t^{*+}) = (1-p_1)x(t^*) < m_3$. Select $n_2, n_3 \in \mathbb{N}$ such that $(n_2-1)T > \dfrac{\ln(\frac{\epsilon_1}{M+q})}{-d+\delta}$ and $(1-p_1)^{n_2}R^{n_3}\exp(n_2\sigma T) > (1-p_1)^{n_2}R^{n_3}\exp((n_2+1)\sigma T) > 1$, where $\sigma = a - \frac{a}{K}m_3 - \frac{c}{b}M < 0$. Let $T' = n_2T + n_3T$. In this case we will show that there exists $t_2 \in (t^*, t^* + T']$ such that $x(t_2) \ge m_3$. Otherwise, by (11) with $u(t^{*+}) = y(t^{*+})$, we have

$$u(t) = (1-p_2)^{n_1+1}\left(u(t^{*+}) - \frac{q\exp((-D+\delta)T)}{1-(1-p_2)\exp((-D+\delta)T)}\right)\exp((-D+\delta)(t-t^*)) + u^*(t)$$

for $(n-1)T < t \leq nT$ and $n_1+1 \leq n \leq n_1+1+n_2+n_3$. So we get $|u(t)-u^*(t)| \leq (M+q)\exp((-D+\delta)(t-t^*)) < \epsilon_1$ and $y(t) \leq u(t) \leq u^*(t)+\epsilon_1$ for $t^*+n_2T \leq t \leq t^*+T'$. Thus we obtain the same equation as (12)

for $t \in [t^*+n_2T, t^*+T']$. As in step 1, we have

$$x(t^*+T') \geq x(t^*+n_2T)R^{n_3}.$$

Since $y(t) \leq M$, we have

$$\begin{cases} x'(t) \geq x(t)\big(a - \frac{a}{K}m_3 - \frac{c}{b}M\big) = \sigma x(t), t \neq nT, \\ x(t^+) = (1-p_1)x(t), t = nT, \end{cases} \tag{13}$$

for $t \in [t^*, t^*+n_2T]$. Integrating (13) on $[t^*, t^*+n_2T]$ we have

$$\begin{aligned} x((t^*+n_2T)) &\geq m_3 \exp(\sigma n_2 T) \\ &\geq m_3(1-p_1)^{n_2}\exp(\sigma n_2 T) > m_3. \end{aligned}$$

Thus $x(t^*+T') \geq m_3(1-p_1)^{n_2}\exp(\sigma n_2 T)R^{n_3} > m_3$ which is a contradiction. Now, let $\bar{t} = \inf_{t>t^*}\{x(t) \geq m_3\}$. Then $x(t) \leq m_3$ for $t^* \leq t < \bar{t}$ and $x(\bar{t}) = m_3$. For $t \in [t^*, \bar{t})$, suppose $t \in (t^*+(k-1)T, t^*+kT]$, $k \in \mathbb{N}$ and $k \leq n_2+n_3$. So, we have, for $t \in [t^*, \bar{t})$, from (13) we obtain $x(t) \geq x(t^{*+})(1-p_1)^{k-1}\exp((k-1)\sigma T)\exp(\sigma(t-(t^*+(k-1)T))) \geq m_3(1-p_1)^k\exp(k\sigma T) \leq m_3(1-p_1)^{n_2+n_3}\exp(\sigma(n_2+n_3)T) \equiv m_1'$.

Case (2) $t^* \neq nT, n \in \mathbb{N}$. Then $x(t) \geq m_3$ for $t \in [t_1, t^*)$ and $x(t^*) = m_3$. Suppose that $t^* \in (n_1'T, (n_1'+1)T)$ for some $n_1' \in \mathbb{N}$. There are two possible cases.

Case(2(a)) $x(t) < m_3$ for all $t \in (t^*, (n_1'+1)T]$. In this case we will show that there exists $t_2 \in [(n_1'+1)T, (n_1'+1)T+T']$ such that $x_2(t_2) \geq m_3$. Suppose not, i.e., $x(t) < m_3$, for all $t \in [(n_1'+1)T, (n_1'+1+n_2+n_3)T]$. Then $x(t) < m_3$ for all $t \in (t^*, (n_1'+1+n_2+n_3)T]$. By (11) with $u((n_1'+1)T^+) = y((n_1'+1)T^+)$, we have

$$u(t) = \Big(u((n_1'+1)T^+) - \frac{q\exp(-D+\delta)}{1-(1-p_2)\exp(-D+\delta)}\Big)\exp((-D+\delta)(t-(n_1'+1)T)) + u^*(t)$$

for $t \in (nT, (n+1)T], n_1'+1 \leq n \leq n_1'+n_2+n_3$. By a similar argument as in (step 1), we have

$$x((n_1'+1+n_2+n_3)T) \geq x_2((n_1'+1+n_2)T)R^{n_3}.$$

It follows from (13) that

$$x((n_1'+1+n_2)T) \geq m_3(1-p)^{n_2+1}\exp(\sigma(n_2+1)T).$$

Thus $x((n_1'+1+n_2+n_3)T) \geq m_3(1-p)^{n_2+1}\exp(\sigma(n_2+1)T)R^{n_3} > m_3$ which is a contradiction. Now, let $\bar{t} = \inf_{t>t^*}\{x(t) \geq m_3\}$. Then $x(t) \leq m_3$ for $t^* \leq t < \bar{t}$ and $x(\bar{t}) = m_3$. For $t \in [t^*, \bar{t})$, suppose $t \in (n_1'T+(k'+1)T, n_1'T+k'T]$, $k' \in \mathbb{N}$,

$k' \leq 1 + n_2 + n_2$, we have $x(t) \geq m_3(1-p)^{1+n_2+n_3} \exp(\sigma(1+n_2+n_3)T) \equiv m_1$. Since $m_1 < m_1'$, so $x(t) \geq m_1$ for $t \in (t^*, \bar{t})$.

Case (2(b)) There is a $t' \in (t^*, (n_1'+1)T]$ such that $x_2(t') \geq m_3$. Let $\hat{t} = \inf_{t>t^*}\{x(t) \geq m_3\}$. Then $x(t) \leq m_3$ for $t \in [t^*, \hat{t})$ and $x(\hat{t}) = m_3$. Also, (13) holds for $t \in [t^*, \hat{t})$. Integrating the equation on $[t^*, t)(t^* \leq t \leq \hat{t})$, we can get that $x(t) \geq x(t^*) \exp(\sigma(t - t^*)) \geq m_3 \exp(\sigma T) \geq m_1$. Thus in both cases a similar argument can be continued since $x(t) \geq m_1$ for some $t > t_1$. This completes the proof. □

Remarks 3.3. *Define* $G(T) = aT + \ln(1-p_2) - \dfrac{cq(1-\exp(-DT))}{bD(1-(1-p_2)\exp(-DT))}$. *Since* $G(0) = \ln(1-p_2) < 0$, $\lim_{T\to\infty} G(T) = \infty$ *and*

$$G''(T) = \frac{cqp_2 \exp(-DT)(1+(1-p_2)\exp(-DT))}{b^4 D(1-(1-p_2)\exp(-DT))^3} > 0, \tag{14}$$

so we have that $G(T) = 0$ *has a unique positive solution* T^*. *From Theorem 3.1 and Theorem 3.2, we know that the prey-free periodic solution is locally asymptotically stable if* $T < T^*$ *and otherwise, the prey and predator can coexist. Thus* T^* *plays the role of a critical value that discriminates between stability and permanence.*

4 Existence and stability of a positive periodic solution

Now, we deal with a problem of the bifurcation of the nontrivial periodic solution of the system (2), near $(0, y^*(t))$. The following theorem establishes the existence of a positive periodic solution of the system (2) near $(0, y^*(t))$.

Theorem 4.1. *The system (2) has a positive periodic solution which is supercritical if* $T > T^*$.

Proof. We will use the results of [6, 9, 11] to prove this theorem. To use theorems of [6, 9, 11], it is convenient for the computation to exchange the variables of x and y and change the period T to τ. Thus the system (2) becomes as follows

$$\left\{\begin{array}{l} \left.\begin{array}{l} x'(t) = -Dx(t) + \dfrac{my(t)x(t)}{b+y^2(t)}, \\ y'(t) = ay(t)(1-\dfrac{y(t)}{K}) - \dfrac{cy(t)x(t)}{b+y^2(t)}, \end{array}\right\} t \neq n\tau, \\ \left.\begin{array}{l} x(t^+) = (1-p_2)x(t) + q, \\ y(t^+) = (1-p_1)y(t), \end{array}\right\} t = n\tau. \end{array}\right. \tag{15}$$

Let Φ be the flow associated with (15). We have $X(t) = \Phi(t, \mathbf{x}_0), 0 < t \leq \tau$, where $\mathbf{x}_0 = (x(0), y(0))$. The flow Φ applies up to time τ. So, $X(\tau) = \Phi(\tau, \mathbf{x}_0)$. We will use all notations in [9]. Note that

$$\begin{aligned}
F_1(x,y) &= -Dx + \frac{myx}{b+y^2}, F_2(x,y) = ay(1-\frac{y}{K}) - \frac{cyx}{b+y^2},\\
\Theta_1(x,y) &= (1-p_2)x + q, \Theta_2(x,y) = (1-p_1)y, \zeta(t) = (y^*(t),0),\\
\frac{\partial\Phi_1(t,\mathbf{x}_0)}{\partial x} &= \exp\left(\int_0^t \frac{\partial F_1(\zeta(r))}{\partial x}dr\right), \frac{\partial\Phi_2(t,\mathbf{x}_0)}{\partial y} = \exp\left(\int_0^t \frac{\partial F_2(\zeta(r))}{\partial y}dr\right),\\
\frac{\partial\Phi_1(t,\mathbf{x}_0)}{\partial y} &= \int_0^t \exp\left(\int_\nu^t \frac{\partial F_1(\zeta(r))}{\partial x}dr\right)\frac{\partial F_1(\zeta(\nu))}{\partial y}\exp\left(\int_0^\nu \frac{\partial F_2(\zeta(r))}{\partial y}dr\right)d\nu.
\end{aligned}$$

Then

$$\frac{\partial\Theta_1}{\partial y} = \frac{\partial\Theta_2}{\partial x} = 0, \frac{\partial\Theta_1}{\partial x} = 1-p_2, \frac{\partial\Theta_2}{\partial y} = 1-p_1, \frac{\partial^2\Theta_i}{\partial x\partial y} = 0, i=1,2 \text{ and } \frac{\partial^2\Theta_2}{\partial y^2} = 0.$$

Now, we can compute

$$d_0' = 1 - \left(\frac{\partial\Theta_2}{\partial y}\cdot\frac{\partial\Phi_2}{\partial y}\right)_{(\tau_0,\mathbf{x}_0)} = 1-(1-p_1)\exp\left(\int_0^{\tau_0} a - \frac{c}{b}y^*(r)dr\right),$$

where τ_0 is the root of $d_0'=0$. Actually, we know that $\tau_0 = T^*$.

$$a_0' = 1 - \left(\frac{\partial\Theta_1}{\partial x}\cdot\frac{\partial\Phi_1}{\partial x}\right)_{(\tau_0,\mathbf{x}_0)} = 1-(1-p_2)\exp(-D\tau_0) > 0,$$

$$\begin{aligned}
b_0' &= -\left(\frac{\partial\Theta_1}{\partial x}\cdot\frac{\partial\Phi_1}{\partial y} + \frac{\partial\Theta_1}{\partial y}\cdot\frac{\partial\Phi_2}{\partial y}\right)_{(\tau_0,\mathbf{x}_0)} = -\left(\frac{\partial\Phi_1}{\partial y}\right)_{(\tau_0,\mathbf{x}_0)}\\
&= -\int_0^{\tau_0}\exp\left(\int_\nu^{\tau_0}\frac{\partial F_1(\zeta(r))}{\partial x}dr\right)\frac{m}{b}y^*(\nu)\exp\left(\int_0^\nu \frac{\partial F_2(\zeta(r))}{\partial y}dr\right)d\nu < 0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2\Phi_2(t,\mathbf{x}_0)}{\partial y\partial x} &= \int_0^t \exp\left(\int_\nu^t \frac{\partial F_2(\zeta(r))}{\partial y}dr\right)\frac{\partial^2 F_2(\zeta(\nu))}{\partial y\partial x}\exp\left(\int_0^\nu \frac{\partial F_2(\zeta(r))}{\partial y}dr\right)d\nu,\\
\frac{\partial^2\Phi_2(\tau_0,\mathbf{x}_0)}{\partial y\partial x} &= -\frac{c}{b}\int_0^{\tau_0}\exp\left(\int_\nu^{\tau_0}\frac{\partial F_2(\zeta(r))}{\partial y}dr\right)\exp\left(\int_0^\nu \frac{\partial F_2(\zeta(r))}{\partial y}dr\right)d\nu < 0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2\Phi_2(t,\mathbf{x}_0)}{\partial y^2} &= \int_0^t \exp\left(\int_\nu^t \frac{\partial F_2(\zeta(r))}{\partial y}dr\right)\frac{\partial^2 F_2(\zeta(\nu))}{\partial y^2}\exp\left(\int_0^\nu \frac{\partial F_2(\zeta(r))}{\partial y}dr\right)d\nu\\
&+ \int_0^t\left[\exp\left(\int_\nu^t \frac{\partial F_2(\zeta(r))}{\partial y}dr\right)\frac{\partial^2 F_2(\zeta(\nu))}{\partial y\partial x}\right]\\
&\times\left[\int_0^\nu \exp\left(\int_\theta^\nu \frac{\partial F_1(\zeta(r))}{\partial x}dr\right)\frac{\partial F_1(\zeta(\theta))}{\partial y}\exp\left(\int_0^\theta \frac{\partial F_2(\zeta(r))}{\partial y}dr\right)d\theta\right]d\nu,
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2\Phi_2(\tau_0,\mathbf{x}_0)}{\partial y^2} =& -\frac{2a}{K}\int_0^{\tau_0}\exp\biggl(\int_\nu^{\tau_0}\frac{\partial F_2(\zeta(r))}{\partial y}dr\biggr)\exp\biggl(\int_0^{\nu}\frac{\partial F_2(\zeta(r))}{\partial y}dr\biggr)d\nu\\
&-\frac{cm}{b^2}\int_0^{\tau_0}\biggl[\exp\biggl(\int_\nu^{\tau_0}\frac{\partial F_2(\zeta(r))}{\partial y}dr\biggr)\biggr]\\
&\times\biggl[\int_0^{\nu}\exp\biggl(\int_\theta^{\nu}\frac{\partial F_1(\zeta(r))}{\partial x}dr\biggr)y^*(\theta)\exp\biggl(\int_0^{\theta}\frac{\partial F_2(\zeta(r))}{\partial y}dr\biggr)d\theta\biggr]d\nu<0,\\
\frac{\partial^2\Phi_2(t,\mathbf{x}_0)}{\partial y\partial\tau} =& \frac{\partial F_2(\zeta(t))}{\partial y}\exp\biggl(\int_0^{t}\frac{\partial F_2(x_p(r),0)}{\partial y}dr\biggr),\\
\frac{\partial^2\Phi_2(\tau_0,\mathbf{x}_0)}{\partial y\partial\tau} =& \frac{\partial F_2(\zeta(\tau_0))}{\partial y}\exp\biggl(\int_0^{\tau_0}\frac{\partial F_2(x_p(r),0)}{\partial y}dr\biggr)\\
=& \Bigl(a-\frac{c}{b}y^*(\tau_0)\Bigr)\exp\biggl(\int_0^{\tau_0}\Bigl(a-\frac{c}{b}y^*(r)\Bigr)dr\biggr),\\
\frac{\partial\Phi_1(\tau_0,\mathbf{x}_0)}{\partial\tau} =& \dot{y}^*(\tau_0)=-\frac{q\exp(-D\tau_0)}{D(1-(1-p_2)\exp(-D\tau_0))}<0,
\end{aligned}
$$

$$
\begin{aligned}
C=&-2\frac{\partial^2\Theta_2}{\partial x\partial y}\biggl(-\frac{b_0'}{a_0'}\cdot\frac{\partial\Phi_1(\tau_0,\mathbf{x}_0)}{\partial x}+\frac{\partial\Phi_1(\tau_0,\mathbf{x}_0)}{\partial y}\biggr)\frac{\partial\Phi_2(\tau_0,\mathbf{x}_0)}{\partial y}\\
&-\frac{\partial^2\Theta_2}{\partial y^2}\biggl(\frac{\partial\Phi_2(\tau_0,\mathbf{x}_0)}{\partial y}\biggr)^2+2\frac{\partial\Theta_2}{\partial y}\cdot\frac{b_0'}{a_0'}\cdot\frac{\partial^2\Phi_2(\tau_0,\mathbf{x}_0)}{\partial y\partial x}-\frac{\partial\Theta_2}{\partial y}\cdot\frac{\partial^2\Phi_2(\tau_0,\mathbf{x}_0)}{\partial y^2}\\
=&\,2(1-p_1)\frac{b_0'}{a_0'}\frac{\partial^2\Phi_2}{\partial x\partial y}-(1-p_1)\frac{\partial^2\Phi_2}{\partial y^2}>0,\\
B=&-\frac{\partial^2\Theta_2}{\partial x\partial y}\biggl(\frac{\partial\Phi_1(\tau_0,\mathbf{x}_0)}{\partial\tau}+\frac{\partial\Phi_1(\tau_0,\mathbf{x}_0)}{\partial x}\cdot\frac{1}{a_0'}\cdot\frac{\partial\Theta_1}{\partial x}\cdot\frac{\partial\Phi_1(\tau_0,X_0)}{\partial\tau}\biggr)\frac{\partial\Phi_2(\tau_0,\mathbf{x}_0)}{\partial y}\\
&-\frac{\partial\Theta_2}{\partial y}\biggl(\frac{\partial^2\Phi_2(\tau_0,\mathbf{x}_0)}{\partial x\partial y}\cdot\frac{1}{a_0'}\cdot\frac{\partial\Theta_1}{\partial x}\cdot\frac{\partial\Phi_1(\tau_0,\mathbf{x}_0)}{\partial\tau}+\frac{\partial^2\Phi_2(\tau_0,\mathbf{x}_0)}{\partial\tau\partial y}\biggr)\\
=&-(1-p_1)\biggl(\frac{\partial^2\Phi_2(\tau_0,\mathbf{x}_0)}{\partial x\partial y}\cdot\frac{1}{a_0'}\cdot\frac{\partial\Phi_1(\tau_0,\mathbf{x}_0)}{\partial\tau}\\
&+\Bigl(a-\frac{c}{b}y^*(\tau_0)\Bigr)\exp\int_0^{\tau_0}\Bigl(a-\frac{c}{b}y^*(r)\Bigr)dr\biggr).
\end{aligned}
$$

To determine the sign of B, let $\phi(t)=a-\dfrac{c}{b}y^*(t)$. Then we obtain that

$$
\phi'(t)=\frac{cDq\exp(-Dt)}{b(1-(1-p_2)\exp(-D\tau_0))}>0
$$

and so $\phi(t)$ is strictly increasing. Since

$$
\int_0^{\tau_0}\phi(t)dt=aT-\frac{cq(1-\exp(-D\tau_0))}{bD(1-(1-p_2)\exp(-D\tau_0))}=-\ln(1-p)>0,
$$

we have $\phi(\tau_0) > 0$. This implies that $B < 0$. Hence $BC < 0$. Thus, from Theorem 2 of [9], the statement follows. □

5 Conclusions

In this paper, we have studied the influences of impulsive perturbations on a predator-prey system with the Monod-Haldane functional response. We have found out that there exists a threshold value that plays a key role on discriminating between the stability of the prey-free periodic solution and the permanence of the system via Floquet theory and the comparison theorem. Furthermore, we have shown that the system has a positive periodic solution which is supercritical under some conditions.

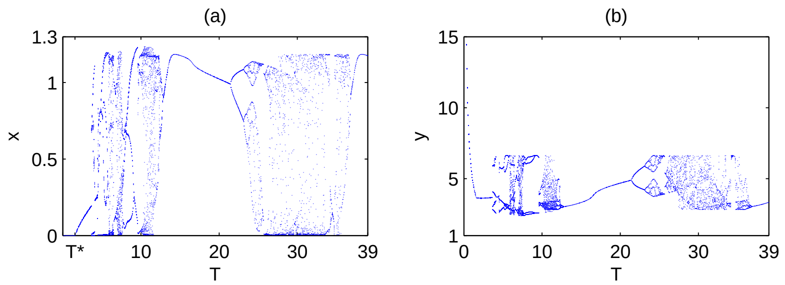

Figure 1: The bifurcation diagrams of the system (2) with an initial value $(2.5, 4)$. (a) x is plotted for T over $[0, 39]$. (b) y is plotted for T over $[0, 39]$.

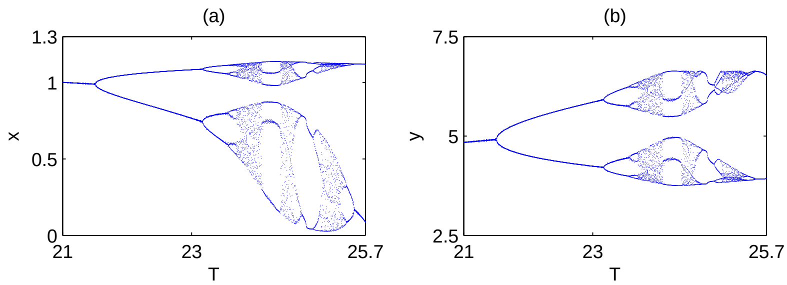

Figure 2: The bifurcation diagrams of the system (2) with an initial value $(2.5, 4)$. (a) x is plotted for T over $[21, 25.7]$. (b) y is plotted for T over $[21, 25.7]$.

In order to illustrate the dynamics of the system by a numerical example, we give bifurcation diagrams in Figures 1 and 2 when the parameters are fixed except the period T as follows:

$$a = 4.0, K = 1.9, b = 0.6, c = 0.75, D = 0.25, m = 0.6, p_1 = 0.3, p_2 = 0.001 \text{ and } q = 1.2.$$

These figures point out that system (2) has various dynamical behaviors such as quasi-periodic, periodic windows, strange attractors and periodic doubling and halving phenomena etc. (see Figure 2). In this case, we can obtain the critical value $T^* \approx 1.58$ suggested in Theorems 3.1 and 3.2. As mentioned in Theorem 4.1, the value T^* plays an important part in the classification for the existence of a positive periodic solution as shown in Figure 1.

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