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Linear impulsive dynamic systems on time scales

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Abstract. The purpose of this paper is to present the fundamental concepts of the basic theory for linear impulsive systems on time scales. First, we introduce the transition matrix for linear impulsive dynamic systems on time scales and we establish some properties of them. Second, we prove the existence and uniqueness of solutions for linear impulsive dynamic systems on time scales. Also we give some sufficient conditions for the stability of linear impulsive dynamic systems on time scales.

1 Introduction

Differential equations with impulse provide an adequate mathematical description of various real-word phenomena in physics, engineering, biology, economics, neutral network, social sciences, etc. Also, the theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects. In the last fifty years the theory of impulsive differential equations has been studied by many authors. We refer to the monographs [10]-[12], [27], [33], [42] and the references therein.

S. Hilger [28] introduced the theory of time scales (measure chains) in order to create a theory that can unify continuous and discrete analysis. The theory of dynamic systems on time scales allows us to study both continuous and discrete dynamic systems simultaneously (see [8], [9], [28], [29]). Since Hilger's initial work [28] there has been significant growth in the theory of dynamic systems on time scales, covering a variety of different qualitative aspects. We refer to the books [9], [15], [16], [32] and the papers [1], [3], [17], [20], [21], [24], [30], [34], [38], [40], [41]. In recent years, some authors studied impulsive dynamic systems on time scales [13], [25], [34], [36], but only few authors have studied linear impulsive dynamic systems on time scales.

In this paper we study some aspects of the qualitative theory of linear impulsive dynamic systems on time scales. In Section 2 we present some preliminary results on linear dynamic systems on time scales and also we give an impulsive inequality on time scales. In Section 3 we prove the existence and uniqueness of solutions for homogeneous linear impulsive dynamic systems on time scales. For this, we introduce the transition matrix for linear impulsive dynamic systems

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on time scales and we give some properties of the transition matrix. In Section 4 we prove the existence and uniqueness of solutions for nonhomogeneous linear impulsive dynamic systems on time scales. In Section 5 we give some sufficient conditions for the stability of linear impulsive dynamic systems on time scales. Finally, in Section 6 we present a brief summary of time scales analysis.

2 Preliminaries

Let $\mathbb{R}^n$ be the space of n-dimensional column vectors $x = \mathrm{col}(x_1, x_2, ...x_n)$ with a norm $||\cdot||$. Also, by the same symbol $||\cdot||$ we will denote the corresponding matrix norm in the space $M_n(\mathbb{R})$ of $n \times n$ matrices. If $A \in M_n(\mathbb{R})$, then we denote by A^T its conjugate transpose. We recall that $||A|| := \sup\{||Ax||; ||x|| \leq 1\}$ and the following inequality $||Ax|| \leq ||A|| \cdot ||x||$ holds for all $A \in M_n(\mathbb{R})$ and $x \in \mathbb{R}^n$. A time scales $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^n)$.

The notations $[a, b]$, $[a, b)$, and so on, will denote time scales intervals such as $[a, b] := \{t \in \mathbb{T}; a \leq t \leq b\}$, where $a, b \in \mathbb{T}$. Also, for any $\tau \in \mathbb{T}$, let $\mathbb{T}_{(\tau)} := [\tau, \infty) \cap \mathbb{T}$ and $\mathbb{T}_+ := \mathbb{T}_{(0)}$. Then

$$BC_{rd}(\mathbb{T}_{(\tau)}, \mathbb{R}^n) := \{f \in C_{rd}(\mathbb{T}_{(\tau)}, \mathbb{R}^n); \sup_{t \in \mathbb{T}_{(\tau)}} ||f(t)|| < +\infty\}$$

is a Banach space with the norm $||f|| := \sup_{t \in \mathbb{T}_{(\tau)}} ||f(t)||$.

We denote by $\mathcal{R}$ (respectively $\mathcal{R}^+$) the set of all regressive (respectively positively regressive) functions from $\mathbb{T}$ to $\mathbb{R}$.

The space of all rd-continuous and regressive functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}\mathrm{R}(\mathbb{T}, \mathbb{R})$. Also,

$$C^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R}) := \{p \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R}); 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

We denote by $C^1_{rd}(\mathbb{T}, \mathbb{R}^n)$ the set of all functions $f : \mathbb{T} \to \mathbb{R}^n$ that are differentiable on $\mathbb{T}$ and its delta-derivative $f^\Delta(t) \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$. The set of rd-continuous (respectively rd-continuous and regressive) functions $A : \mathbb{T} \to M_n(\mathbb{R})$ is denoted by $C_{rd}(\mathbb{T}, M_n(\mathbb{R}))$ (respectively by $C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$). We recall that a matrix-valued function A is said to be regressive if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$, where I is the $n \times n$ identity matrix.

Now consider the following dynamic system on time scales

$$x^\Delta = A(t)x \tag{2.1}$$

where $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$. This is a *homogeneous linear dynamic system* on time scales that is *nonautonomous*, or *time-variant*. The corresponding *nonhomogeneous linear dynamic system* is given by

$$x^\Delta = A(t)x + h(t) \tag{2.2}$$

where $h \in C_{rd}(\mathbb{T}_+, \mathbb{R}^n)$.

A function $x \in C^1_{rd}(\mathbb{T}_+, \mathbb{R}^n)$ is said to be a solution of (2.2) on $\mathbb{T}_+$ provided $x^\Delta(t) = A(t)x(t) + h(t)$ for all $t \in \mathbb{T}_+$.

Theorem 2.1. (Existence and Uniqueness Theorem [15, Theorem 5.8])
If $A \in C_{rd}\mathbb{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $h \in C_{rd}(\mathbb{T}_+, \mathbb{R}^n)$, then for each $(\tau, \eta) \in \mathbb{T}_+ \times \mathbb{R}^n$ the initial value problem

$$x^\Delta = A(t)x + h(t), \quad x(\tau) = \eta$$

has a unique solution $x : \mathbb{T}_{(\tau)} \to \mathbb{R}^n$. $\square$

A matrix $X_A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ is said to be a *matrix solution* of (2.1) if each column of X_A satisfies (2.1). A *fundamental matrix* of (2.1) is a matrix solution X_A of (2.1) such that $\det X_A(t) \neq 0$ for all $t \in \mathbb{T}_+$. A *transition matrix* of (2.1) at initial time $\tau \in \mathbb{T}_+$ is a fundamental matrix such that $X_A(\tau) = I$. The transition matrix of (2.1) at initial time $\tau \in \mathbb{T}_+$ will be denoted by $\Phi_A(t, \tau)$. Therefore, the transition matrix of (2.1) at initial time $\tau \in \mathbb{T}_+$ is the unique solution of the following matrix initial value problem

$$Y^\Delta = A(t)Y, \quad Y(\tau) = I,$$

and $x(t) = \Phi_A(t, \tau)\eta$, $t \geq \tau$, is the unique solution of initial value problem

$$x^\Delta = A(t)x, \quad x(\tau) = \eta.$$

Theorem 2.2. ([15, Theorem 5.21]) *If $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ then*

(i) $\Phi_A(t, t) = I$;

(ii) $\Phi_A(\sigma(t), s) = [I + \mu(t)A(t)]\Phi_A(t, s)$;

(iii) $\Phi_A^{-1}(t, s) = \Phi^T_{\ominus A^T}(t, s)$;

(iv) $\Phi_A(t, s) = \Phi_A^{-1}(s, t) = \Phi^T_{\ominus A^T}(s, t)$;

(v) $\Phi_A(t, s)\Phi_A(s, r) = \Phi_A(t, r)$, $t \geq s \geq r$. $\square$

Theorem 2.3. ([15, Theorem 5.24]) *If $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $h \in C_{rd}(\mathbb{T}_+, \mathbb{R}^n)$, then for each $(\tau, \eta) \in \mathbb{T}_+ \times \mathbb{R}^n$ the initial value problem*

$$x^\Delta = A(t)x + h(t), \quad x(\tau) = \eta$$

has a unique solution $x : \mathbb{T}_{(\tau)} \to \mathbb{R}^n$ given by

$$x(t) = \Phi_A(t, \tau)\eta + \int_\tau^t \Phi_A(t, \sigma(s))h(s)\Delta s, \quad t \geq \tau. \ \square$$

As in the scalar case, along with (2.1), we consider its *adjoint equation*

$$y^{\Delta} = -A^T(t)x^{\sigma}. \tag{2.3}$$

If $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, then the initial value problem $y^{\Delta} = -A^T(t)x^{\sigma}$, $x(\tau) = \eta$, has a unique solution $y : \mathbb{T}_{(\tau)} \to \mathbb{R}^n$ given by $y(t) = \Phi_{\ominus A^T}(t, \tau)\eta$, $t \geq \tau$.

Theorem 2.4. ([15, Theorem 5.27]) *If $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $h \in C_{rd}(\mathbb{T}_+, \mathbb{R}^n)$, then for each $(\tau, \eta) \in \mathbb{T}_+ \times \mathbb{R}^n$ the initial value problem*

$$y^{\Delta} = -A^T(t)y^{\sigma} + h(t), \quad x(\tau) = \eta$$

has a unique solution $x : \mathbb{T}_{(\tau)} \to \mathbb{R}^n$ given by

$$y(t) = \Phi_{\ominus A^T}(t, \tau)\eta + \int_{\tau}^{t} \Phi^T_{\ominus A}(t, s)h(s)\Delta s, \quad t \in \mathbb{T}_{(\tau)}.$$

Lemma 2.1. *Let $\tau \in \mathbb{T}_+$, $y, b \in C_{rd}\mathbb{R}(\mathbb{T}_+, \mathbb{R})$, $p \in C^+_{rd}\mathbb{R}(\mathbb{T}_+, \mathbb{R})$ and $c, b_k \in \mathbb{R}_+$, $k = 1, 2, \ldots.$ Then*

$$y(t) \leq c + \int_{\tau}^{t} p(s)y(s)\Delta s + \sum_{\tau < t_k < t} b_k y(t_k),\ t \in \mathbb{T}_{(\tau)} \tag{2.4}$$

implies

$$y(t) \leq c \prod_{\tau < t_k < t} (1 + b_k)e_p(t, \tau),\ t \geq \tau. \tag{2.5}$$

Proof. Let $v(t) := c + \int_{\tau}^{t} p(s)y(s)\Delta s + \sum_{\tau < t_k < t} b_k y(t_k)$, $t \geq \tau$. Then

$$\begin{cases} v^{\Delta}(t) = p(t)y(t),\ t \neq t_k,\ v(\tau) = c \\ v(t_k^+) = v(t_k) + b_k y(t_k),\ k = 1, 2, \ldots. \end{cases}$$

Since $y(t) \leq v(t)$, $t \geq \tau$, we then have

$$\begin{cases} v^{\Delta}(t) \leq p(t)v(t),\ t \neq t_k,\ v(\tau) = c \\ v(t_k^+) = (1 + b_k)v(t_k),\ k = 1, 2, \ldots. \end{cases}$$

An application of Lemma A.4 yields

$$v(t) \leq c \prod_{\tau < t_k < t} (1 + b_k)e_p(t, \tau),\ t \geq \tau,$$

which implies (2.5). $\square$

3 Homogeneous linear impulsive dynamic system on time scales

Consider the following homogeneous linear impulsive dynamic system on time scales

$$\begin{cases} x^{\Delta} = A(t)x, \;\; t \in \mathbb{T}_+, \, t \neq t_k \\ x(t_k^+) = x(t_k^-) + B_k x(t_k), \, k = 1, 2, ... \end{cases} \tag{3.1}$$

where $B_k \in M_n(\mathbb{R})$, $k = 1, 2, ...$, $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, $0 = t_0 < t_1 < t_2 < ... < t_k < ...$, with $\lim\limits_{k\to\infty} t_k = \infty$, $x(t_k^-)$ represent the left limit of $x(t)$ at $t = t_k$ (with $x(t_k^-) = x(t)$ if t_k is left-scattered) and $x(t_k^+)$ represents the right limit of $x(t)$ at $t = t_k$ (with $x(t_k^+) = x(t)$ if t_k is right-scattered). We assume for the remainder of the paper that, for $k = 1, 2, ...$, the points of impulse t_k are rigth-dense.

Along with (3.1) we consider the following initial value problem

$$\begin{cases} x^{\Delta} = A(t)x, \;\; t \in \mathbb{T}_{(\tau)}, \, t \neq t_k \\ x(t_k^+) = x(t_k^-) + B_k x(t_k), \, k = 1, 2, ... \\ x(\tau^+) = \eta, \;\; \tau \geq 0. \end{cases} \tag{3.2}$$

We note that, instead of the usual initial condition $x(\tau) = \eta$, we impose the limiting condition $x(\tau^+) = \eta$ which, in general case, is natural for (3.2) since (τ, η) may be such that $\tau = t_k$ for some $k = 1, 2, ...$ In the case when $\tau \neq t_k$ for any k, we shall understand the initial condition $x(\tau^+) = \eta$ in the usual sense, that is, $x(\tau) = \eta$.

In order to define the solution of (3.2), we introduce the following spaces

$$\Omega := \{x : \mathbb{T}_+ \to \mathbb{R}^n; \; x \in C((t_k, t_{k+1}), \mathbb{R}^n), \, k = 0, 1, ..., \, x(t_k^+) \text{ and } x(t_k^-) \text{ exist with } x(t_k^-) = x(t_k), \, k = 1, 2, ...\}$$

and

$$\Omega^{(1)} := \{x \in \Omega; \; x \in C^1((t_k, t_{k+1}), \mathbb{R}^n), \, k = 0, 1, ...\},$$

where $C((t_k, t_{k+1}), R^n)$ is the set of all continuous functions on (t_k, t_{k+1}) and $C^1((t_k, t_{k+1})$ is the set of all continuously differentiable functions on (t_k, t_{k+1}), $k = 0, 1, ...$

A function $x \in \Omega^{(1)}$ is said to be a solution of (3.1), if it satisfies $x^{\Delta}(t) = A(t)x(t)$, everywhere on $\mathbb{T}_{(\tau)} \setminus \{\tau, t_{k(\tau)}, t_{k(\tau)+1}, ...\}$ and for each $j = k(\tau), k(\tau) + 1, ...$ satisfies the impulsive conditions $x(t_j^+) = x(t_j) + B_j x(t_j)$ and the initial condition $x(\tau^+) = \eta$, where $k(\tau) := \min\{k = 1, 2, ...; \tau < t_k\}$.

Theorem 3.1. *If $A \in C_{rd}\mathbb{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $B_k \in M_n(\mathbb{R})$, $k = 1, 2, ...$, then any solution of (3.2) is also a solution of the impulsive integral equation*

$$x(t) = x(\tau) + \int_{\tau}^{t} A(s)x(s)\Delta s + \sum_{\tau < t_j \leq t} B_j x(t_j), \, t \geq \tau. \tag{3.3}$$

and conversely.

Proof. There exists $i \in \{1, 2, ...\}$ such that $\tau \in [t_{i-1}, t_i)$. Then any solution of (3.2) on $[\tau, t_i)$ is also a solution of integral equation $x(t) = x(\tau) + \int_\tau^t A(s)x(s)\Delta s$, $t \in [\tau, t_i]$. Further, any solution of the initial value problem

$$\begin{cases} x^\Delta = A(t)x, \;\; t \in (t_i, t_{i+1}) \\ x(t_i^+) = x(t_i) + B_i x(t_i). \end{cases}$$

is a solution of integral equation $x(t) = x(t_i^+) + \int_{t_i}^t A(s)x(s)\Delta s$, $t \in [t_i, t_{i+1})$. It follows that

$$\begin{aligned} x(t) &= x(t_i^+) + \int_{t_i}^t A(s)x(s)\Delta s = x(t_i) + B_i x(t_i) + \int_{t_i}^t A(s)x(s)\Delta s \\ &= x(\tau) + \int_\tau^{t_i} A(s)x(s)\Delta s + \int_{t_i}^t A(s)x(s)\Delta s + B_i x(t_i) \\ &= x(\tau) + \int_\tau^t A(s)x(s)\Delta s + B_i x(t_i),\; t \in [t_i, t_{i+1}). \end{aligned}$$

Next, we suppose that, for any $k > i + 2$, any solution of (3.2) on $[t_{k-1}, t_k)$ is a solution of (3.3). Then any solution of the initial value problem

$$\begin{cases} x^\Delta = A(t)x, \;\; t \in (t_k, t_{k+1}] \\ x(t_k^+) = x(t_k) + B_k x(t_k) \end{cases}$$

is a solution of integral equation $x(t) = x(t_k^+) + \int_{t_k}^t A(s)x(s)\Delta s$, $t \in [t_k, t_{k+1})$. It follows that

$$\begin{aligned} x(t) &= x(t_k^+) + \int_{t_k}^t A(s)x(s)\Delta s = x(t_k) + B_k x(t_k) + \int_{t_k}^t A(s)x(s)\Delta s \\ &= x(\tau) + \int_\tau^{t_k} A(s)x(s)\Delta s + \sum_{\tau < t_j < t_k} B_j x(t_j) + B_k x(t_k) + \int_{t_k}^t A(s)x(s)\Delta s \\ &= x(\tau) + \int_\tau^t A(s)x(s)\Delta s + \sum_{\tau < t_j \le t} B_j x(t_j),\; t \in [t_k, t_{k+1}). \end{aligned}$$

Therefore, by the Mathematical Induction Principle, (3.3) is proved. The converse statement follows trivially and the proof is complete. □

Theorem 3.2. *If $A \in C_{rd}\mathbb{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $B_k \in M_n(\mathbb{R})$, $k = 1, 2, ...$, then the solution of (3.2) satisfies the following estimate*

$$||x(t)|| \le ||x(\tau)|| \prod_{\tau < t_k \le t} (1 + ||B_k||) \exp\left(\int_\tau^t ||A(s)||\Delta s \right), \tag{3.4}$$

for $\tau, t \in \mathbb{T}_+$ with $t \ge \tau$.

Proof. From (3.3) we obtain that

$$||x(t)|| \le ||x(\tau)|| + \int_\tau^t ||A(s)|| \cdot ||x(s)||\Delta s + \sum_{\tau < t_j \le t} ||B_j|| \cdot ||x(t_j)||,\; t \ge \tau.$$

Then, by Lemma 2.1, it follows that

$$||x(t)|| \leq ||x(\tau)|| \prod_{\tau<t_k\leq t}(1+||B_k||)\exp_{||A(\cdot)||}(t,\tau),\ t\geq\tau.$$

Since for any $a \geq 0$,

$$\lim_{u\searrow\mu(s)}\frac{\ln(1+au)}{u} = \begin{cases} a & \text{if } \mu(s)=0 \\ \frac{\ln(1+a\mu(s))}{\mu(s)} \leq a & \text{if } \mu(s)>0, \end{cases}$$

then explicit estimation of the modulus of the exponential function on time scales (see [24]) gives

$$\begin{aligned}\exp_{||A(\cdot)||}(t,\tau) &= \exp\left(\int_\tau^t \lim_{u\searrow\mu(s)}\frac{\ln(1+u||A(s)||)}{u}\Delta s\right)\\ &\leq \exp\left(\int_\tau^t ||A(s)||\Delta s\right),\ t\geq\tau.\end{aligned}$$

Thus we obtain (3.4). □

Along with (3.1) we consider the impulsive transition matrix $S_A(t,s)$, $0 \leq s \leq t$, associated with $\{B_k,t_k\}_{k=1}^{\infty}$, given by

$$S_A(t,s)=\begin{cases} \Phi_A(t,s) & \text{if } t_{k-1}\leq s\leq t\leq t_k; \\ \Phi_A(t,t_k^+)(I+B_k)\Phi_A(t_k,s) & \text{if } t_{k-1}\leq s<t_k<t<t_{k+1}; \\ \Phi_A(t,t_k^+)[\prod_{s<t_j\leq t}(I+B_j)\Phi_A(t_j,t_{j-1}^+)](I+B_i)\Phi_A(t_i,s) & \\ \qquad\qquad \text{if } t_{i-1}\leq s<t_i<...<t_k<t<t_{k+1}, \end{cases} \tag{3.5}$$

where $\Phi_A(t,s)$, $0\leq s\leq t$, is the transition matrix of (2.1) at initial time $s\in\mathbb{T}_+$.

Remark 3.1. Since

$$S_A(t,s) = \\ \Phi_A(t,t_k^+)(I+B_k)\Phi_A(t_k,t_{k-1}^+)[\prod_{s<t_j<t_k}(I+B_j)\Phi_A(t_j,t_{j-1}^+)](I+B_i)\Phi_A(t_i,s),$$

it follows that

$$S_A(t,s) = \Phi_A(t,t_k^+)(I+B_k)S_A(t_k,s) \text{ for } t_{i-1}\leq s<t_i<...<t_k\leq t<t_{k+1}.$$

In the following, we will assume that $I+B_k$ is invertible for each $k=1,2,....$

Theorem 3.3. *If $A\in C_{rd}\mathcal{R}(\mathbb{T}_+,M_n(\mathbb{R}))$ and $B_k\in M_n(\mathbb{R})$, $k=1,2,...$, then for each $(\tau,\eta)\in\mathbb{T}_+\times\mathbb{R}^n$ the initial value problem (3.2) has a unique solution given by*

$$x(t)=S_A(t,\tau)\eta,\ \ t\geq\tau.$$

Proof. Let $(\tau,\eta)\in\mathbb{T}_+\times\mathbb{R}^n$. Then there exists $i\in\{1,2,...\}$ such that $\tau\in[t_{i-1},t_i)$. Then the unique solution of (3.2) on $[\tau,t_i)$ is given by $x(t)=\Phi_A(t,\tau)\eta=S_A(t,\tau)\eta$, $t\in[\tau,t_i)$.

Further, we consider the initial value problem

$$\begin{cases} x^\Delta=A(t)x, & t\in(t_i,t_{i+1})\\ x(t_i^+)=x(t_i)+B_ix(t_i). \end{cases}$$

This initial value problem has the unique solution given by $x(t)=\Phi_A(t,t_i^+)x(t_i^+)$, $t\in[t_i,t_{i+1})$. It follows that

$$\begin{aligned} x(t) &= \Phi_A(t,t_i^+)x(t_i^+)=\Phi_A(t,t_i^+)(I+B_i)x(t_i)\\ &= \Phi_A(t,t_i^+)(I+B_i)\Phi_A(t_i,\tau)\eta\\ & S_A(t,\tau)\eta, \end{aligned}$$

and so $x(t)=S_A(t,\tau)\eta$, $t\in[t_i,t_{i+1})$. Next, we suppose that, for any $k>i+2$, the unique solution of (3.2) on $[t_{k-1},t_k]$ is given by

$$x(t)=S_A(t,\tau)\eta=\Phi_A(t,t_{k-1}^+)[\prod_{s<t_j<t}(I+B_j)\Phi_A(t_j,t_{j-1}^+)](I+B_i)\Phi_A(t_i,\tau)\eta.$$

Then the initial value problem

$$\begin{cases} x^\Delta=A(t)x, & t\in(t_k,t_{k+1})\\ x(t_k^+)=x(t_k)+B_kx(t_k) \end{cases}$$

has the unique solution $x(t)=\Phi_A(t,t_k^+)x(t_k^+)$, $t\in[t_k,t_{k+1})$. It follows that

$$x(t)=\Phi_A(t,t_k^+)x(t_i^+)=\Phi_A(t,t_k^+)(I+B_k)x(t_k^+)=$$

$$\Phi_A(t,t_k^+)(I+B_k)\Phi_A(t,t_{k-1}^+)[\prod_{s<t_j\le t}(I+B_j)\Phi_A(t_j,t_{j-1}^+)](I+B_i)\Phi_A(t_i,\tau)\eta$$

$$=\Phi_A(t,t_k^+)[\prod_{s<t_j<t}(I+B_j)\Phi_A(t_j,t_{j-1}^+)](I+B_i)\Phi_A(t_i,\tau)$$

$$=S_A(t,\tau)\eta,$$

and so $x(t)=S_A(t,\tau)\eta$, $t\in[t_k,t_{k+1}]$. Therefore, by the Mathematical Induction Principle, the theorem is proved. $\square$

Corollary 3.1. *If $A\in C_{rd}\mathbb{R}(\mathbb{T}_+,M_n(\mathbb{R}))$ and $B_k\in M_n(\mathbb{R})$, $k=1,2,...$, then the impulsive transition matrix $S_A(t,s)$, $0\le s\le t$, is the unique solution of the following matrix initial value problem*

$$\begin{cases} Y^\Delta=A(t)Y, & t\in\mathbb{T}_{(s)},\ t\ne t_k\\ Y(t_k^+)=(I+B_k)Y(t_k), & k=1,2,...\\ Y(s^+)=I, & s\ge 0. \end{cases}\tag{3.6}$$

Moreover, the following properties hold:

(i) $S_A(t_k^+, s) = (I + B_k)S_A(t_k, s)$, $t_k \geq s$, $k = 1, 2, ...;$

(ii) $S_A(t, t_k^+) = S_A(t, t_k)(I + B_k)^{-1}$, $t_k \leq t$, $k = 1, 2, ...;$

(iii) $S_A(t, t_k^+)S_A(t_k^+, s) = S_A(t, s)$, $0 \leq s \leq t_k \leq t$, $k = 1, 2, ...$ $\square$

From the Theorems 3.2 and 3.3, we obtain the following result.

Corollary 3.2. *If $A \in C_{rd}\mathbb{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $B_k \in M_n(\mathbb{R})$, $k = 1, 2, ...,$ then we have the following estimate*

$$||S_A(t, \tau)|| \leq \prod_{\tau < t_k \leq t} (1 + ||B_k||) \exp\left(\int_\tau^t ||A(s)||\Delta s\right), \tag{3.7}$$

for $\tau, t \in \mathbb{T}_+$ with $t \geq \tau$. Moreover, for any $\tau, t \in \mathbb{T}_+$ the function $(r, s) \to S_A(r, s)$ is bounded on set $\{(r, s) \in \mathbb{T}_+ \times \mathbb{T}_+; \tau \leq s \leq r \leq t\}$.

Proof. Using the Theorems 3.2 and 3.3, for all $\tau, t \in \mathbb{T}_+$ with $t \in \mathbb{T}_{(\tau)}$, it follows

$$||x(t)|| = ||S_A(t, \tau)x(\tau)|| \leq ||x(\tau)|| \prod_{\tau < t_k \leq t} (1 + ||B_k||) \exp\left(\int_\tau^t ||A(s)||\Delta s\right),$$

which implies (3.7). $\square$

Let $X_A(t)$, $t \in \mathbb{T}_+$, be the unique solution of (3.6) with the initial condition $Y(0) = I$, i.e., $X_A(t) := S_A(t, 0)$, $t \in \mathbb{T}_+$.

Theorem 3.4. *If $A \in C_{rd}\mathbb{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $B_k \in M_n(\mathbb{R})$, then the impulsive transition matrix $S_A(t, s)$ has the following properties*

(i) $S_A(t, s) = X_A(t)X_A^{-1}(s)$, $0 \leq s \leq t$;

(ii) $S_A(t, t) = I$, $t \geq 0$;

(iii) $S_A(t, s) = S_A^{-1}(s, t)$, $0 \leq s \leq t$;

(iv) $S_A(\sigma(t), s) = [I + \mu(t)A(t)]S_A(t, s)$, $0 \leq s \leq t$;

(v) $S_A(t, s)S_A(s, r) = S_A(t, r)$, $0 \leq r \leq s \leq t$.

Proof. (i). Let $Y(t) := X_A(t)X_A^{-1}(s)$, $0 \leq s \leq t$. Then we have that

$$Y^\Delta(t) = X_A^\Delta(t)X_A^{-1}(s) = A(t)X_A(t)X_A^{-1}(s) = A(t)Y(t),\ t \neq t_k.$$

Also, $Y(s) = X_A(s)X_A^{-1}(s) = I$, and

$$\begin{aligned} Y(t_k^+) - Y(t_k) &= X_A(t_k^+)X_A^{-1}(s) - X_A(t_k)X_A^{-1}(s) \\ &= [X_A(t_k^+) - X_A(t_k)]X_A^{-1}(s) = B_kX_A(t_k)X_A^{-1}(s) \\ &= B_kY(t_k) \end{aligned}$$

for each $t_k \geq s$. Therefore, $Y(t) = X_A(t)X_A^{-1}(s)$ solves the initial value problem (3.6), which has exactly one solution. Therefore, $S_A(t,s) = X_A(t)X_A^{-1}(s)$, $0 \leq s \leq t$.

The properties (ii) and (iii) follows from (i). Now, from Theorem A.2 and Corollary 3.1, we have that

$$S_A(\sigma(t), s) = S_A(t,s) + \mu(t)S_A^{\Delta}(t,s)$$

$$= S_A(t,s) + \mu(t)A(t)S_A(t,s)$$

$$= [I + \mu(t)A(t)]S_A(t,s),$$

and so (iv) is true.

Further, let $Y(t) := S_A(t,s)S_A(s,r)$, $0 \leq r \leq s \leq t$. Then we have

$$Y^{\Delta}(t) = S_A^{\Delta}(t,s)S_A(s,r) = A(t)S_A(t,s)S_A(s,r) = A(t)Y(t), \; t \neq t_k,$$

and $Y(r^+) = S_A(r^+, s)S_A(s, r^+) = S_A(r^+, s)S_A^{-1}(r^+, s) = I$ according to (iii). Also,

$$Y(t_k^+) = S_A(t_k^+, s)S_A(s,r) = (I + B_k)S_A(t_k, s)S_A(s,r)$$

$$= (I + B_k)S_A(t_k, r) = (I + B_k)Y(t_k)$$

for each $t_k \geq s$. Therefore, $Y(t)$ solves (3.6) with initial condition $Y(r^+) = I, \; r \in \mathbb{T}_+$. By the uniqueness of solution, it follows that $S_A(t,r) = Y(t) = S_A(t,s)S_A(s,r)$, $0 \leq r \leq s \leq t$, and so (v) is true. $\square$

Theorem 3.5. *If* $A \in C_{rd}\mathbb{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, $B_k \in M_n(\mathbb{R})$ *and* $a, b, \tau \in \mathbb{T}_+$, *then*

$$\frac{\partial}{\Delta s} S_A(t,s) = -S_A(t, \sigma(s))A(s), \; s \neq t_k.$$

Proof. Indeed, from Theorem 3.4 and Theorem A.2, we have

$$\begin{aligned}
\frac{\partial}{\Delta t} S_A(t,s) &= \frac{\partial}{\Delta t} S_A^{-1}(s,t) = -S_A^{-1}(\sigma(s), t)[\frac{\partial}{\Delta t} S_A(s,t)]S_A^{-1}(s,t) \\
&= -S_A^{-1}(\sigma(s), t)A(s)S_A(s,t)S_A^{-1}(s,t) \\
&= -S_A(t, \sigma(s))A(s).
\end{aligned}$$

Therefore, $\frac{\partial}{\Delta t} S_A(t,s) = -S_A(t, \sigma(s))A(s)$ for all $s \in \mathbb{T}_+$, $s \neq t_k$, $k = 1, 2, ...$
$\square$

Theorem 3.6. *If* $A \in C_{rd}\mathbb{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ *and* $B_k \in M_n(\mathbb{R})$, *then*

$$S_A^{-1}(t,s) = S_{\ominus A^T}^T(t,s), \; 0 \leq s \leq t. \tag{3.8}$$

Proof. Let $Y(t) := (S_A^{-1}(t,s))^T$, $0 \le s \le t$. According to Theorem A.2 and (6.5), we have

$$\begin{aligned} Y^\Delta(t) &= -\left(S_A^{-1}(\sigma(t),s) S_A^\Delta(t,s) S_A^{-1}(t,s)\right)^T \\ &= -\left(S_A^{-1}(\sigma(t),s) A(t) S_A(t,s) S_A^{-1}(t,s)\right)^T \\ &= -\left(S_A^{-1}(\sigma(t),s) A(t)\right)^T \\ &= -(S_A^{-1}(t,s)[I+\mu(t)A(t)]^{-1}A(t))^T \\ &= (S_A^{-1}(t,s)(\ominus A)(t))^T = (\ominus A)^T(t))(S_A^{-1}(t,s))^T \\ &= (\ominus A)^T(t)Y(t), \end{aligned}$$

and hence

$$Y^\Delta = (\ominus A)^T(t)Y,\ t \ge s,\ t \ne t_k.$$

Also, $Y(s^+) = (S_A^{-1}(s^+,s^+))^T = (I^{-1})^T = I$ and

$$Y(t_k^+) = (S_A^{-1}(t_k^+,s))^T = (X_A(s)X_A^{-1}(t_k^+))^T = ((X_A^T)^{-1}(s)X_A^T(t_k^+))^{-1}$$

$$= ((X_A^T)^{-1}(s)X_A^T(t_k)(I+B_k^T))^{-1} = (I+B_k^T)^{-1}(X_A^T)^{-1}(t_k)X_A^T(s)$$

$$= (I+B_k^T)^{-1}(X_A(s)X_A^{-1}(t_k))^T = (I+B_k^T)^{-1}(S_A^{-1}(t_k,s))^T$$

$$= (I+C_k)Y(t_k)$$

for each $t_k \ge s$, where $C_k := -B^T(I+B_k^T)^{-1}$, $k = 1,2,....$ Therefore, $Y(t) = (S_A^{-1}(t,s))^T$, $0 \le s \le t$, solve the initial value problem

$$\begin{cases} Y^\Delta = (\ominus A)^T(t)Y,\ t \ge s,\ t \ne t_k. \\ Y(t_k^+) = (I+C_k)Y(t_k),\ k = 1,2,... \\ Y(s^+) = I, \end{cases}$$

which has exactly one solution. It follows that $S_{\ominus A^T}(t,s) = Y(t) = (S_A^{-1}(t,s))^T$, and so $S_A^{-1}(t,s) = S_{\ominus A^T}^T(t,s)$, $0 \le s \le t$. □

Remark 3.2. The matrix equation $Y^\Delta = -A^T(t)Y^\sigma$ is equivalent to the equation $Y^\Delta = (\ominus A)^T(t)Y$.

Indeed, we have

$$\begin{aligned} y^\Delta &= -A^T(t)y^\sigma = -A^T(t)[y+\mu(t)y^\Delta] \\ &= -A^T(t)y - \mu(t)A^T(t)y^\Delta, \end{aligned}$$

that is,

$$[I+\mu(t)A^T(t)]y^\Delta = -A^T(t)y.$$

Since A is regressive, then A^T is also regressive. Then, by (6.5), the above equality is equivalent to

$$y^\Delta = -[I+\mu(t)A^T(t)]^{-1}A^T(t)y = (\ominus A^T)(t)y.$$

Remark 3.3. If $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $B_k \in M_n(\mathbb{R})$, then $y(t) = S^T_{\ominus A^T}(t,\tau)\eta$, $0 \le \tau \le t$, is the unique solution of initial value problem

$$\begin{cases} y^\Delta = -A^T(t)y^\sigma,\ t \ge \tau,\ t \ne t_k. \\ y(t_k^+) = (I + C_k)y(t_k),\ k = 1, 2, ... \\ y(\tau^+) = \eta, \end{cases} \tag{3.9}$$

where $C_k := -B^T(I + B_k^T)^{-1}$, $k = 1, 2, ...$

The homogeneous linear impulsive dynamic system on time scales (3.9) is called the *adjoint dynamic system* of (3.2).

Corollary 3.3. *If $A \in C_{rd}R(T_+, M_n(\mathbb{R}))$ and $B_k \in M_n(\mathbb{R})$, $k = 1, 2, ...$, then any solution of (3.9) is also a solution of the impulsive integral equation*

$$y(t) = y(\tau) - \int_\tau^t A^T(s)y^\sigma(s)\Delta s + \sum_{\tau<t_j<t} B_j^T y(t_j),\ t \in \mathbb{T}_{(\tau)}.$$

and conversely.

4 Nonhomogeneous linear impulsive dynamic system on time scales

Let $\boldsymbol{l}^\infty(\mathbb{R}^n)$ be the space of all sequences $c := \{c_k\}_{k=1}^\infty$, $c_k \in \mathbb{R}^n$, $k = 1, 2, ...$,such that $\sup_{k\ge1} ||c_k|| < \infty$. Then $\boldsymbol{l}^\infty(\mathbb{R}^n)$ is a Banach space with the norm $||c|| := \sup_{k\ge1} ||c_k||$.

Consider the following nonhomogeneous initial value problem

$$\begin{cases} x^\Delta = A(t)x + h(t),\ \ t \in \mathbb{T}_{(\tau)},\ t \ne t_k \\ x(t_k^+) = x(t_k) + B_k x(t_k) + c_k,\ k = 1, 2, ... \\ x(\tau^+) = \eta,\ \ \tau \ge 0. \end{cases} \tag{4.1}$$

where $B_k \in M_n(\mathbb{R})$, $k = 1, 2, ...$, $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, $c := \{c_k\}_{k=1}^\infty \in \boldsymbol{l}^\infty(\mathbb{R}^n)$ and $h : \mathbb{T}_+ \to \mathbb{R}^n$ is a given function.

Theorem 4.1. *If $B_k \in M_n(\mathbb{R})$, $k = 1, 2, ...$, $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, $c := \{c_k\}_{k=1}^\infty \in \boldsymbol{l}^\infty(\mathbb{R}^n)$ and $h \in C_{rd}\mathcal{R}(\mathbb{T}_+, \mathbb{R}^n)$ then, for each $(\tau, \eta) \in \mathbb{T}_+ \times \mathbb{R}^n$, the initial value problem (4.1) has a unique solution given by*

$$x(t) = S_A(t,\tau)\eta + \int_\tau^t S_A(t, \sigma(s))h(s)\Delta s + \sum_{\tau<t_j<t} S_A(t, t_j^+)c_j,\ t \ge \tau. \tag{4.2}$$

Proof. Let $(\tau, \eta) \in \mathbb{T}_+ \times \mathbb{R}^n$. Then there exists $i \in \{1, 2, ...\}$ such that $\tau \in [t_{i-1}, t_i)$. Then the unique solution of (4.1) on $[\tau, t_i]$ is given by

$$\begin{aligned} x(t) &= \Phi_A(t,\tau)\eta + \int_\tau^t \Phi_A(t, \sigma(s))h(s)\Delta s \\ &= S_A(t,\tau)\eta + \int_\tau^t S_A(t, \sigma(s))h(s)\Delta s, t \in [\tau, t_i]. \end{aligned}$$

For $t \in (t_i, t_{i+1}]$ the Cauchy problem

$$\begin{cases} x^{\Delta} = A(t)x + h(t), \; t \in (t_i, t_{i+1}) \\ x(t_i^+) = x(t_i) + B_i x(t_i) + c_i, \end{cases}$$

has the unique solution

$$x(t) = \Phi_A(t, t_i^+) x(t_i^+) + \int_{t_i}^{t} \Phi_A(t, \sigma(s)) h(s) \Delta s, \; t \in (t_i, t_{i+1}].$$

It follows that

$$x(t) = \Phi_A(t, t_i^+)[(I + B_i)x(t_i) + c_i] + \int_{t_i}^{t} \Phi_A(t, \sigma(s))h(s)\Delta s$$

$$= \Phi_A(t, t_i^+)(I + B_i)[\Phi_A(t_i, \tau)\eta + \int_{\tau}^{t_i} S_A(t_i, \sigma(s))h(s)\Delta s]$$

$$+ \Phi_A(t, t_i^+)c_i + \int_{t_i}^{t} \Phi_A(t, \sigma(s))h(s)\Delta s =$$

$$\Phi_A(t, t_i^+)(I + B_i)\Phi_A(t_i, \tau)\eta + \int_{\tau}^{t_i} \Phi_A(t, t_i^+)(I + B_i)\Phi_A(t_i, \sigma(s))h(s)\Delta s$$

$$+ \Phi_A(t, t_i^+)c_i + \int_{t_i}^{t} \Phi_A(t, \sigma(s))h(s)\Delta s$$

Using (3.5) we get that

$$x(t) =$$
$$S_A(t, \tau)\eta + \int_{\tau}^{t_k} S_A(t, \sigma(s))h(s)\Delta s + \int_{t_k}^{t} S_A(t, \sigma(s))h(s)\Delta s + S_A(t, t_i^+)c_i$$

and so

$$x(t) = S_A(t, \tau)\eta + \int_{\tau}^{t} S_A(t, \sigma(s))h(s)\Delta s + S_A(t, t_i^+)c_i, \; t \in (t_i, t_{i+1}].$$

Next, we suppose that, for any $k > i + 2$, the unique solution of (4.1) on $[t_{k-1}, t_k]$ is given by

$$x(t) = S_A(t, \tau)\eta + \int_{\tau}^{t} S_A(t, \sigma(s))h(s)\Delta s + \sum_{\tau < t_j < t} S_A(t, t_j^+)c_j, \; t \in [t_k, t_{k+1}].$$

Then the initial value problem

$$\begin{cases} x^{\Delta} = A(t)x + h(t), \; t \in (t_k, t_{k+1}] \\ x(t_k^+) = x(t_k) + B_k x(t_k) + c_k \end{cases}$$

has the unique solution

$$x(t) = \Phi_A(t, t_k^+) x(t_k^+) + \int_{t_k}^{t} \Phi_A(t, \sigma(s)) h(s) \Delta s, \; t \in [t_k, t_{k+1}].$$

It follows that

$x(t)= \Phi_A(t,t_k^+)[(I+B_k)x(t_k)+c_k]+ \int_{t_k}^t \Phi_A(t,\sigma(s))h(s)\Delta s =$

$\Phi_A(t,t_k^+)(I+B_k)[S_A(t_k,\tau)\eta+ \int_\tau^{t_k} S_A(t_k,\sigma(s))h(s)\Delta s+ \sum_{\tau<t_j<t_k} S_A(t_k,t_j^+)c_j]$

$+\Phi_A(t,t_k^+)c_k+ \int_{t_k}^t \Phi_A(t,\sigma(s))h(s)\Delta s,$

hence

$x(t)=$

$\Phi_A(t,t_k^+)(I+B_k)S_A(t_k,\tau)\eta+ \int_\tau^{t_k} \Phi_A(t,t_k^+)(I+B_k)S_A(t_k,\sigma(s))h(s)\Delta s$

$+\sum_{\tau<t_j<t_k} \Phi_A(t,t_k^+)(I+B_k)S_A(t_k,t_j^+)c_j+S_A(t,t_k^+)c_k$

$+\int_{t_k}^t S_A(t,\sigma(s))h(s)\Delta s.$

Using the Remark 3.1, we obtain that

$$\begin{aligned} x(t) &= S_A(t,\tau)\eta + \int_\tau^{t_k} S_A(t,\sigma(s))h(s)\Delta s + \int_{t_k}^t S_A(t,\sigma(s))h(s)\Delta s \\ &\quad + \sum_{\tau<t_j<t} S_A(t,t_j^+)c_j, \end{aligned}$$

and so,

$$x(t) = S_A(t,\tau)\eta + \int_\tau^t S_A(t,\sigma(s))h(s)\Delta s + \sum_{\tau<t_j<t} S_A(t,t_j^+)c_j,$$

Therefore, by the Mathematical Induction Principle, (4.2) is proved. □

Corollary 4.1. *If* $B_k \in M_n(\mathbb{R})$, $k=1,2,...$, $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, *and* $h \in C_{rd}\mathcal{R}(\mathbb{T}_+,\mathbb{R}^n)$ *then, for each* $(\tau,\eta)\in \mathbb{T}_+\times\mathbb{R}^n$, *the initial value problem*

$$\begin{cases} x^\Delta = A(t)x + h(t), \ \ t\in\mathbb{T}_{(\tau)},\, t\neq t_k \\ x(t_k^+) = x(t_k) + B_k x(t_k),\, k=1,2,... \\ x(\tau^+) = \eta, \ \ \tau\geq 0. \end{cases}$$

has a unique solution given by

$$x(t) = S_A(t,\tau)\eta + \int_\tau^t S_A(t,\sigma(s))h(s)\Delta s,\, t\in\mathbb{T}_{(\tau)}.$$

Theorem 4.2. *If $B_k \in M_n(\mathbb{R})$, $k = 1, 2, ...$, $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, $c := \{c_k\}_{k=1}^{\infty} \in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$ and $h \in C_{rd}\mathcal{R}(\mathbb{T}_+, \mathbb{R}^n)$ then, for each $(\tau, \eta) \in \mathbb{T}_+ \times \mathbb{R}^n$, the initial value problem*

$$\begin{cases} y^{\Delta} = -A^T(t)y^{\sigma} + h(t),\ t \geq \tau,\ t \neq t_k. \\ y(t_k^+) = (I + C_k)y(t_k) + c_k,\ k = 1, 2, ... \\ y(\tau^+) = \eta. \end{cases} \tag{4.3}$$

has a unique solution given by

$$y(t) = S_{\ominus A^T}(t, \tau)\eta + \int_{\tau}^{t} S_{\ominus A^T}(t, s)h(s)\Delta s + \sum_{\tau < t_j < t} S_{\ominus A^T}(t, t_j^+)c_j,\ t \geq \tau, \tag{4.4}$$

where $C_k := -B^T(I + B_k^T)^{-1}$, $k = 1, 2, ...$

Proof. We have

$$\begin{aligned} y^{\Delta} &= -A^T(t)y^{\sigma} + h(t) = -A^T(t)[y + \mu(t)y^{\Delta}] + h(t) \\ &= -A^T(t)y - \mu(t)A^T(t)y^{\Delta} + h(t), \end{aligned}$$

that is,

$$[I + \mu(t)A^T(t)]y^{\Delta} = -A^T(t)y + h(t).$$

Since A is regressive, then A^T is also regressive, and thus the above inequality is equivalent to

$$\begin{aligned} y^{\Delta} &= -[I + \mu(t)A^T(t)]^{-1}A^T(t)y + [I + \mu(t)A^T(t)]^{-1}h(t) \\ &= (\ominus A^T)(t)y + [I + \mu(t)A^T(t)]^{-1}h(t). \end{aligned}$$

From Theorem 4.1 it follows that (4.3) has the unique solution

$$\begin{aligned} y(t) = S_{\ominus A^T}(t, \tau)\eta &+ \textstyle\int_{\tau}^{t} S_{\ominus A^T}(t, \sigma(s))[I + \mu(s)A^T(s)]^{-1}h(s)\Delta s \\ &+ \textstyle\sum_{\tau < t_j < t} S_{\ominus A^T}(t, t_j^+)c_j,\ t \geq \tau. \end{aligned} \tag{4.5}$$

Since, by Theorem 3.4 and (3.8), we have

$$\begin{aligned} &S_{\ominus A^T}(t, \sigma(s))[I + \mu(s)A^T(s)]^{-1} = \{[I + \mu(s)A^T(s)]^{-1}S_{\ominus A^T}^T(t, \sigma(s))\}^T \\ &= \{[I + \mu(s)A^T(s)]^{-1}S_A^{-1}(t, \sigma(s))\}^T = \{[I + \mu(s)A^T(s)]^{-1}S_A(\sigma(s), t)\}^T \\ &= S_A^T(s, t) = (S_A^{-1}(s, t))^T = S_{\ominus A^T}(t, s), \end{aligned}$$

then from (4.5) we obtain (4.4). $\square$

5 Boundedness and stability of linear impulsive dynamic system on time scales

Definition 5.1. The dynamic system (3.2) is said to be *exponentially stable* (**e.s.**) if there exists a positive constant λ with $-\lambda \in \mathcal{R}^+$ such that for every $\tau \in \mathbb{T}_+$, there exists $N = N(\tau) \geq 1$ such that the solution of (3.2) through $(\tau, x(\tau))$ satisfies

$$||x(t)|| \leq N||x(\tau)||e_{-\lambda}(t,\tau) \text{ for all } t \in \mathbb{T}_{(\tau)}.$$

The dynamic system (3.2) is said to be *uniformly exponentially stable* (**u.e.s.**) if it is **e.s.** and the constant N can be chosen independently of $\tau \in \mathbb{T}_+$.

Theorem 5.1. *Suppose that $B_k \in M_n(\mathbb{R})$, $k = 1,2,...$, $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, and there exists a positive constant θ such that $t_{k+1} - t_k < \theta$, $k = 1,2,....$ If the solution of initial value problem*

$$\begin{cases} x^{\Delta} = A(t)x, \ \ t \in \mathbb{T}_{(\tau)}, \, t \neq t_k \\ x(t_k^+) = x(t_k) + B_k x(t_k) + c_k, \, k = 1,2,... \\ x(\tau^+) = 0, \ \ \tau \geq 0, \end{cases} \tag{5.1}$$

is bounded for any $c := \{c_k\}_{k=1}^{\infty} \in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$, then there exists a positive constants $N = N(\tau) \geq 1$, λ with $-\lambda \in \mathcal{R}^+$ such that

$$||S_A(t,\tau|| \leq Ne_{-\lambda}(t,\tau) \text{ for all } t \in \mathbb{T}_{(\tau)}.$$

Proof. From Theorem 4.1, the solution of (5.1) is given by

$$x(t) = \sum_{\tau < t_j < t} S_A(t, t_j^+)c_j, \, t \in \mathbb{T}_{(\tau)}. \tag{5.2}$$

For each fixed $t \in \mathbb{T}_{(\tau)}$, by the Corollary 3.2, the operator $U_t : \boldsymbol{l}^{\infty}(\mathbb{R}^n) \to \mathbb{R}^n$, given by $U_t(c) := x(t)$, is a bounded linear operator. In fact, $||U_t(c)|| \leq \sum_{\tau<t_j<t} ||S_A(t,t_j^+)|| \cdot ||c||_{\boldsymbol{l}^\infty} < \infty$ for any $c \in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$. Since the solution $x(t)$ of (5.1) is bounded for any $c \in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$, then uniform boundedness principle implies that there exists a constant $K > 0$ such that

$$||x(t)|| \leq K||c||_{l^\infty} \text{ for all } c \in \boldsymbol{l}^{\infty}(\mathbb{R}^n) \text{ and } \tau \in \mathbb{T}_+,$$

that is,

$$|| \sum_{\tau<t_j<t} S_A(t,t_j^+)c_j|| \leq K||c||_{\boldsymbol{l}^\infty} \text{ for all } c \in \boldsymbol{l}^{\infty}(\mathbb{R}^n) \text{ and } \tau \in \mathbb{T}_+. \tag{5.3}$$

Let $\tau \in \mathbb{T}_+$ be fixed. Then there exists $i \in \{1,2,...\}$ such that $\tau \in [t_{i-1}, t_i)$. We define the sequences $\{\beta_j\}_{j=1}^{\infty}$ and $\{c_j\}_{j=1}^{\infty} \in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$ given by

$$\beta_j := \begin{cases} 0 & \text{if } j < i \\ ||S_A(t_j^+,\tau)||, & \text{if } j \geq i \end{cases} \quad \text{and} \quad c_j := \begin{cases} 0 & \text{if } j < i \\ \frac{1}{\beta_j} S_A(t_j^+,\tau)y & \text{if } j \geq i \end{cases}$$

respectively. Here y is an arbitrary fixed element of $\mathbb{R}^n\backslash\{0\}$. Also, we observe that $||c||_{l^\infty} = 1$. Hence, from (5.3) we obtain

$$||S_A(t,\tau)y|| \sum_{\tau<t_j<t} \frac{1}{\beta_j} \leq K||y|| \text{ for all } t \in \mathbb{T}_{(\tau)}.$$

Since y is an arbitrary element of $\mathbb{R}^n\backslash\{0\}$ then it follows that

$$||S_A(t,\tau)|| \sum_{\tau<t_j<t} \frac{1}{\beta_j} \leq K \text{ for all } t \in \mathbb{T}_{(\tau)}. \tag{5.4}$$

Now, if $t_k > t_i$ and $t \in [\tau, t_{k+1})$ then, from (5.4), we obtain that

$$||S_A(t_k^+,\tau)|| \sum_{\tau<t_j\leq t_k} \frac{1}{\beta_j} \leq K \quad \text{for all } t \in [\tau, t_{k+1}),$$

that is,

$$\beta_k \sum_{\tau<t_j\leq t_k} \frac{1}{\beta_j} \leq K \quad \text{for all } t \in [\tau, t_{k+1}). \tag{5.5}$$

If $t \in [\tau, t_{i+1})$ then, from (5.4), we obtain

$$||S_A(t,\tau)|| \leq K\beta_i \text{ for all } t \in (t_i, t_{i+1}).$$

Without loss of generality we can assume that $K > 1$.

Further, if $t \in [\tau, t_{i+2})$ then, from (5.5), we obtain $\beta_{i+1}(\frac{1}{\beta_i} + \frac{1}{\beta_{i+1}}) \leq K$, and so $\beta_{i+1} \leq (K-1)\beta_i$.

If $t \in [\tau, t_{i+3})$ then, from (5.5), we obtain $\beta_{i+2}(\frac{1}{\beta_i} + \frac{1}{\beta_{i+1}} + \frac{1}{\beta_{i+2}}) \leq K$. Then

$$1 + \frac{\beta_{i+2}}{(K-1)\beta_i} + \frac{\beta_{i+2}}{\beta_i} \leq 1 + \frac{\beta_{i+2}}{\beta_{i+1}} + \frac{\beta_{i+2}}{\beta_i} \leq K,$$

that is, $\frac{\beta_{i+2}}{\beta_i}(\frac{1}{K-1} + 1) \leq K - 1$. It follows that $\beta_{i+2} \leq \frac{(K-1)^2}{K}\beta_i$.

Further, we prove by induction that if $t \in (\tau, t_{i+j})$ then

$$\beta_{i+j} \leq \frac{(K-1)^j}{K^{j-1}}\beta_i \text{ for all } j \geq 1. \tag{5.6}$$

Suppose that (5.6) is true for all $j \leq l-1$. If $t \in [\tau, t_{i+l})$ then, from (5.5) we have that

$$\beta_{i+l}(\frac{1}{\beta_i} + \frac{1}{\beta_{i+1}} + \ldots + \frac{1}{\beta_{i+l}}) \leq K.$$

It follows that

$$\begin{aligned} & 1 + \beta_{i+l}[\frac{1}{\beta_i} + \frac{1}{(K-1)\beta_i} + \frac{K}{(K-1)^2\beta_i} + \ldots + \frac{K^{l-2}}{(K-1)^{l-1}\beta_i}] \\ \leq \quad & 1 + \beta_{i+l}(\frac{1}{\beta_i} + \frac{1}{\beta_{i+1}} + \ldots + \frac{1}{\beta_{i+l-1}}) \leq K, \end{aligned}$$

hence

$$\frac{\beta_{i+l}}{\beta_i}[1+\frac{1}{K-1}+\frac{K}{(K-1)^2}+...+\frac{K^{l-2}}{(K-1)^{l-1}}]\leq K-1$$

and so, $\beta_{i+l}\leq\frac{(K-1)^l}{K^{l-1}}\beta_i$. Now, let $j\geq 1$ be fixed and let $\{c_l\}_{l=1}^{\infty}\in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$ be the sequence defined by

$$c_l:=\begin{cases}0 & \text{if } l\neq i+j\\ S_A(t_l^+,\tau)y & \text{if } l=i+j\end{cases}$$

where y is an arbitrary fixed element of $\mathbb{R}^n\backslash\{0\}$. Then from (5.3) and using (5.6) we obtain that $||S_A(t,\tau)||\leq K||S_A(t_{i+j}^+,\tau)||=K\beta_{i+j}\leq\frac{(K-1)^j}{K^{j-2}}\beta_i$, hence

$$||S_A(t,\tau)||\leq\frac{(K-1)^j}{K^{j-2}}\beta_i \text{ for } t\in[\tau,t_{i+j+1}).$$

Since $t_{k+1}-t_k<\theta$, $k=1,2,...$, it follows that, for $t\in[t_{i+j},t_{i+j+1})$, we have $t-\tau\leq t_{i+j+1}-t_i<(j+2)\theta$, that is, $j>\frac{1}{\theta}(t-\tau)-2$. Therefore, we have that

$$||S_A(t,\tau)||\leq\frac{K^4}{(K-1)^2}\beta_i(1-\frac{1}{K})^{\frac{1}{\theta}(t-\tau)} \text{ for } t\in[t_{i+j},t_{i+j+1}).$$

Further, we define the positive function $\lambda(t)$, with $-\lambda(t)\in\mathcal{R}^+$, as the solution of the inequality $e_{-\lambda}(t,\tau)\geq(1-\frac{1}{K})^{\frac{1}{\theta}(t-\tau)}$, for $t\in[t_{i+j},t_{i+j+1})$. Let

$$N=\max\{\frac{K^4}{(K-1)^2}\beta_i,\sup_{\tau\leq t<t_i}\frac{||S_A(t,\tau)||}{e_{-\lambda}(t,\tau)}\}.$$

Then for all $t,\tau\in\mathbb{T}_+$ with $t\in\mathbb{T}_{(\tau)}$, we obtain that

$$||S_A(t,\tau)||\leq Ne_{-\lambda}(t,\tau),$$

and so the theorem is proved. □

Remark 5.1. For example, when $\mathbb{T}=\mathbb{R}$, the solution of the inequality $e_{-\lambda}(t,\tau)=e^{-\lambda(t-\tau)}\geq(1-\frac{1}{K})^{\frac{1}{\theta}(t-\tau)}$, for $\tau\in[t_i,t_{i+1})$, $t\in[t_{i+j},t_{i+j+1}]$ and $j\geq 0$, is $0\leq\lambda\leq-\frac{1}{\theta}\ln(1-\frac{1}{K})$, $K>1$.

When

$$\mathbb{T}=\mathbb{P}_{1,1}=\bigcup_{k=0}^{\infty}[2k,2k+1],$$

then $e_{-\lambda}(t,\tau)=(1-\lambda)^je^{-\lambda(t-\tau)}e^{\lambda j}$ for $\tau\in[2i,2i+1)$ and $t\in[2(i+j),2(i+j)+1]$ with $j\geq 0$. In this case, $\mu(t)=0$ if $t\in\bigcup_{k=0}^{\infty}[2k,2k+1)$ and $\mu(t)=1$ if $t\in\bigcup_{k=0}^{\infty}\{2k+1\}$. It follows that $-\lambda\in\mathcal{R}^+$ if and only if $\lambda\in[0,1)$. Next, we consider the function

$$f(\lambda)=(1-\lambda)^je^{-\lambda(t-\tau)}e^{\lambda j}-(1-\frac{1}{K})^{\frac{1}{\theta}(t-\tau)},\ \lambda\in[0,1),\ K>1.$$

Since $f'(\lambda) = -[j + (1-\lambda)(t-\tau-j)](1-\lambda)^{j-1}e^{-\lambda(t-\tau)}e^{\lambda j}$ it follows that $f(\lambda)$ is decreasing on $[0,1)$. Then, $f(0) = 1 - (1-\frac{1}{K})^{\frac{1}{\theta}(t-\tau)} > 0$ and $\lim_{\lambda \nearrow 1} f(\lambda) = -(1-\frac{1}{K})^{\frac{1}{\theta}(t-\tau)} < 0$, implies that there exists a unique $\lambda_0 \in (0,1)$ such that $f(\lambda_0) = 0$. Therefore, the solution of the inequality

$$e_{-\lambda}(t,\tau) = (1-\lambda)^j e^{-\lambda(t-\tau)}e^{\lambda j} \geq (1-\frac{1}{K})^{\frac{1}{\theta}(t-\tau)}$$

for $t \in [2(i+j), 2(i+j)+1]$ with $j \geq 0$ is $0 \leq \lambda \leq \lambda_0$.

Corollary 5.1. *Suppose that* $B_k \in M_n(\mathbb{R})$*,* $k = 1,2,...$*,* $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$*, and there exists a positive constant* θ *such that* $t_{k+1} - t_k < \theta$*,* $k = 1,2,....$ *If the solution of initial value problem (5.1) is bounded for any* $c := \{c_k\}_{k=1}^{\infty} \in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$*, then the dynamic system (3.2) is* **e.s.**

Proof. From Theorem 5.1, exists a positive constants $N = N(\tau) \geq 1$, λ with $-\lambda \in R^+$ such that $||S_A(t,\tau)|| \leq Ne_{-\lambda}(t,\tau)$ for all $t \in T_{(\tau)}$. For any $\tau \in \mathbb{T}_+$, the solution of (3.2) satisfies

$$||x(t)|| = ||S_A(t,\tau)x(\tau)|| \leq ||S_A(t,\tau)||\cdot||x(\tau)|| \leq N||x(\tau)||e_{-\lambda}(t,\tau) \text{ for all } t \in T_{(\tau)}$$

and thus exponential stability is proved. □

Lemma 5.1. *If there exist a constant* $\theta > 0$ *such that* $t_{k+1} - t_k < \theta$, for $k = 1,2,...$*, then for each constant* $\lambda > 0$ *with* $\frac{1}{\lambda} > \theta$ *we have that* $-\lambda \in \mathcal{R}^+$*.*

Proof. Since t_k and t_{k+1} are right-dense points then, for $t \in [t_k, t_{k+1}]$, we have that $t_k \leq t \leq \sigma(t) \leq t_{k+1}$. It follows that $\mu(t) = \sigma(t) - t \leq t_{k+1} - t_k < \theta$ for $t \in [t_k, t_{k+1}]$. Therefore, $\mu(t) < \theta$ for $t \in \mathbb{T}_+$. If $\frac{1}{\lambda} > \theta$ then we have that $1 - \lambda\mu(t) > 1 - \frac{1}{\theta}\mu(t) > 0$ and thus $-\lambda \in \mathcal{R}^+$. □

Theorem 5.2. *Suppose that* $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$*,* $B_k \in M_n(\mathbb{R})$*,* $k = 1,2,...$*, and there exist positive constants* θ*,* b*,* and M *such that*

$$t_{k+1} - t_k < \theta, \quad \sup_{k\geq 1}||B_k|| \leq b, \quad \int_{t_k}^{t_{k+1}} ||A(s)||\Delta s \leq M, \ k = 1,2,... \tag{5.7}$$

If the solution of initial value problem (5.1) is bounded for any $c := \{c_k\}_{k=1}^{\infty} \in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$*, then there exists a positive constants* N*,* λ *with* $-\lambda \in \mathcal{R}^+$ *such that*

$$||S_A(t,\tau)|| \leq Ne_{-\lambda}(t,\tau) \textit{ for all } t \in \mathbb{T}_{(\tau)}.$$

Proof. Let $k \in \{1,2,...\}$ be fixed. For an arbitrary fixed $y \in \mathbb{R}^n \backslash \{0\}$, we define the sequence $\{c_j\}_{j=1}^{\infty} \in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$ given by $c_k = y$ and $c_j = 0$ if $j \neq k$. Then $x(t) = S_A(t,t_k^+)y$, $t > t_k$, is the solution of (5.1) with initial condition $x(t_k^+) = 0$.

From Theorem 5.1 we obtain that

$$||S_A(t,t_k^+)|| \leq N_k e_{-\lambda}(t,t_k^+) \textit{ for all } t > t_k,$$

where

$$N_k = \max\{\frac{K^4}{(K-1)^2}||S_A(t_{k+1}^+, t_k^+)||, \sup_{t_k \le t < t_{k+1}} \frac{||S_A(t, t_k^+)||}{e_{-\lambda}(t, t_k^+)}\}.$$

Now we have to show that N_k can be chosen independently of k.

From Corollary 3.2 and (5.7), we obtain that $||S_A(t, t_k^+)|| \le e^M$, $t \in [t_k, t_{k+1}]$. It follows that $||S_A(t_{k+1}^+, t_k^+)|| \le ||I+B_k||\cdot||S_A(t_{k+1}, t_k^+)|| \le (b+1)e^M$. On the other hand, since $t - t_k^+ \le t_{k+1} - t_k < \theta$, $t \in [t_k, t_{k+1}]$ then, by Lemma 5.1, we can choose a constant λ with $\frac{1}{\lambda} > \theta$ such that that $-\lambda \in \mathcal{R}^+$. Using the Bernoulli's inequality, we have that

$$e_{-\lambda}(t, t_k^+) \ge 1 - \lambda(t - t_k^+) \ge 1 - \lambda\theta > 0,$$

and thus $\frac{1}{e_{-\lambda}(t,t_k^+)} \le \frac{1}{1-\lambda\theta}$. Hence $||S_A(t, t_k^+)|| \le \widetilde{N} e_{-\lambda}(t, \tau)$, where $\widetilde{N} = \max\{\frac{(b+1)K^4}{(K-1)^2}e^M, \frac{b+1}{1-\lambda\theta}e^M\}$.

Next, let $\tau \in \mathbb{T}_+$ be arbitrary. Then there exists $k \in \{1, 2, ...\}$ such that $\tau \in [t_k, t_{k+1})$. Then we have

$$\begin{aligned} ||S_A(t, \tau)|| &= ||S_A(t, t_k^+)S_A(t_k^+, \tau)|| \le ||S_A(t, t_k^+)|| \cdot ||S_A(t_k^+, \tau)|| \le \\ &\le \widetilde{N} e_{-\lambda}(t, t_k^+)\widetilde{N} e_{-\lambda} S_A(t_k^+, \tau) = \widetilde{N}^2 e_{-\lambda}(t, \tau). \end{aligned}$$

Therefore $||S_A(t, \tau)|| \le N e_{-\lambda}(t, \tau)$, with $N = \widetilde{N}^2$, and the theorem is proved. □

Corollary 5.2. *Suppose that* $B_k \in M_n(\mathbb{R})$, $k = 1, 2, ...$, $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, *and there exist positive constants* θ, b, and M *such that*

$$t_{k+1} - t_k < \theta, \quad \sup_{k\ge1} ||B_k|| \le b, \quad \int_{t_k}^{t_{k+1}} ||A(s)||\Delta s \le M, \ k = 1, 2, ...$$

If the solution of initial value problem (5.1) is bounded for any $c := \{c_k\}_{k=1}^{\infty} \in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$, *then the dynamic system (3.2) is* **u.e.s.** □

Lemma 5.2. *For any constant* $\lambda > 0$ *with* $-\lambda \in \mathcal{R}^+$, *we have that*

$$e_{-\lambda}(t, \tau) \le e^{-\lambda(t-\tau)}, \text{ for all } \tau, t \in \mathbb{T} \text{ with } t \in \mathbb{T}_{(\tau)}. \tag{5.8}$$

Proof. Indeed, $-\lambda \in \mathcal{R}^+$ implies that $1 - \lambda\mu(t) > 0$ for all$t \in \mathbb{T}_+$. Since

$$\lim_{u \searrow \mu(s)} \frac{\ln(1 - \lambda u)}{u} = \begin{cases} -\lambda & \text{if } \mu(s) = 0 \\ \frac{\ln(1-\lambda\mu(s))}{\mu(s)} \le -\lambda & \text{if } \mu(s) \in (0, \frac{1}{\lambda}) \end{cases}$$

then, using the explicit estimation of the modulus of the exponential function on time scales (see [24]), we have that

$$\begin{aligned} e_{-\lambda}(t, \tau) &= \exp(\int_{\tau}^{t} \lim_{u \searrow \mu(s)} \frac{\ln(1 - \lambda u)}{u} \Delta s) \\ &\le \exp(-\int_{\tau}^{t} \lambda \Delta s) \le e^{-\lambda(t-\tau)}, \end{aligned}$$

and the inequality (5.8) is proved. $\square$

Theorem 5.3. *Suppose that $B_k \in M_n(\mathbb{R})$, $k = 1, 2, ...$, $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, and there exist positive constants γ, θ, b, and M such that*

$$\gamma < t_{k+1} - t_k < \theta, \quad \sup_{k\geq 1} ||B_k|| \leq b, \quad \int_{t_k}^{t_{k+1}} ||A(s)||\Delta s \leq M, \ k = 1, 2, ...$$

If the solution of initial value problem (5.1) is bounded for any $c := \{c_k\}_{k=1}^{\infty} \in \boldsymbol{l}^{\infty}(\mathbb{R}^n)$, then the solution of (4.1) is bounded for each $h \in BC_{rd}\mathcal{R}(\mathbb{T}_+, \mathbb{R}^n)$.

Proof. Let $i \in \{1, 2, ...\}$ be such that $\tau \in [t_i, t_{i+1})$ By Theorem 5.2 there exist positive constants N and λ with $-\lambda \in \mathcal{R}^+$ such that

$$||S_A(t, \tau)|| \leq Ne_{-\lambda}(t, \tau) \text{ for all } t \in \mathbb{T}_{(\tau)}.$$

For every function $h \in C_{rd}\mathcal{R}(\mathbb{T}_+, \mathbb{R}^n)$, the corresponding solution x_h of (4.1) is given by (4.2). From (4.2) we obtain that

$$\begin{aligned} ||x_h(t)|| &\leq ||S_A(t, \tau)|| \cdot ||\eta|| + \int_{\tau}^{t} ||S_A(t, \sigma(s))|| \cdot ||h(s)||\Delta s \\ &+ \sum_{\tau < t_j < t} ||S_A(t, t_j^+)|| \cdot ||c_j|| \leq N||\eta||e_{-\lambda}(t, \tau) \\ &+ \int_{\tau}^{t} N||h(s)||e_{-\lambda}(t, \sigma(s))\Delta s + \sum_{\tau < t_j < t} e_{-\lambda}(t, t_j^+)||c||_{\boldsymbol{l}^{\infty}}. \end{aligned}$$

We have that

$$\begin{aligned} & \int_{\tau}^{t} N||h(s)||e_{-\lambda}(t, \sigma(s))\Delta s \\ \leq\ & N||h|| \int_{\tau}^{t} e_{-\lambda}(t, \sigma(s))\Delta s = -\frac{N||h||}{\lambda}(e_{-\lambda}(t, \tau) - e_{-\lambda}(t, t)) \\ =\ & \frac{N||h||}{\lambda}(1 - e_{-\lambda}(t, \tau)) \leq \frac{N||h||}{\lambda}. \end{aligned}$$

Further, let t_k be the greatest of all $t_j < t$. Then

$$t - t_{k-1} \geq t_k - t_{k-1} > \gamma, \, t - t_{k-2} > 2\gamma, \, t - t_i > (k - i)\gamma$$

and, by Lemma 5.2, we have that

$$\sum_{\tau < t_j < t} e_{-\lambda}(t, t_j^+) \leq \sum_{j=i}^{\infty} e^{-\lambda\gamma j} = \frac{1}{1 - e^{-\lambda\gamma}}.$$

Therefore,

$$||x_h(t)|| \leq N||\eta||e_{-\lambda}(t, \tau) + \frac{N||h||}{\lambda} + \frac{N||c||_{\boldsymbol{l}^{\infty}}}{1 - e^{-\lambda\gamma}}.$$

Since $\lim_{t\to\infty} e_{-\lambda}(t,\tau) = 0$ it follows that x_h is bounded. $\square$

Example 5.1. We consider the linear impulsive dynamic system

$$\begin{cases} x^{\Delta} = A(t)x, \ \text{if } t \in \mathbb{T}_{(\tau)},\ t \neq t_{2k} \\ x(t_{2k}^{+}) = (I + B_{2k})x(t_{2k}),\ k = 1, 2, ... \\ x(\tau) = \eta, \end{cases} \tag{5.9}$$

where $A(t) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix}$, $I + B_{2k} = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix}$, $\beta > \alpha > 0$ and $t_k = k$. Also, we choose $(a_{2k})_{k\geq 1}$ and $(b_{2k})_{k\geq 1}$ such that $a_{ij} := a_{2(i+1)}a_{2(i+2)}...a_{2(i+j)} \leq a$ and $b_{ij} := b_{2(i+1)}b_{2(i+2)}...b_{2(i+j)} \leq b$ for each fixed $i \geq 0$ and for $j = 1, 2, ...$ If $\mathbb{T} = \mathbb{R}$, then then impulsive transition matrix associated with $\{B_{2k}, t_{2k}\}_{k=1}^{\infty}$ is given by

$$S_A(t,\tau) = \begin{pmatrix} a_{ij}e^{-\alpha(t-\tau)} & 0 \\ 0 & b_{ij}e^{-\beta(t-\tau)} \end{pmatrix}$$

if $\tau \in [t_{2i}, t_{2i+2})$ and $t \in [t_{2(i+j)}, t_{2(i+j+1)}]$, where $i \geq 0$ is fixed and $j = 1, 2, ...$

It follows that

$$\|S_A(t,\tau)\| \leq Ne^{-\alpha(t-\tau)} \text{ for } t \geq \tau,$$

where $N = \max\{a, b\}$.

If $\mathbb{T} = \mathbb{P}_{1,1}$, then then impulsive transition matrix associated with $\{B_{2k}, t_{2k}\}_{k=1}^{\infty}$ is given by

$$\begin{aligned} S_A(t,\tau) &= \begin{pmatrix} a_{ij}(1-\alpha)^j e^{-\alpha(t-\tau)}e^{\alpha j} & 0 \\ 0 & b_{ij}(1-\beta)^j e^{-\beta(t-\tau)}e^{\beta j} \end{pmatrix} \\ &= \begin{pmatrix} a_{ij}e_{-\alpha}(t,\tau) & 0 \\ 0 & b_{ij}e_{-\beta}(t,\tau) \end{pmatrix} \end{aligned}$$

if $\tau \in [t_{2i}, t_{2i+1})$ and $t \in [t_{2(i+j)}, t_{2(i+j)+1}]$, where $i \geq 0$ is fixed and $j = 1, 2, ...$

It follows that

$$\|S_A(t,\tau)\| \leq Ne_{-\alpha}(t,\tau) \text{ for } t \geq \tau,$$

where $N = \max\{a, b\}$.

Next, the solution of the initial value problem

$$\begin{cases} x^{\Delta} = A(t)x, \ \ t \in \mathbb{T}_{(\tau)},\ t \neq t_{2k} \\ x(t_{2k}^{+}) = x(t_{2k}) + B_{2k}x(t_{2k}) + c_{2k},\ k = 1, 2, ... \\ x(\tau^{+}) = 0, \ \ \tau \geq 0, \end{cases} \tag{5.10}$$

is given by

$$x(t) = \sum_{l=1}^{j} S_A(t, t_{2(i+l)}^{+})c_{2(i+l)}$$

when $\tau \in [t_{2i}, t_{2i+2})$ and $t \in [t_{2(i+j)}, t_{2(i+j+1)})$ if $\mathbb{T} = \mathbb{R}$ or when $\tau \in [t_{2i}, t_{2i+1})$ and $t \in [t_{2(i+j)}, t_{2(i+j)+1}]$ if $\mathbb{T} = \mathbb{P}_{1,1}$, where $i \geq 0$ is fixed and $j = 1, 2, ...$ Then

$$\begin{aligned} \|x(t)\| &\leq \sum_{l=1}^{\infty} \left\| S_A(t, t^+_{2(i+l)}) \right\| \cdot ||c||_{l^\infty} \\ &\leq \sum_{l=1}^{\infty} \max\{a_{il}, b_{il}\} e^{-2\alpha(l-1)} ||c||_{l^\infty}, \end{aligned}$$

where $i \geq 0$ is fixed.

Therefore, if for each fixed $i \geq 0$ we have that $\sum_{l=1}^{\infty} \max\{a_{il}, b_{il}\} e^{-2\alpha(l-1)} < \infty$, then the solution of the initial value problem (5.10) is bounded for any $c := \{c_k\}_{k=1}^{\infty} \in \boldsymbol{l}^\infty(\mathbb{R}^n)$. Moreover, that $\prod_{l=1}^{\infty} a_{il} < \infty$, $\prod_{l=1}^{\infty} b_{il} < \infty$ for each fixed $i \geq 0$, then

$$\|S_A(t, \tau)\| \leq N e_{-\alpha}(t, \tau) \text{ for } t \geq \tau,$$

where $N = \max\{a, b\}$ and

$$e_{-\alpha}(t, \tau) = \begin{cases} e^{-\alpha(t-\tau)} & \text{if } t \in \mathbb{R} \\ (1-\alpha)^l e^{-\alpha(t-\tau)} e^{\alpha l} & \text{if } t \in \mathbb{P}_{1,1} = \bigcup_{l=0}^{\infty} [2l, 2l+1]. \end{cases}$$

Consequently, the impulsive dynamic system (5.9) is uniformly exponentially stable.

6 Appendix on time scales analysis

We recall some basic definitions and results in the calculus on time scales analysis. We refer to [15, 16], and also to the paper [1, 2, 3, 8], for more information on analysis on time scales. A time scales $\mathbb{T}$ is a nonempty closed subset of R, and the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T}; s > t\}$ (supplemented by $\inf \emptyset = \sup \mathbb{T}$), the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T}; s < t\}$ (supplemented by $\sup \emptyset = \inf \mathbb{T}$), while the *graininess* $\mu : \mathbb{T} \to \mathrm{R}_+$ is given by $\mu(t) := \sigma(t) - t$. For our purpose, we will assume that the time scales $\mathbb{T}$ is unbounded above, i.e., $\sup \mathbb{T} = \infty$. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. A time scales $\mathbb{T}$ is said to be discrete if t is left-scattered and right-scattered for each $t \in \mathbb{T}$. The notations $[a, b]$, $[a, b)$, and so on, will denote time scales intervals such as $[a, b] := \{t \in \mathbb{T}; a \leq t \leq b\}$, where $a, b \in \mathbb{T}$. Let R^n be the space of n-dimensional column vectors $x = \mathrm{col}(x_1, x_2, ...x_n)$ with a norm $|| \cdot ||$. Also, by the same symbol $|| \cdot ||$ we will denote the corresponding matrix norm in the space $M_n(\mathrm{R})$ of $n \times n$ matrices. We recall that $||A|| := \sup\{||Ax||; ||x|| \leq 1\}$ and the following inequality $||Ax|| \leq ||A|| \cdot ||x||$ holds for all $A \in M_n(\mathrm{R})$ and $x \in \mathrm{R}^n$.

Definition A.1. A function $f : \mathbb{T} \to \mathrm{R}^n$ is said to be *rd-contionous* if

(i) f is continuous at every right-dense point $t \in \mathbb{T}$,

(ii) $f(t^-) := \lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

The set of all rd-continuous functions $f : \mathbb{T} \to \mathrm{R}^n$ will be denoted by $C_{rd}(\mathbb{T}, \mathrm{R}^n)$.

A function $p : \mathbb{T} \to \mathrm{R}$ is said to be *regressive* (respectively *positively regressive*) if $1 + \mu(t)p(t) \neq 0$ (respectively $1 + \mu(t)p(t) > 0$) for all $t \in \mathbb{T}$. The set $\mathcal{R}$ (respectively $\mathcal{R}^+$) of all regressive (respectively positively regressive) functions from $\mathbb{T}$ to R is an Abelian group with respect to the circle addition operation $\oplus$, given by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t). \tag{6.1}$$

The inverse element of $p \in \mathcal{R}$ is given by

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)} \tag{6.2}$$

and so, the circle subtraction operation $\ominus$ is defined by

$$(p \ominus q)(t) = (p \oplus (\ominus q))(t) = -\frac{p(t) - q(t)}{1 + \mu(t)q(t)}. \tag{6.3}$$

The space of all rd-continuous and regressive functions from $\mathbb{T}$ to R is denoted by $C_{rd}\mathcal{R}(\mathbb{T}, \mathrm{R})$. Also,

$$C^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R}) := \{p \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R}); 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

The set of rd-continuous (respectively rd-continuous and regressive) functions $A : \mathbb{T} \to M_n(\mathrm{R})$ is denoted by $C_{rd}(\mathbb{T}, M_n(\mathrm{R}))$ (respectively by $C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathrm{R}))$). We recall that a matrix-valued function A is said to be regressive if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$, where I is the $n \times n$ identity matrix. Moreover, the set $\mathcal{R}(\mathbb{T}, M_n(\mathrm{R}))$ of all regressive matrix-valued functions is a group with respect to the addition operation $\oplus$ define

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t) \tag{6.4}$$

for all $t \in \mathbb{T}$. The inverse element of $A \in \mathcal{R}(\mathbb{T}, M_n(\mathrm{R}))$ is given by

$$(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) = -A(t)[I + \mu(t)A(t)]^{-1} \tag{6.5}$$

for all $t \in \mathbb{T}$.

Definition A.2. A function $f : \mathbb{T} \to \mathrm{R}^n$ is said to be *differentiable* at $t \in \mathbb{T}$, with *delta-derivative* $f^\Delta(t) \in \mathrm{R}^n$ if given $\varepsilon > 0$ there exists a neighborhood U of t such that, for all $s \in \mathbb{T}$,

$$||f^\sigma(t) - f(s) - f^\Delta(t)[\sigma(t) - s]|| \leq \varepsilon|(t) - s|,$$

where $f^\sigma(t) := f(\sigma(t))$ for all $t \in \mathbb{T}$.

We denote by $C^1_{rd}(\mathbb{T}, \mathrm{R}^n)$ the set of all functions $f : \mathbb{T} \to \mathrm{R}^n$ that are differentiable on $\mathbb{T}$ and its delta-derivative $f^\Delta(t) \in C_{rd}(\mathbb{T}, \mathrm{R}^n)$.

Theorem A.1. ([1, 15]) *Assume that $f : \mathbb{T} \to \mathrm{R}^n$ and let $t \in \mathbb{T}$.*

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t}.$$

(iii) *If f is differentiable at t and t is right-dense, then*

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f^{\sigma}(t) - f(s)}{t - s}.$$

(iv) *If f is differentiable at t, then $f^{\sigma}(t) = f(t) + \mu(t)f^{\Delta}(t)$.*

(v) *If $f, g : \mathbb{T} \to \mathrm{R}^n$ are both differentiable at t, then the product $f^T g$ is also differentiable at t and*

$$(f^T g)^{\Delta}(t) = f^T(t)g^{\Delta}(t) + (f^T)^{\Delta}(t)g^{\sigma}(t).$$

Theorem A.2. ([15]) If $A, B : \mathbb{T} \to M_n(\mathrm{R})$ *are differentiable, then*

(i) $A^{\sigma}(t) = A(t) + \mu(t)A^{\Delta}(t)$ *for all* $t \in \mathbb{T}$*;*

(ii) $(A^T)^{\Delta} = (A^{\Delta})^T$;

(iii) $(A + B)^{\Delta} = A^{\Delta} + B^{\Delta}$, *and* $(AB)^{\Delta} = A^{\Delta}B^{\sigma} + AB^{\Delta} = A^{\sigma}B^{\Delta} + A^{\Delta}B$*;*

(iv) $(A^{-1})^{\Delta} = -(A^{\sigma})^{-1}A^{\Delta}A^{-1} = -A^{-1}A^{\Delta}(A^{\sigma})^{-1}$*if* AA^{σ} *is invertible;*

(v) $(AB^{-1})^{\Delta} = (A^{\Delta} - AB^{-1}B^{\Delta})(B^{\sigma})^{-1} = [A^{\Delta} - (AB^{-1})^{\sigma}B^{\Delta}]B^{-1}$ *if* BB^{σ} *is invertible.*

Definition A.3. Let $f \in C_{rd}(\mathbb{T},\mathrm{R}^n)$. A function $g : \mathbb{T} \to \mathbb{R}^n$ is called the *antiderivative* of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. In this cases, we define

$$\int_a^t f(s)\Delta s = g(t) - g(a), \quad a, t \in \mathbb{T}.$$

Theorem 3. (Existence of Antiderivatives,) *Every function $f \in C_{rd}(T,R^n)$ has an antiderivative. In particular if $\tau \in T$, then the function $F : T \to R^n$ defined by*

$$F(t) := \int_{\tau}^{t} f(s)\Delta s \text{ for } t \in \mathbb{T}$$

is an antiderivative of f.

Theorem A.4. ([2, 15]) If $a, b, c \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$, and $f, g \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$*, then*

(i) $\int_a^b (\alpha f + \beta g)(t)\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$;

(ii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iii) $\int_a^{\sigma(a)} f(t)\Delta t = \mu(a)f(a)$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(v) $\int_a^b f^T(t)g^\Delta(t)\Delta t = (f^T g)(b) - (f^T g)(a) - \int_a^b (f^T)^\Delta(t)g(\sigma(t))\Delta t$;

(vi) $\left\| \int_a^b f(t)\Delta t \right\| \leq \int_a^b ||f(t)||\Delta t$;

Let $p \in C_{rd}\mathcal{R}(\mathbb{T},\mathrm{R})$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. Then the *exponential function* on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(t)}(p(\tau))\Delta\tau\right) \text{ with } \xi_h(z) := \begin{cases} \frac{\ln(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0, \end{cases}$$

and it is the unique solution of the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$.

Theorem A.5. ([15]) *If* $p, q \in C_{rd}\mathcal{R}(\mathbb{T},\mathrm{R})$ *then the following hold:*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$*;*

(ii) $e_p(\sigma(t),s) = [1 + \mu(t)p(t)]e_p(t,s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_p(s,t) = e_{\ominus p}(t,s)$*;*

(iv) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*

(v) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$ *and* $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$*;*

(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$*;*

(vii) *If* $p \in \mathcal{R}^+$ *then* $e_p(t,s) > 0$ *for all* $t, s \in \mathbb{T}$*;*

(viii) $\int_a^b p(s)e_p(c,\sigma(s))\Delta s = e_p(c,a) - e_p(c,b)$.

Lemma A.2. ([15, Theorem 6.1]) *Let* $\tau \in \mathbb{T}$*,* $y, b \in C_{rd}\mathcal{R}(\mathbb{T},\mathrm{R})$ *and* $p \in C_{rd}^+\mathcal{R}(\mathbb{T},\mathrm{R})$*. Then*

$$y^\Delta(t) \leq p(t)y(t) + b(t) \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq y(\tau)e_p(t,\tau) + \int_\tau^t e_p(t,\sigma(s))b(s)\Delta s \text{ for all } t \in \mathbb{T}.$$

Lemma A.3. (Bernoulli's inequality, [15, Theorem 6.2]) *Let* $\alpha \in R$ *with* $\alpha \in \mathcal{R}^+$. *Then*

$$e_\alpha(t,\tau) \geq 1 + \alpha(t-\tau), \text{ for all } t \in \mathbb{T}_{(\tau)}.$$

Lemma A.4. (Gronwall's inequality, [15, Theorem 6.4]) *Let* $\tau \in \mathbb{T}$, $y, b \in C_{rd}\mathcal{R}(\mathbb{T},\mathrm{R})$ *and* $p \in C^+_{rd}\mathcal{R}(\mathbb{T},\mathrm{R})$, $p \geq 0$. *Then*

$$y(t) \leq b(t) + \int_\tau^t y(s)p(s)\Delta s \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq b(t) + \int_\tau^t e_p(t,\sigma(s))b(s)\Delta s \text{ for all } t \in \mathbb{T}.$$

Lemma A.5. ([31]) *Let* $\tau \in \mathbb{T}_+$, $y \in C_{rd}\mathcal{R}(\mathbb{T}_+,\mathrm{R})$, $p \in C^+_{rd}\mathcal{R}(\mathbb{T}_+,\mathrm{R})$ *and* $c_k, d_k \in \mathrm{R}_+$, $k = 1, 2, ...$ *Then*

$$\begin{cases} y^\Delta(t) \leq p(t)y(t) + b(t)\ , \ \ t \in \mathbb{T}_{(\tau)},\, t \neq t_k \\ y(t_k^+) \leq c_k y(t_k) + d_k,\ k = 1, 2, ... \end{cases}$$

implies

$$\begin{aligned} y(t) \quad \leq \quad & y(\tau) \prod_{\tau<t_k<t} c_k e_p(t,\tau) + \sum_{\tau<t_k<t} \Big(\prod_{t_k<t_j<t} c_j e_p(t,t_k)\Big) d_k \\ & + \int_\tau^t \prod_{s<t_k<t} c_k e_p(t,\sigma(s))b(s)\Delta s. \end{aligned}$$

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