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On the viscous Burgers equation in unbounded domain

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Abstract

In this paper we investigate the existence and uniqueness of global solutions, and a rate stability for the energy related with a Cauchy problem to the viscous Burgers equation in unbounded domain $\mathbb{R} \times (0, \infty)$. Some aspects associated with a Cauchy problem are presented in order to employ the approximations of Faedo-Galerkin in whole real line $\mathbb{R}$. This becomes possible due to the introduction of weight Sobolev spaces which allow us to use arguments of compactness in the Sobolev spaces.

Key words: Unbounded domain, global solvability, uniqueness, a rate decay estimate for the energy.

AMS subject classifications codes: 35B40, 35K15, 35K55, 35R35.

1 Introduction and Formulation of the Problem

We are concerned with the existence of global solutions – precisely, global weak solutions, global strong solutions and regularity of the strong solutions –, uniqueness of the solutions and the asymptotic stability of the energy for the nonlinear Cauchy problem related to the classic viscous Burgers equation

$$u_t + uu_x - u_{xx} = 0$$

established in $\mathbb{R} \times (0, T)$, for an arbitrary $T > 0$. More precisely, we consider the real valued function $u = u(x, t)$ defined for all $(x, t) \in \mathbb{R} \times (0, T)$ which is the solution of the Cauchy problem

$$\begin{cases} u_t + uu_x - u_{xx} = 0 \quad \text{in} \quad \mathbb{R} \times (0, T), \\ u(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R}. \end{cases} \tag{1.1}$$

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The Burgers equation has a long history. We briefly sketch this history by citing one of the pioneer work by Bateman [2] about an approximation of the flux of fluids. Later, Burgers published the works [5] and [6] which are also about flux of fluids or turbulence. In the classic fashion the Burgers equation has been studied by several authors, mainly in the last century, and excellent papers and books can be found in the literature specialized in PDE. One can cite, for instance, Courant & Friedrichs [8], Courant & Hilbert [9], Hopf [10], Lax [13] and Stoker [17].

Today, the equation $(1.1)_1$ ($(1.1)_1$ refers to the first equation in (1.1)) is known as *viscous Burgers equations* and perhaps it is the simplest nonlinear equation associating the nonlinear propagation of waves with the effect of the heat conduction.

The existence of global solutions for the Cauchy problem (1.1) will be obtained employing the Faedo-Galerkin and Compactness methods. The Faedo-Galerkin method is probably one of the most effective methods to establish existence of solutions for non-linear evolution problems in domains whose spatial variable x lives in bounded sets. To spatial unbounded sets, there exist few results about existence of solutions established by the referred method. Thus, as the non-linear problem (1.1) is defined in $\mathbb{R}$, in order to reach our goal through this method we will also need to use compactness' argument, as in Aubin [1] or Lions [15]. In order to apply the Compactness method we employ a suitable theory on weight Sobolev spaces to be set as follows. In fact, in the sequel $H^m(\mathbb{R})$ represents the Sobolev space of order m in $\mathbb{R}$, with $m \in \mathbb{N}$. The space $L^2(\mathbb{R})$ is the Lebesgue space of the classes of functions $u : \mathbb{R} \to \mathbb{R}$ with square integrable on $\mathbb{R}$. Assuming that X is a Banach space, T is a positive real number or $T = +\infty$ and $1 \leq p \leq \infty$, we will denote by $L^p(0, T; X)$ the Banach space of all measurable mapping $u :]0, T[\longrightarrow X$, such that $t \mapsto \|u(t)\|_X$ belongs to $L^p(0, T)$. For more details on the functional spaces above cited the reader can consult, for instance, the references [3] and [15]. In this work we will also use the following weight vectorial spaces

$$\mathcal{L}^2(K) = \Big\{ \phi \in L^2(\mathbb{R}); \;\; \int_{\mathbb{R}} |\phi(y)|^2_{\mathbb{R}} K(y) dy < \infty \Big\},$$
$$\mathcal{H}^m(K) = \Big\{ \phi \in H^m(\mathbb{R}); \;\; D^i\phi \in \mathcal{L}^2(K) \Big\} \text{ with } i = 1, 2, \ldots, m, \;\; m \in \mathbb{N},$$

where K is a weight function given for

$$K(y) = \exp\{y^2/4\}, \quad y \in \mathbb{R}. \tag{1.2}$$

The inner product and norm of $\mathcal{L}^2(K)$ and $\mathcal{H}^m(K)$ are defined by

$$(\phi,\psi)=\int_{\mathbb{R}}\phi(y)\psi(y)K(y)dy, \qquad |\phi|^2=\int_{\mathbb{R}}|\phi(y)|^2_{\mathbb{R}}K(y)dy,$$
$$((\phi,\psi))_m=\sum_{i=1}^{m}\int_{\mathbb{R}}D^i\phi(y)D^i\psi(y)K(y)dy, \qquad \|\phi\|_m^2=\sum_{i=1}^{m}\left|D^i\phi\right|^2,$$

respectively. The vector spaces $\mathcal{L}^2(K)$ and $\mathcal{H}^m(K)$ are Hilbert spaces with the above inner products. By $D(\mathbb{R})$ it denotes the class of C^∞ functions in $\mathbb{R}$ with compact support and convergence in the Laurent Schwartz sense, see [16].

We will also use the functional structure of the spaces $L^p(0,T;H)$ with $1\le p\le\infty$, where H is one of the spaces: $\mathcal{L}^2(K)$ or $\mathcal{H}^m(K)$.

Some properties of the spaces $\mathcal{L}^2(K)$ and $\mathcal{H}^m(K)$ as the compactness of the inclusion $\mathcal{H}^m(K)\hookrightarrow\mathcal{L}^2(K)$ and Poincaré inequality with the weight (1.2) has been proven in Escobedo-Kavian [11]. Results on compactness of space of spherically symmetric functions that vanishes at infinity were proven by Strauss [18]. In this direction one can see some results in Kurtz [12].

The method used to prove the existence of solutions for the Cauchy problem (1.1) is to transform it to another equivalent one proposed in the suitable functional spaces by using a change of variables defined by

$$z(y,s)=(t+1)^{1/2}u(x,t) \text{ where } y=\frac{x}{(t+1)^{1/2}} \text{ and } s=\ln(t+1). \tag{1.3}$$

The changing of variable (1.3) defines a diffeomorphism $\sigma\colon\mathbb{R}_x\times(0,T)\to\mathbb{R}_y\times(0,S)$ with $\sigma(x,t)=(y,s)$ and $S=\ln(T+1)$. From (1.3) we have $t=e^s-1$ and $x=e^{s/2}y$. Therefore,

$$z(y,s)=e^{s/2}u(e^{s/2}y,e^s-1) \quad\text{and}\quad u(x,t)=(t+1)^{-1/2}z\Big(x/(t+1)^{1/2},\ln(t+1)\Big).$$

Differentiating u with respect to t, it yields

$$\begin{aligned}
u_t &= \frac{-1}{2}(t+1)^{-3/2}z+(t+1)^{-1/2}\Big(\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial z}{\partial s}\frac{\partial s}{\partial t}\Big)\\
&= (t+1)^{-3/2}\Big(-\frac{z}{2}-\frac{yz_y}{2}+z_s\Big)\\
&= e^{-3s/2}\Big(-\frac{z}{2}-\frac{yz_y}{2}+z_s\Big).
\end{aligned}$$

Differentiating u with respect to x, it yields

$$u_x = (t+1)^{-1/2}\frac{\partial z}{\partial y}\frac{\partial y}{\partial x} = (t+1)^{-1/2}\frac{\partial z}{\partial y}\frac{1}{(t+1)^{1/2}} = (t+1)^{-1}z_y = e^{-s}z_y.$$

Differentiating again with respect to x, it yields

$$u_{xx} \;=\; (t+1)^{-1}\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial x} = (t+1)^{-3/2}z_{yy} = e^{-3s/2}z_{yy}.$$

Inserting the three last identities in $(1.1)_1$, we obtain

$$z_s - z_{yy} - \frac{yz_y}{2} - \frac{z}{2} + zz_y = 0 \quad \text{in} \quad \mathbb{R}\times(0,S). \tag{1.4}$$

Moreover, for $t=0$, we have by definition of y that $x=y$. Thus the initial data becomes

$$u_0(x) = u(x,0) = z(y,0) = z_0(y). \tag{1.5}$$

For use later and a better understanding we will modify the equation (1.4) as follows: one defines the operator $L : \mathcal{H}^2(K) \longrightarrow \mathbb{R}$ by $\phi \longmapsto L\phi = -\phi_{yy} - \frac{y\phi_y}{2}$, which satisfies:

$$L\phi = -\frac{1}{K}(K\phi_y)_y \quad \text{and} \quad (L\phi,\psi) = (\phi_y,\psi_y) = ((\phi,\psi))_2 \tag{1.6}$$

for all $\phi \in \mathcal{H}^2(K)$ and $\psi \in \mathcal{H}^1(K)$. Therefore, from (1.4), (1.5) and $(1.6)_1$ the Cauchy problem (1.1) is equivalent by σ to

$$\begin{cases} z_s + Lz - \dfrac{z}{2} + zz_y = 0 \quad \text{in} \quad \mathbb{R}\times(0,S), \\ z(y,0) = u_0(y) \quad \text{in} \quad \mathbb{R}. \end{cases} \tag{1.7}$$

The purpose of this work is: in Section 2, we investigate the existence of global weak solutions of (1.1), its uniqueness and as well as analysis of the decay of these solutions. In Section 3 we establish the same properties of Section 2 for the strong solutions. In Section 4, we study the regularity of the strong solutions.

2 Weak Solution

Setting the initial data $u_0 \in \mathcal{L}^2(K)$ we are able to show that the Cauchy problem (1.1) has a unique global weak solution $u = u(x,t)$ defined in $\mathbb{R}\times(0,\infty)$ with real values and the energy associated with this solution is asymptotically stable.

The concept of the solutions for (1.1) is established in the following sense

Definition 2.1. *A global weak solution for the Cauchy problem (1.1) is a real valued function* $u = u(x,t)$ *defined in* $\mathbb{R} \times (0,\infty)$ *such that*

$$u \in L^2_{loc}(0,\infty; H^1(\mathbb{R})), \quad u_t \in L^2_{loc}\left(0,\infty; [H^1(\mathbb{R})]'\right),$$

the function u *satisfies the identity integral*

$$-\int_0^T\int_{\mathbb{R}}[uv]\varphi_t dxdt + \int_0^T\int_{\mathbb{R}}[uu_x v]\varphi dxdt + \int_0^T\int_{\mathbb{R}}[u_x v_x]\varphi dxdt = 0, \tag{2.1}$$

for all $v \in H^1(\mathbb{R})$ *and for all* $\varphi \in D(0,T)$. *Moreover,* u *satisfies the initial condition*

$$u(x,0) = u_0(x) \text{ for all } x \in \mathbb{R}.$$

The existence of solution of (1.1) in the precedent sense is guaranteed by the following theorem

Theorem 2.1. *Suppose* $u_0 \in \mathcal{L}^2(K)$, *then there exists a unique global solution* u *of (1.1) in the sense of Definition 2.1. Moreover, energy* $E(t) = \frac{1}{2}|u(t)|^2$ *associated with this solution satisfies*

$$E(t) \leq E(0)(t+1)^{-3/4}. \tag{2.2}$$

The following proposition, whose proof has have been done in Escobedo & Kavian [11], will be useful throughout this paper.

Proposition 2.1. *One has the results*

(1) $\displaystyle\int_{\mathbb{R}} |y|^2|v(y)|^2_{\mathbb{R}} K dy \leq 16\int_{\mathbb{R}} |v_y(y)|^2_{\mathbb{R}} K dy$ *for all* $v \in \mathcal{H}^1(K)$;

(2) *The immersion* $\mathcal{H}^1(K) \hookrightarrow \mathcal{L}(K)$ *is compact*;

(3) $L : \mathcal{H}^1(K) \longrightarrow [\mathcal{H}^1(K)]'$ *is an isomorphism*;

(4) *The eigenvalues of* L *are positive real numbers* $\lambda_j = j/2$ *for* $j = 1, 2\ldots,$ *and the related space with* λ_j *is* $N(L - \lambda_j I) = \left[D^j\omega_1\right]$ *with*

$$\omega_1(y) = \frac{1}{(4\pi)^{1/4}}[K(y)]^{-1} = \frac{1}{(4\pi)^{1/4}}\exp\{-y^2/4\}.$$

(5) *Finally, one has the Poincaré inequality* $|v| \leq \sqrt{2}\,|v_y|$ *for* $v \in \mathcal{H}^1(K)$ □

As the two Cauchy problems (1.1) and (1.7) are equivalent the Definition 2.1 and Theorem 2.1 are also equivalent to Definition 2.2 and Theorem 2.2.

Definition 2.2. *A global weak solution for the Cauchy problem (1.7) is a real valued function* $z = z(y,s)$ *defined in* $\mathbb{R} \times (0,\infty)$ *such that*

$$z \in L^2_{loc}(0,\infty;\mathcal{H}^1(K)), \quad z_s \in L^2_{loc}\left(0,\infty;[\mathcal{H}^1(K)]'\right),$$

the function z *satisfies the identity integral*

$$-\int_0^S \int_{\mathbb{R}} [zv]\varphi_s K dyds + \int_0^S \int_{\mathbb{R}} [zz_y v]\varphi K dyds + \int_0^S \int_{\mathbb{R}} [z_y v_y]\varphi K dyds - \frac{1}{2}\int_0^S \int_{\mathbb{R}} [zv]\varphi K dyds = 0, \tag{2.3}$$

for all $v \in \mathcal{H}^1(K)$ *and for all* $\varphi \in D(0,S)$*. Moreover,* z *satisfies the initial condition*

$$z(y,0) = z_0(y) \ \text{ for all } \ y \in \mathbb{R}.$$

The existence of solutions for system (1.7) will be shown by means of Faedo-Galerkin method. In fact, as $\mathcal{L}^2(K)$ is a separable Hilbert space there exists a orthogonal hilbertian basis $(\omega_j)_{j\in\mathbb{N}}$ of $\mathcal{L}^2(K)$. Moreover, since $\mathcal{H}^1(K) \hookrightarrow \mathcal{L}^2(K)$ is compactly imbedding there exist ω_j solutions of the spectral problem associated with the operator L in $\mathcal{H}^1(K)$. This means that

$$(L\omega_j, v) = \lambda_j(\omega_j, v) \ \text{ for all } \ v \in \mathcal{H}^1(K) \ \text{ and } \ j \in \mathbb{N}. \tag{2.4}$$

Fixed the first eigenfunction ω_1 of L we set $\{\omega_1\}^\perp = \left\{v \in \mathcal{L}^2(K);\ (\omega_1, v) = 0\right\}$. In these conditions one defines V_N as the subspace of $\mathcal{L}^2(K)$ spanned by the $N-$ eigenfunction $\omega_1, \omega_2, \ldots, \omega_N$ of $(\omega_j)_{j\in\mathbb{N}}$, being ω_j with $j \in \mathbb{N}$ defined by (2.4).

Now, we are ready to state the following result.

Theorem 2.2. *Suppose* $z_0 \in \mathcal{L}^2(K) \cap \{\omega_1\}^\perp$*, then there exists a unique solution* z *of (1.7) in the sense of Definition 2.2, provided* $|z_0| < \frac{1}{4\sqrt{3C_1}}$ *holds, where* C_1 *is a positive real constant defined below in the Proposition 2.2-item (b). Moreover, the energy* $E(s) = \frac{1}{2}|z(s)|^2$ *satisfies*

$$E(s) \leq E(0)\exp\left[-s/4\right]. \tag{2.5}$$

Since Theorems 2.1 and 2.2 are equivalent, it suffices to prove the Theorem 2.2. Before this, we first introduce the following property, which will be useful later:

Proposition 2.2. *Considering v in $\mathcal{H}^1(K)$ we have*

$$(a)\;\; K^{1/2}v \in L^\infty(\mathbb{R}) \;\; and \;\; \left|K^{1/2}v\right|_{L^\infty(\mathbb{R})} \le C_1\|v\|_1;$$

$$(b)\;\; |v|_{L^\infty(\mathbb{R})} \le C_1\|v\|_1;$$

$$(c)\;\; \|v\|_1 \le \sqrt{3}|v_y|,$$

where $C_1 = 4C$ and $C > 0$ is defined by $\left|K^{1/2}v\right|_{L^\infty(\mathbb{R})} \le C\left|K^{1/2}v\right|_{H^1(\mathbb{R})}$.
Moreover, if $v \in \mathcal{H}^2(K)$ then

$$(d)\;\; |v_y| \le \sqrt{2}|Lv|;$$

$$(e)\;\; \|v\|_2 \le \widetilde{C}_1|Lv|,$$

for some $\widetilde{C}_1 > 0$ established to follow at the end of the proof below.

Proof - As

$$\left|K^{1/2}v\right|^2_{H^1(\mathbb{R})} \le \int_{\mathbb{R}} |v(y)|^2_{\mathbb{R}} K dy + \int_{\mathbb{R}} \left[\frac{|y|^2}{8}|v(y)|^2_{\mathbb{R}}K + 2|v_y(y)|^2_{\mathbb{R}}K\right] dy,$$

then from Proposition 2.1 one has

$$\left|K^{1/2}v\right|^2_{H^1(\mathbb{R})} \le \int_{\mathbb{R}} |v(y)|^2_{\mathbb{R}} K dy + 4\int_{\mathbb{R}} |v_y(y)|^2_{\mathbb{R}} K dy \le 4\|v\|_1^2.$$

As the continuous immersion $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ holds, we have $K^{1/2}v \in L^\infty(\mathbb{R})$ and there exists $C > 0$ such that $\left|K^{1/2}v\right|_{L^\infty(\mathbb{R})} \le C\left|K^{1/2}v\right|_{H^1(\mathbb{R})}$. This proves the statement (a). As $K^{1/2} \ge 1$, then from (a) one gets (b). The statement (c) is an immediate consequence from Proposition 2.1-item (5). Notice that for all $v \in \mathcal{H}^2(K)$ one has $(v_y, v_y) = (Lv, v)$. From this and Proposition 2.1-item (5) one gets (d). Finally, let $v \in \mathcal{H}^2(K)$ and $Lv = f$ with $f \in \mathcal{L}^2(K)$. Defining $w = K^{1/2}v$ one can write

$$w_{yy} = \left(\frac{1}{4} + \frac{y^2}{16}\right) w - K^{1/2}Lv.$$

From this, Proposition 2.1-item (5) and Proposition 2.2-item (c), one has

$$\int_{\mathbb{R}} \left[|w_{yy}|^2_{\mathbb{R}} + \frac{1}{16}|w|^2_{\mathbb{R}} + \frac{y^4}{256}|w|^2_{\mathbb{R}} + \frac{y^2}{32}|w|^2_{\mathbb{R}} + \frac{1}{2}|w_y|^2_{\mathbb{R}} + \frac{y^2}{8}|w_y|^2_{\mathbb{R}}\right] dy \;=\; \int_{\mathbb{R}} \left[K|Lv|_{\mathbb{R}} - \frac{1}{4}yw_yw\right] dy \le \frac{3}{2}|Lv|.$$

On the other hand, one has

$$|v_{yy}|^2_{\mathbb{R}} \le C_2\Big[|w|^2_{\mathbb{R}} + y^2|w|^2_{\mathbb{R}} + y^4|w|^2_{\mathbb{R}} + y^2|w_y|^2_{\mathbb{R}} + |w_{yy}|^2_{\mathbb{R}}\Big]K^{-1}.$$

From these two above inequalities, Proposition 2.1-item (5) and (d) one obtains (e) ∎

Proof of Theorem 2.2 - We will employ the Faedo-Galerking approximate method to prove the existence of solutions. In fact, the approximate system is obtained from (2.4) and this consists in finding $z^N(s,y) = \sum_{i=1}^{N} g_{iN}(s)\omega_i(y) \in V_N$, the solution of the system of ordinary differential equations

$$\begin{cases} \left(z_s^N(s),\omega\right) + \left(Lz^N(s),\omega\right) - 1/2\left(z^N(s),\omega\right) + \left(z^N(s)z_y^N(s),\omega\right) = 0, \\ z^N(0) = z_0^N = \sum_{j=1}^{N}(z_0,\omega_j)\omega_j, \end{cases} \tag{2.6}$$

for all ω belong to V_N. The System (2.6) has local solution z^N in $0 \le s < s_N$, see for instance, Coddington-Levinson [7]. The estimates to be proven later allow us to extend the solutions z^N to whole interval $[0,S[$ for all $S>0$ and to obtain subsequences that converge, in convenient spaces, to the solution of (1.7) in the sense of Definition 2.2.

Estimate 1. Setting $\omega = z^N(s) \in V_N$ in $(2.6)_1$, it yields

$$\frac{1}{2}\frac{d}{ds}\left|z^N(s)\right|^2 + \left|z_y^N(s)\right|^2 - \frac{1}{2}\left|z^N(s)\right|^2 + \int_{\mathbb{R}} z^N(s)z_y^N(s)z^N(s)K dy = 0.$$

The integral above is upper bounded. In fact, by using Hölder inequality, Proposition 2.1-item (5) and Proposition 2.2-item (b) we can write

$$\left|\int_{\mathbb{R}} z^N(s)z_y^N(s)Lz^N(s)K dy\right|_{\mathbb{R}} \le \sqrt{3}C_1\left|z_y^N(s)\right|^2\left|z^N(s)\right|.$$

From this and from precedent identity we get

$$\frac{1}{2}\frac{d}{ds}\left|z^N(s)\right|^2 + \frac{1}{2}\left|z_y^N(s)\right|^2 + \frac{1}{2}\Big(\left|z_y^N(s)\right|^2 - \left|z^N(s)\right|^2\Big) \le \sqrt{3}C_1\left|z_y^N(s)\right|^2\left|z^N(s)\right|. \tag{2.7}$$

By using $(1.6)_2$, the fact that basis (ω_j) is orthonormal and (2.4) we have

$$\left|z_y^N(s)\right|^2 = \sum_{j=1}^{N}\left(g_{jN}(s)\right)^2\lambda_j \quad \text{and} \quad \left|z^N(s)\right|^2 = \sum_{j=1}^{N}\left(g_{jN}(s)\right)^2.$$

By using these two identities we are able to prove that

$$\frac{1}{2}\Big(|z_y^N(s)|^2 - |z^N(s)|^2\Big) \ge 0 \quad \text{for all} \quad N \in \mathbb{N}. \tag{2.8}$$

In fact, note that

$$\frac{1}{2}\Big(|z_y^N(s)|^2 - |z^N(s)|^2\Big) = \frac{1}{2}\left(g_{1N}(s)\right)^2(\lambda_1 - 1) + \frac{1}{2}\sum_{j=2}^{N}\left(g_{jN}(s)\right)^2(\lambda_j - 1).$$

Next one can prove that

$$g_{1N}(s) = 0 \ \text{ and } \ \sum_{j=2}^{N} (g_{jN}(s))^2 (\lambda_j - 1) \geq 0 \ \text{ for all } \ s \ \text{ and } \ N. \tag{2.9}$$

From Proposition 2.1-item (4) the second statement in (2.9) is obvious. Therefore, it suffices to prove that $g_{1N}(s) = 0$ for all s and N. In fact, first, note that

$$g_{1N}(s) = \Big(\sum_{j=1}^{N} g_{jN}(s)\omega_j, \omega_1 \Big) = (z_N(s), \omega_1) \, .$$

Thus, we will show that $(z_N(s), \omega_1) = 0$. Setting $\omega = \omega_1 \in V_N$ in $(2.6)_1$, it yields

$$\big(z_s^N(s), \omega_1\big) + \big(Lz^N(s), \omega_1\big) - \frac{1}{2}\big(z^N(s), \omega_1\big) + \big(z^N(s) z_y^N(s), \omega_1\big) = 0. \tag{2.10}$$

By using (2.4) and Proposition 2.1-item (4) one can writes

$$\big(Lz^N(s), \omega_1\big) = \frac{1}{2}\big(z^N(s), \omega_1\big) \, .$$

The non-linear term of (2.10) is null, because

$$\big(z^N(s) z_y^N(s), \omega_1\big) \ = \ \frac{1}{2}\frac{1}{(4\pi)^{1/4}} \int_{\mathbb{R}} \Big[\Big(z^N(s) \Big)^2 \Big]_y dy.$$

we have used above Proposition 2.1-item (4), that is,

$$\omega_1(y) = \frac{1}{(4\pi)^{1/4}} \exp\{-y^2/4\}.$$

From this, as $\omega_j \in \mathcal{H}^1(K)$ then $\big(z^N\big)^2$ and $\big(z_y^N\big)^2$ belong to $L^1(\mathbb{R})$ and consequently $\lim_{|y| \longrightarrow \infty} z^N(y,\, t) = 0$. Thus, $\big(z^N(s) z_y^N(s), \omega_1\big) = 0$ for all N and s. Taking into account these facts in (2.10), it yields $\big(z_s^N(s), \omega_1\big) = 0$. Thus, by using $(2.6)_2$ and hypothesis on z_0 we get $\big(z^N(s), \omega_1\big) = \big(z^N(0), \omega_1\big) = 0$. Therefore, this completes the proof of statement of (2.9) □

Since (2.8) is true, the inequality (2.7) is reduced to

$$\frac{1}{2}\frac{d}{ds}\left|z^N(s)\right|^2 + \frac{1}{4}\left|z_y^N(s)\right|^2 + \left|z_y^N(s)\right|^2 \Big(\frac{1}{4} - \sqrt{3} C_1 \left|z^N(s)\right| \Big) \leq 0. \tag{2.11}$$

Next, we will prove that

$$|z^N(s)| < \frac{1}{4\sqrt{3} C_1} \ \text{ for all } \ s \geq 0. \tag{2.12}$$

In fact, suppose it is not true. Then there exists s^* such that

$$|z^N(s)| < \frac{1}{4\sqrt{3}C_1} \quad \text{for all} \quad 0 \le s < s^* \quad \text{and} \quad |z^N(s^*)| = \frac{1}{4\sqrt{3}C_1}.$$

Integrating (2.11) from 0 to s^*, it yields

$$\frac{1}{2}\,|z^N(s^*)|^2 + \frac{1}{4}\int_0^{s^*} |z_y^N(s)|^2 ds + \int_0^{s^*} |z_y^N(s)|^2\Big(\frac{1}{4} - \sqrt{3}C_1|z^N(s)|\Big)ds \le \frac{1}{2}\,|z_0|^2.$$

From hypothesis on z_0 we have

$$|z^N(s^*)| < 1/4\sqrt{3}C_1.$$

This contradicts $\left|z^N(s^*)\right| = 1/4\sqrt{3}C_1$. Thus, (2.12) it is true. Therefore, integrating (2.11) from 0 to s and by using (2.4) and $(2.6)_2$, it yields

$$|z^N(s)|^2 + \frac{1}{2}\int_0^{s} |z_y^N(\tau)|^2 d\tau \le |z_0|^2 \le \frac{1}{4\sqrt{3}C_1}. \tag{2.13}$$

Estimate 2. In this estimate we will use the projection operator

$$P_N : \mathcal{L}^2(K) \longrightarrow V_N \quad \text{defined by} \quad v \longmapsto P_N(v) = \sum_{i=1}^{N}(v,\omega_i)\omega_i.$$

Thus, from $(2.6)_1$ we have

$$\sum_{i=1}^{N}(z_s^N,\omega_i)\omega_i + \sum_{i=1}^{N}(Lz^N,\omega_i)\omega_i - \frac{1}{2}\sum_{i=1}^{N}(z^N,\omega_i)\omega_i + \sum_{i=1}^{N}(z^N(s)z_y^N(s),\omega_i)\omega_i = 0.$$

From this and definition of P_N one can write

$$P_N\left(z_s^N\right) + P_N\left(Lz^N\right) - \frac{1}{2}P_N\left(z^N\right) + P_N\left(z^N(s)z_y^N(s)\right) = 0.$$

As $P_N V_N \subset V_N$ and z^N; z_s^N; $Lz^N \in V_N$, then

$$z_s^N = -Lz^N + \frac{1}{2}z^N - P_N\left(z^N(s)z_y^N(s)\right). \tag{2.14}$$

The identity (2.14) is verified in the $L^\infty\left(0,S;\left[\mathcal{H}^1(K)\right]'\right)$ sense. In fact, analyzing each term on the right-hand side of (2.14) we prove this statement as one can see:

$$
\begin{aligned}
|Lz^N(s)|_{[\mathcal{H}^1(K)]'} &= \sup_{\substack{v\in\mathcal{H}^1(K)\\ ||v||_1\le 1}} \left|\left(z_y^N(s),v_y\right)_{\mathcal{L}^2(K)}\right|_{\mathbb{R}} \le \left|z_y^N(s)\right|. && (2.15)\\
|z^N(s)|_{[\mathcal{H}^1(K)]'} &= \sup_{\substack{v\in\mathcal{H}^1(K)\\ ||v||_1\le 1}} \left|\left(z^N(s),v\right)_{\mathcal{L}^2(K)}\right|_{\mathbb{R}} \le \left|z^N(s)\right|. && (2.16)\\
\left|P_N\left(z^N(s)z_y^N(s)\right)\right|_{[\mathcal{H}^1(K)]'} &\le C_1 \sup_{\substack{v\in\mathcal{H}^1(K)\\ ||v||_1\le 1}} \left|z^N(s)\right|\left|z_y^N(s)\right| \|(P_N v)\|_{\mathcal{H}^1(K)}\\
&\le \left|z^N(s)\right|\left|z_y^N(s)\right|. && (2.17)
\end{aligned}
$$

As the proof of the three identities (2.15), (2.16) and (2.17) are similar, we will just make the last one. In fact,

$$
\begin{aligned}
\left|P_N\left(z^N(s)z_y^N(s)\right)\right|_{[H^1(K)]'} &= \sup_{\substack{v\in\mathcal{H}^1(K)\\ ||v||\le 1}} \left|\left\langle P_N\left(z^N(s)z_y^N(s)\right),v\right\rangle_{[H^1(K)]'\times H^1(K)}\right|_{\mathbb{R}}\\
&= \sup_{\substack{v\in\mathcal{H}^1(K)\\ ||v||\le 1}} \left|\left(P_N\left(z^N(s)z_y^N(s)\right),v\right)_{\mathcal{L}^2(K)}\right|_{\mathbb{R}}\\
&= \sup_{\substack{v\in\mathcal{H}^1(K)\\ ||v||\le 1}} \left|\left(z^N(s)z_y^N(s),P_N v\right)_{\mathcal{L}^2(K)}\right|_{\mathbb{R}}\\
&\le C_1 \sup_{\substack{v\in\mathcal{H}^1(K)\\ ||v||\le 1}} \left|z^N(s)\right|\left|z_y^N(s)\right| \|(P_N v)\|_{\mathcal{H}^1(K)}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\|(P_N v)\|^2_{\mathcal{H}^1(K)} &= \left\|\sum_{i=1}^N (v,\omega_i)\omega_i\right\|^2 = \sum_{i=1}^N (v,\omega_i)^2(\omega_{iy},\omega_{iy})\\
&= \sum_{i=1}^N (v,\omega_i)^2(L\omega_i,\omega_i) = \sum_{i=1}^N (v,\omega_i)^2\lambda_i\\
&= \sum_{i=1}^N \left(v,\frac{L\omega_i}{\sqrt{\lambda_i}}\right)^2 = \sum_{i=1}^N \left(v_y,\frac{\omega_{iy}}{\sqrt{\lambda_i}}\right)^2\\
&\le \sum_{i=1}^\infty \left(v_y,\frac{\omega_{iy}}{\sqrt{\lambda_i}}\right)^2 = \|v\|^2_{\mathcal{H}^1(K)} \le 1.
\end{aligned}
$$

Inserting this inequality in the precedent one we get (2.17). By using (2.15)-(2.17) in (2.14), we get

$$\begin{aligned}
|z_s^N(s)|^2_{[\mathcal{H}^1(K)]'} &\leq \left[\frac{1}{2}|z^N(s)| + \left(1 + C_1|z^N(s)|\right)|z_y^N(s)|\right]^2 \\
&\leq \left[\left(\frac{\sqrt{2}}{2} + 1 + C_1|z^N(s)|\right)|z_y^N(s)|\right]^2 \\
&\leq \left[\left(\frac{\sqrt{2}}{2} + 1 + \frac{C_1}{2\sqrt{\sqrt{3}C_1}}\right)|z_y^N(s)|\right]^2,
\end{aligned}$$

where we have used in the two last step the Poincaré inequality and Estimate (2.13). Integrating this inequality from 0 to S and again using Estimate (2.13), we obtain

$$\int_0^S |z_s^N(s)|^2_{[\mathcal{H}^1(K)]'} ds \leq C, \tag{2.18}$$

where

$$C = \left(\frac{\sqrt{2}}{2} + 1 + \frac{C_1}{2\sqrt{\sqrt{3}C_1}}\right)^2 \frac{1}{2\sqrt{3}C_1}.$$

The limit in the approximate problem (2.6): By Estimates 1 and 2, more precisely, from (2.13) and (2.18) we can extract subsequences of (z^N), which one will denote by (z^N), and a function $z : \mathbb{R} \times (0, S) \to \mathbb{R}$ satisfying

$$\left|\begin{array}{lll}
z^N \rightharpoonup z & \text{weak star in} & L^\infty\left(0, S; \mathcal{L}^2(K)\right), \\
z^N \rightharpoonup z & \text{weak in} & L^2\left(0, S; \mathcal{H}^1(K)\right), \\
z_s^N \rightharpoonup z_s & \text{weak in} & L^2\left(0, S; \left[\mathcal{H}^1(K)\right]'\right).
\end{array}\right. \tag{2.19}$$

From these convergence we are able to pass to the limits in the linear terms of (2.6). The nonlinear term needs careful analysis. In fact, from $(2.19)_{1,3}$ and Aubin's compactness result, see Aubin [1], Browder [4], Lions [15] or Lions [14], we can extract a subsequences of (z^N), which one will denote by (z^N), such that

$$z^N \to z \quad \text{strongly in} \quad L^2(0, S; \mathcal{L}^2(K)). \tag{2.20}$$

On the other hand, for all $\phi(x, s) = v(x)\theta(s)$ with $v \in \mathcal{H}^1(K)$ and $\theta \in D(0, S)$ we have

$$\begin{aligned}
\int_0^S \left(z^N(s)z_y^N(s), \phi(s)\right) ds &= \int_0^S \left(z_y^N, z^N\phi\right) ds \qquad (2.21)\\
&= \int_0^S \left(z_y^N, \left[z^N - z\right]\phi\right) ds + \int_0^S \left(z_y^N, z\phi\right) ds.
\end{aligned}$$

Next, we will show that the last two integrals on the right-hand side of (2.21) converge. In fact, the first one can be upper bounded as follows

$$\Big|\int_0^S \left(z_y^N, \left[z^N - z\right]\phi\right) ds\Big|_{\mathbb{R}} \leq$$
$$\int_0^S \left|z_y^N\right| |\phi|_{L^\infty(\mathbb{R})} \left|z^N - z\right| ds \leq$$
$$C_1|\phi|_{L^\infty(0,S;\mathcal{H}^1(K))} \left|z_y^N\right|_{L^2(0,S;\mathcal{L}^2(K))} \left|z^N - z\right|_{L^2(0,S;\mathcal{L}^2(K))}.$$

From this, (2.13) and (2.20) we have

$$\int_0^S \left(z_y^N(s), \left[z^N(s) - z(s)\right]\phi(s)\right) ds \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

The second integral also converges because from $(2.19)_2$ we have, in particular, that

$$z_y^N \rightharpoonup z_y \quad \text{weak in} \quad L^2\left(0, S; \mathcal{L}^2(K)\right)$$

and because $\phi z \in L^2\left(0, S; \mathcal{L}^2(K)\right)$. Therefore, we have

$$\int_0^S \left(z_y^N(s), z(s)\phi(s)\right) ds \longrightarrow \int_0^S \left(z_y(s), z(s)\phi(s)\right) ds \text{ as } N \longrightarrow \infty.$$

Taking these two limits in (2.21) we get

$$\int_0^S \left(z^N(s) z_y^N(s), \phi(s)\right) ds \longrightarrow \int_0^S \left(z(s) z_y(s), \phi(s)\right) ds \text{ as } N \longrightarrow \infty \quad \blacksquare$$

Uniqueness of solutions of (1.7): The global weak solutions of the initial value problem (1.7) is unique for all $s \in [0, S]$, $S > 0$. In fact, from $(2.19)_1$ and $(2.19)_3$ the duality $\langle z_s, z\rangle_{[\mathcal{H}^1(K)]' \times \mathcal{H}^1(K)}$ makes sense. Thus, suppose z and $\widehat{z}$ are two solutions of (1.7) and let $\varphi = z - \widehat{z}$, then φ satisfies

$$\varphi_s + L\varphi - \frac{\varphi}{2} = -\left(z z_y - \widehat{z}\,\widehat{z}_y\right), \qquad \varphi(0) = 0. \tag{2.22}$$

Taking the duality paring on the both sides of $(2.22)_1$ with φ we obtain

$$\frac{1}{2}\frac{d}{ds}|\varphi(s)|^2 + |\varphi_y(s)|^2 = \frac{1}{2}|\varphi(s)|^2 - \left(z(s)\varphi_y(s), \varphi(s)\right) - \left(\widehat{z}_y(s)\varphi(s), \varphi(s)\right). \tag{2.23}$$

From Proposition 2.2-item (b) and (c) one obtains

$$\begin{gathered}
|(z(s)\varphi_y(s),\varphi(s))+(\widehat{z}_y(s)\varphi(s),\varphi(s))|_{\mathbb{R}}\leq\\
|z(s)|_{L^\infty(\mathbb{R})}\,|\varphi_y(s)|\,|\varphi(s)|+|\varphi(s)|_{L^\infty(\mathbb{R})}\,|\widehat{z}_y(s)|\,|\varphi(s)|\leq\\
C_1\sqrt{3}\,|\varphi_y(s)|\,\Big(\,|z_y(s)|+|\widehat{z}_y(s)|\,\Big)\,|\varphi(s)|\leq\\
\frac{1}{2}\,|\varphi_y(s)|^2+3C_1^2\left(|z_y(s)|^2+|\widehat{z}_y(s)|^2\right)|\varphi(s)|^2\,.
\end{gathered}$$

Substituting this inequality in (2.23) yields

$$\frac{1}{2}\frac{d}{ds}|\varphi(s)|^2+\frac{1}{2}|\varphi_y(s)|^2\leq\left[\frac{1}{2}+3C_1^2\left(|z_y(s)|^2+|\widehat{z}_y(s)|^2\right)\right]|\varphi\,(s)|^2\,.$$

From $(2.19)_2$ one has that z and $\widehat{z}$ belong to $L^2\left(0,S;\mathcal{H}^1(K)\right)$. Therefore, applying the Gronwall inequality one gets $\varphi(s)=0$ in $[0,S]$ ■

Asymptotic behavior: The asymptotic behavior, as $s\to\infty$, of $E(s)=\frac{1}{2}|z(s)|^2$ given by the unique solution of the Cauchy problem (1.7) is established as a consequence of inequality (2.11). In fact, from (2.11), (2.12) and Banach-Steinhauss theorem we get that the limit function z defined by (2.19) satisfies the inequality

$$\frac{1}{2}\,\frac{d}{ds}\,|z(s)|^2+\frac{1}{4}\,|z_y(s)|^2\leq 0.$$

From Proposition 2.1-item (5) we obtain

$$\frac{d}{ds}\,|z(s)|^2+\frac{1}{4}\,|z(s)|^2\leq 0.$$

As a consequence from this inequality we get the inequality (2.5) directly ■

Remark 2.1. *The inequality (2.2) is a consequence of (2.5). In fact, from (1.3) we obtain* $|z(s)|^2=(t+1)^{1/2}|u(t)|^2$ *and* $|z_0|^2=|u_0|^2$*. Moreover, as* $s=\ln(t+1)$*, then* $\exp[-s/4]=(t+1)^{-1/4}$*. Therefore, from (2.5) we have*

$$|u(t)|^2=\frac{1}{t+1}|z(s)|^2\leq|u_0|^2(t+1)^{-3/4}\quad\blacksquare$$

3 Strong Solution

Setting the initial data $u_0 \in \mathcal{H}^1(K)$ we are able to show that the Cauchy problem (1.1) has a unique real valued strong solution $u = u(x,t)$ defined in $\mathbb{R} \times (0,T)$ for all $T > 0$. Precisely, the strong solution of (1.1) is defined as follows.

Definition 3.1. *A global strong solution for the initial value problem (1.1) is a real valued function $u = u(x,t)$ defined in $\mathbb{R} \times (0,T)$ for an arbitrary $T > 0$, such that*

$$u \in L^\infty_{loc}(0,\infty; H^2(\mathbb{R})), \quad u_t \in L^2_{loc}(0,\infty; H^1(\mathbb{R})),$$

$$\int_0^T \int_{\mathbb{R}} u_t \varphi dxdt + \int_0^T \int_{\mathbb{R}} u u_x \varphi dxdt + \int_0^T \int_{\mathbb{R}} u_x \varphi_x dxdt = 0, \tag{3.1}$$

for all $\varphi \in L^2\left(0,T; H^1(\mathbb{R})\right)$. Moreover, u satisfies the initial condition

$$u(x,0) = u_0(x) \text{ for all } x \in \mathbb{R}.$$

The existence of solution of (1.1) in the precedent sense is guaranteed by the following theorem.

Theorem 3.1. *Suppose $u_0 \in \mathcal{H}^1(K)$, then there exists a unique global solution u of (1.1) in the sense of Definition 3.1. Moreover, energy $E(t) = \frac{1}{2}|u(t)|^2$ associated with this solution satisfies*

$$E(t) \leq E(0)(t+1)^{-3/4}.$$

As the Cauchy problems (1.1) and (1.7) are equivalent, the Definition 3.1 and Theorem 3.1 are also equivalent to Definition 3.2 and Theorem 3.2.

Definition 3.2. *A global strong solution of the initial-boundary value problem (1.7) is a real valued function $z = z(y,s)$ defined in $\mathbb{R} \times (0,S)$ for arbitrary $S > 0$, such that*

$$z \in L^\infty_{loc}(0,\infty; \mathcal{H}^2(K)), \quad z_s \in L^2_{loc}(0,\infty; \mathcal{H}^1(K)) \text{ for } S > 0,$$

$$\int_0^S \int_{\mathbb{R}} z_s \varphi K dyds + \int_0^S \int_{\mathbb{R}} Lz\varphi K dyds + \tag{3.2}$$

$$-\frac{1}{2}\int_0^S \int_{\mathbb{R}} z\varphi K dyds + \int_0^S \int_{\mathbb{R}} z z_y \varphi K dyds = 0,$$

for all $\varphi \in L^2\left(0,S; \mathcal{H}^1(K)\right)$. Moreover, z satisfies the initial condition

$$z(y,0) = z_0(y) \text{ for all } y \in \mathbb{R}.$$

The existence of solutions of the system (1.7) will be also shown by means of Faedo-Galerkin method and by using the special basis defined as solutions of spectral problem (2.4) and the first eigenfunction ω_1 of L such that $\{\omega_1\}^{\perp} = \big\{v \in \mathcal{L}^2(K);\, (\omega_1, v) = 0\big\}$. Under these conditions one defines V_N as in Section 2. Now we state the following theorem.

Theorem 3.2. *Suppose $z_0 \in \mathcal{H}^1(K) \cap \{\omega_1\}^{\perp}$, then there exists a unique solution z of (1.7) in the sense of Definition 2.2, provided $|z_{0y}| < 1/4\sqrt{6}C_1$ holds. Moreover, the energy $E(s) = \frac{1}{2}|z(s)|^2$ satisfies*

$$E(s) \leq E(0) \exp\left[-s/4\right].$$

Since Theorems 3.1 and 3.2 are equivalent it is suffices to prove Theorem 3.2.

Proof of Theorem 3.2 - We need to establish two estimates. In fact,

Estimate 3. Setting $\omega = Lz^N(s) \in V_N$ in $(2.6)_1$, it yields

$$\frac{1}{2}\frac{d}{ds}|z_y^N(s)|^2 + |Lz^N(s)|^2 \leq$$
$$\frac{1}{8}|z^N(s)|^2 + \frac{1}{2}|Lz^N(s)|^2 + \Big|\int_{\mathbb{R}} z^N(s) z_y^N(s) Lz^N(s) K dy\Big|_{\mathbb{R}}.$$

Next, we will find the upper bound of the last term on the right-hand side of the above inequality. In fact, by using Hölder inequality, Proposition 2.2-items (b), (c) and (d) we can write

$$\Big|\int_{\mathbb{R}} z^N(s) z_y^N(s) Lz^N(s) K dy\Big|_{\mathbb{R}} \leq C_1\sqrt{6}\,|z_y^N(s)|\,\big|Lz^N(s)\big|^2.$$

From this we have

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}|z_y^N(s)|^2 + \frac{1}{8}|Lz^N(s)|^2 + \frac{1}{8}\Big(|Lz^N(s)|^2 - |z^N(s)|^2\Big) + \\ |Lz^N(s)|^2\Big(\frac{1}{4} - \sqrt{6}C_1|z_y^N(s)|\Big) \leq 0.\end{aligned} \tag{3.3}$$

Use a similar argument as in Estimate 1 we are able to prove that

$$\frac{1}{8}\Big(|Lz^N(s)|^2 - |z^N(s)|^2\Big) \geq 0 \text{ for all } N \in \mathbb{N}. \tag{3.4}$$

In fact, note that

$$\frac{1}{8}\Big(|Lz^N(s)|^2-|z^N(s)|^2\Big)=\frac{1}{8}\left(g_{1N}(s)\right)^2\left(\lambda_1^2-1\right)+\frac{1}{8}\sum_{j=2}^{N}\left(g_{jN}(s)\right)^2\left(\lambda_j^2-1\right).$$

From (2.9) we have

$$\sum_{j=2}^{N}\left(g_{jN}(s)\right)^2\left(\lambda_j^2-1\right)\geq 0 \ \text{ and } \ g_{1N}(s)=0 \ \text{ for all } \ s \ \text{ and } \ N.$$

From this we obtain (3.4), see Estimate 1. Since (3.4) is true, the inequality (3.3) is reduced to

$$\frac{1}{2}\frac{d}{ds}|z_y^N(s)|^2+\frac{1}{8}|Lz^N(s)|^2+|Lz^N(s)|^2\Big(\frac{1}{4}-\sqrt{6}C_1|z_y^N(s)|\Big)\leq 0. \tag{3.5}$$

Next, proceeding as (2.12) we will prove that

$$|z_y^N(s)|<1/4\sqrt{6}C_1 \ \text{ for all } \ s\geq 0. \tag{3.6}$$

Therefore, by using (3.6) in (3.5), it yields

$$\frac{1}{2}\frac{d}{ds}|z_y^N(s)|^2+\frac{1}{8}|Lz^N(s)|^2\leq 0.$$

Integrating from 0 to s and using the hypothesis on the initial data we obtain

$$|z_y^N(s)|^2+\frac{1}{4}\int_0^s|Lz^N(\tau)|^2d\tau\leq|z_{0y}|^2<\frac{1}{4\sqrt{6}C_1}. \tag{3.7}$$

Estimate 4. Setting $\omega=z_s^N(s)\in V_N$ in $(2.6)_1$, it yields

$$|z_s^N(s)|^2=-\left(Lz^N(s),z_s^N(s)\right)+\frac{1}{2}\left(z^N(s),z_s^N(s)\right)-\left(z^N(s)z_y^N(s),z_s^N(s)\right).$$

Next, we will estimate the three inner product on the right-hand side of the above identity. In fact, by usual inequalities and Proposition 2.2 we have

$$|z_s^N(s)|^2 \ \leq \ C_2^2\left[\left|Lz^N(s)\right|+\frac{1}{2}\left|z^N(s)\right|+\left|z_y^N(s)\right|^2\right]^2,$$

where $C_2=\max\{1,\ \sqrt{3}C_1\}$. From this we get a constant $C>0$ independent of N and s such that

$$\int_0^S\left|z_s^N(s)\right|^2ds\leq C, \tag{3.8}$$

where C depends on the constant of Estimate (3.7), that is, $1/4\sqrt{6}C_1$ and of the constants of immersions established in Proposition 2.2.

From (3.7) and (3.8) we can take the limit on the approximate system (2.6). In fact, the analysis of the limit as $N \longrightarrow \infty$ in the linear terms of (2.6) is similar to those of Section 2. However, the nonlinear term is made as follows. From (3.7), (3.8) and Aubin-Lions theorem one extracts subsequences of (z^N), which will be denoted by (z^N), such that

$$\left| \begin{array}{lllll} z^N \rightharpoonup z & \text{weak in} & L^2\left(0,S;\mathcal{H}^2(K)\right) & \text{as} & N \longrightarrow \infty, \\ z^N \rightarrow z & \text{strong in} & L^2\left(0,S;\mathcal{L}^2(K)\right) & \text{as} & N \longrightarrow \infty, \\ z_y^N \rightarrow z_y & \text{strong in} & L^2\left(0,S;\mathcal{L}^2(K)\right) & \text{as} & N \longrightarrow \infty, \\ z_s^N \rightharpoonup z_s & \text{weak in} & L^2\left(0,S;\mathcal{L}^2(K)\right) & \text{as} & N \longrightarrow \infty. \end{array} \right. \tag{3.9}$$

From usual inequalities and Proposition 2.2 one has

$$\begin{gathered} \int_{\mathbb{R}} \left| z_y^N(s) z^N(s) - z_y(s) z(s) \right|_{\mathbb{R}}^2 K dy = \\ \int_{\mathbb{R}} \left| \left[z_y^N(s) - z_y(s) \right] z^N(s) + z_y(s) \left[z^N(s) - z(s) \right] \right|_{\mathbb{R}}^2 K dy \leq \\ 2 \left| z^N(s) \right|_{L^\infty(\mathbb{R})}^2 \left| z_y^N(s) - z_y(s) \right|^2 + 2 \left| z_y(s) \right|^2 \left| z^N(s) - z(s) \right|^2 \leq \\ C \left[\left| z_y^N(s) - z_y(s) \right|^2 + \left| z^N(s) - z(s) \right|^2 \right]. \end{gathered}$$

Taking (3.9) in this inequality, it yields

$$\begin{gathered} \int_0^S \left| z_y^N(s) z^N(s) - z_y(s) z(s) \right|^2 ds \leq \\ C \int_0^S \left[\left| z_y^N(s) - z_y(s) \right|^2 + \left| z^N(s) - z(s) \right|^2 \right] ds \longrightarrow 0 \ \text{ as } \ N \longrightarrow \infty. \end{gathered} \tag{3.10}$$

Therefore, one has

$$z_y^N z^N \longrightarrow z_y z \ \text{ strong in } \ L^2\left(0,S;\mathcal{L}^2(K)\right) \ \text{ as } \ N \longrightarrow \infty \quad \square$$

Thus, the proof of Theorem 3.2 is completed by using a similar argument as in Section 2.

Finally, the uniqueness of solutions and the exponential decay rate of the energy are established in a similar way as in Section 2 ■

4 Regularity of Strong Solutions

Our goal here is to prove a result of regularity for strong solutions established in Section 3. We will achieve this goal by means of the following regularity result

Theorem 4.1. *Let $z = z(y, s)$ be a strong solution of problem (1.7), which is guaranteed by Theorem 3.2, then $z \in C^0\left([0, T]; \mathcal{H}^1(K)\right)$.*

Proof: We will show that z is the limit of a Cauchy sequence. In fact, suppose M, $N \in \mathbb{N}$ fixed with $N > M$ and z^N, z^M are two solutions of (1.7). Thus, $v^N = z^N - z^M$ satisfies

$$v_s^N(s) + Lv^N(s) - \frac{1}{2}v^N(s) = P_M\left(z^M(s)z_y^M(s)\right) - P_N\left(z^N(s)z_y^N(s)\right) \quad \text{in} \quad \mathcal{L}^2(K).$$

Therefore, from (2.6), one has that

$$\begin{gathered}\left(v_s^N(s), \omega\right) + \left(Lv^N(s), \omega\right) - 1/2\left(v^N(s), \omega\right) = \\ \left(P_M\left(z^M(s)z_y^M(s)\right) - \left(P_N\left(z^N(s)z_y^N(s)\right), \omega\right), \omega\right),\end{gathered} \tag{4.1}$$

for all $\omega \in V^N \subset \mathcal{L}^2(K)$, where P_N, P_M are projection operators defined in $\mathcal{L}^2(K)$ with values in V^N, V^M respectively.

Estimate 5. Setting $\omega = v^N(s) \in V_N$ in (4.1) we get

$$\begin{gathered}\frac{1}{2}\frac{d}{ds}\left|v^N(s)\right|^2 + \left|v_y^N(s)\right|^2 - \frac{1}{2}\left|v^N(s)\right|^2 \leq \\ \Big|P_N\left(z^N(s)z_y^N(s) - z(s)z_y(s)\right) + P_N\left(z(s)z_y(s)\right) - \\ P_M\left(z(s)z_y(s)\right) + P_M\left(z(s)z_y(s) - z^M(s)z_y^M(s)\right)\Big|\left|v^N(s)\right| \leq \\ \left|z^N(s)z_y^N(s) - z(s)z_y(s)\right|^2 + \left|P_N\left(z(s)z_y(s)\right) - P_M\left(z(s)z_y(s)\right)\right|^2 + \\ \left|z(s)z_y(s) - z^M(s)z_y^M(s)\right|^2 + \frac{3}{4}\left|v^N(s)\right|^2.\end{gathered}$$

Integrating form 0 to S one has

$$\begin{gathered}\frac{1}{2}\left|v^N(s)\right|^2 + \int_0^S \left|v_y^N(s)\right|^2 ds \leq \frac{1}{2}\left|v_0^N\right|^2 + \int_0^S \left|z^N(s)z_y^N(s) - z(s)z_y(s)\right|^2 ds + \\ \int_0^S \left|P_N\left(z(s)z_y(s)\right) - P_M\left(z(s)z_y(s)\right)\right|^2 ds + \\ \int_0^S \left|z(s)z_y(s) - z^M(s)z_y^M(s)\right|^2 ds + \frac{5}{4}\int_0^S \left|v^N(s)\right|^2 ds.\end{gathered} \tag{4.2}$$

The task now is to show that

$$\int_0^S |P_N(z(s)z_y(s)) - P_M(z(s)z_y(s))|^2 \, ds \longrightarrow 0 \text{ as } N \longrightarrow \infty. \tag{4.3}$$

In fact, as $zz_y \in L^2\left(0, S; \mathcal{L}^2(K)\right)$, then $z(s)z_y(s) \in \mathcal{L}^2(K)$ a. e. in $[0, S]$. Therefore, $P_N(z(s)z_y(s)) \longrightarrow z(s)z_y(s)$ in a. e. in $[0, S]$ as $N \longrightarrow \infty$. That is,

$|P_N(z(s)z_y(s)) - z(s)z_y(s)| \longrightarrow 0$ in a. e. in $[0, S]$ as $N \longrightarrow \infty$. Moreover,

$|P_N(z(s)z_y(s)) - z(s)z_y(s)| \leq 2\,|z(s)z_y(s)|$ and $|z(s)z_y(s)| \in L^2(0, S)$.

Thus, applying Lebesgue's dominated convergence theorem, it yields

$$\int_0^S |P_N(z(s)z_y(s)) - z(s)z_y(s)|^2 \, ds \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

In other words

$$P_N(zz_y) \longrightarrow zz_y \text{ in } L^2\left(0, S; \mathcal{L}^2(K)\right) \text{ as } N \longrightarrow \infty.$$

Therefore, $(P_N(zz_y))_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2\left(0, S; \mathcal{L}^2(K)\right)$. Hence we have that (4.3) is true □

On the other hand, from (4.2) and Granwall inequality one gets

$$\begin{gathered}\frac{1}{2}\left|v^N(s)\right|^2 + \int_0^S \left|v_y^N(s)\right|^2 ds \leq \\ \left[\frac{1}{2}\left|v_0^N\right|^2 + \int_0^S \left|z^N(s)z_y^N(s) - z(s)z_y(s)\right|^2 ds + \right. \\ \int_0^S |P_N(z(s)z_y(s)) - P_M(z(s)z_y(s))|^2 \, ds + \\ \left. \int_0^S \left|z(s)z_y(s) - z^M(s)z_y^M(s)\right|^2 ds\right] \exp\left\{(5/4)S\right\}.\end{gathered}$$

From this, hypothesis on the initial data, (3.10) and (4.3) one gets that $\left(z^N\right)_{N\in\mathbb{N}}$ is a Cauchy sequence in $C^0\left(0, S; \mathcal{L}^2(K)\right)$.

To obtain the desired regularity one needs one more estimate as follows.

Estimate 6. Setting $\omega = Lv^N(s) \in V_N$ in (4.1) we get

$$\frac{1}{2}\frac{d}{ds}\left|v_y^N(s)\right|^2 + \left|Lv^N(s)\right|^2 - \frac{1}{2}\left|v_y^N(s)\right|^2 \leq$$
$$\Big|P_N\left(z^N(s)z_y^N(s) - z(s)z_y(s)\right) + P_N\left(z(s)z_y(s)\right) -$$
$$P_M\left(z(s)z_y(s)\right) + P_M\left(z(s)z_y(s) - z^M(s)z_y^M(s)\right)\Big|\left|Lv^N(s)\right| \leq$$
$$\left|z^N(s)z_y^N(s) - z(s)z_y(s)\right|^2 + \left|P_N\left(z(s)z_y(s)\right) - P_M\left(z(s)z_y(s)\right)\right|^2 +$$
$$\left|z(s)z_y(s) - z^M(s)z_y^M(s)\right|^2 + \frac{3}{4}\left|Lv^N(s)\right|^2.$$

Integrating from 0 to S and using Granwall inequality, one gets

$$\frac{1}{2}\left|v_y^N(s)\right|^2 + \int_0^S \left|Lv_y^N(s)\right|^2 ds \leq$$
$$\Big[\frac{1}{2}\left|v_{0y}^N\right|^2 + \int_0^S \left|z^N(s)z_y^N(s) - z(s)z_y(s)\right|^2 ds +$$
$$\int_0^S \left|P_N\left(z(s)z_y(s)\right) - P_M\left(z(s)z_y(s)\right)\right|^2 ds +$$
$$\int_0^S \left|z(s)z_y(s) - z^M(s)z_y^M(s)\right|^2 ds\Big] \exp\left\{(5/4)S\right\}.$$

From this, hypothesis on the initial data, (3.10) and (4.3) one gets that $\left(z^N\right)_{N\in\mathbb{N}}$ is a Cauchy sequence in $C^0\left(0,S;\mathcal{H}^1(K)\right)$. Thus, we have the desired regularity. And therefore the proof of Theorem 4.1 is ended ■

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