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FUNCTION BOUNDS FOR SOLUTIONS OF VOLTERRA INTEGRO DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. Introducing shift operators on time scales we construct the integro-dynamic equation corresponding to the convolution type Volterra differential and difference equations in particular cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Extending the scope of time scale variant of Gronwall's inequality we determine function bounds for the solutions of the integro dynamic equation.

1. INTRODUCTION

In this paper, we are concerned with the investigation of function bounds for the solutions of integro dynamic equation of type

$$x^{\Delta}(t) = -a(t)x(t) + \int_{t_0}^{t} b(\delta_-(s,t))x(s)\Delta s, \quad t \in [t_0, \infty) \cap \mathbb{T}, \tag{1.1}$$

which includes the following Volterra equations in particular cases:

- *Volterra integro differential equation of convolution type*: For $\mathbb{T} = \mathbb{R}$ with $\delta_-(s,t) = t - s$ and $t_0 = 0$

$$x'(t) = -a(t)x(t) + \int_{0}^{t} b(t-s)x(s)ds, \quad t \in [0, \infty). \tag{1.2}$$

- *Volterra integral equation with nonconvolutional kernel:* For $\mathbb{T} = \mathbb{R}$ with $\delta_-(s,t) = t/s$ and $t_0 = 1$

$$x'(t) = -a(t)x(t) + \int_{1}^{t} b(\frac{t}{s})x(s)ds, \quad t \in [1, \infty). \tag{1.3}$$

- *Volterra integro difference equation of convolution type*: For $\mathbb{T} = \mathbb{Z}$ with $\delta_-(s,t) = t - s + \lambda$ and $t_0 = \lambda$

$$\Delta x(t) = -a(t)x(t) + \sum_{k=\lambda}^{t-1} b(t-k+\lambda)x(k), \quad t \in [\lambda, \infty) \cap \mathbb{Z}_+, \tag{1.4}$$

  where Δ is the forward difference operator.
- *Volterra integro $q-$difference equation*: For $\mathbb{T} = q^{\mathbb{Z}}$ with $\delta_-(s,t) = t/s$ and $t_0 = 1$

$$\Delta_q x(t) = -a(t)x(t) + \sum_{s\in[1,t)\cap q^{\mathbb{Z}}} \mu(s)b(\frac{t}{s})x(s), \quad t \in [1, \infty) \cap q^{\mathbb{Z}}, \tag{1.5}$$

  where Δ_q is the q-difference operator given by $\Delta_q x(t) = \frac{x(qt)-x(t)}{(q-1)t}$.

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Many papers have appeared in the literature on Volterra equations on particular time scales such as $\mathbb{R}$, $\mathbb{Z}$ and $q^{\mathbb{N}}$. An early contribution to integro q-difference equations was made by Trjitzinsky [19]. In [15] and [16] Elaydi dealt with stability analysis of convolution type Volterra integro difference equations of the form (1.4). In [4], Becker derived an extension of Gronwall's inequality to find function bounds for the solutions of Eq. (1.2), where $a, b : [0, \infty) \to \mathbb{R}$ are continuous functions and b is nonnegative. Since the time scale theory provides a wide perspective for the unification of discrete and continuous analyses, Volterra integro dynamic equations on general time scales became topic of several research papers. For instance, boundedness of the solutions of nonlinear Volterra integro-dynamic equations on time scales has been investigated in [3] by means of nonnegative definite Lyapunov functionals on time scales. Furthermore, in [2], existence of periodic solutions of nonlinear system of Volterra type integro-dynamic equations has been shown using the topological degree method and Schaefer's fixed point theorem. However, to the best of our knowledge, function bounds for the solutions of Volterra integral equations of the form (1.3) has not been treated elsewhere before.

Motivated by the results of [4], we bring the integro dynamic equation (1.1) under investigation to obtain more general results which are not known even for the above mentioned particular cases. Some applications are also given to illustrate the usefulness of our results.

The remaining part of this paper is organized as follows: In the second section, we propose an extension of Gronwall's inequality ([7, Corollary 6.7, p.257]) on time scales. In the third section, we introduce the shift operators $\delta_{\pm}$ to construct the kernel of integro dynamic equation (1.1). In the last section, we give several theorems and corollaries regarding the function bounds for the solutions of (1.1). Hence, it turns out that the results in Sections III and Section IV are valid only for the time scales containing an initial point t_0 so that there exist shift operators $\delta_{\pm}(s, t)$ on $[t_0, \infty)_{\mathbb{T}}$.

For the sake of brevity, we assume familiarity with time scale calculus. For a comprehensive review on fundamental aspects of the theory we refer the reader to [7] and [8].

Throughtout the paper, we denote by σ and ρ the forward and backward jump operators, $\sigma : \mathbb{T} \to \mathbb{T}$ and $\rho : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$, respectively. A point $t \in \mathbb{T}$ is said to be right dense (right scattered) if $\sigma(t) = t$ ($\sigma(t) > t$). We say $t \in \mathbb{T}$ is left dense (left scattered) if $\rho(t) = t$ ($\rho(t) < t$). The graininness (step size) function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right dense points and its left sided limits exists (finite) at left dense points. We use the notation C_{rd} to indicate the set of rd-continuous function on $\mathbb{T}$. Hereafter, we shall denote by $[u, v)_{\mathbb{T}}$ the time scale interval $[u, v) \cap \mathbb{T}$. The intervals $(u, v)_{\mathbb{T}}$, $[u, v]_{\mathbb{T}}$, and $(u, v]_{\mathbb{T}}$ are defined similarly.

We list the following theorems which will be needed at several occasions throughout this study.

Theorem 1 (First Mean Value Theorem)**.** [8, Theorem 5.41. p. 142] *Let f and g be bounded and Δ integrable functions on $[u, v]_{\mathbb{T}}$, and let g be nonnegative (or nonpositive) on $[u, v]_{\mathbb{T}}$. Let us set*

$$m = \inf \{f(t) : t \in [u, v)_{\mathbb{T}}\} \text{ and } M = \sup \{f(t) : t \in [u, v)_{\mathbb{T}}\}.$$

Then there exists a real number Λ satisfying the inequalities $m \leq \Lambda \leq M$ such that

$$\int_{t_1}^{t_2} f(t)g(t)\Delta t = \Lambda \int_{t_1}^{t_2} g(t)\Delta t.$$

Theorem 2 (Intermediate Value Theorem)**.** [7, Theorem 1.115] *Assume* $x : \mathbb{T} \to \mathbb{R}$ *is continuous,* $u < v$ *are points in* $\mathbb{T}$*, and*

$$x(u)x(v) < 0.$$

Then there exists $c \in [u, v)_{\mathbb{T}}$ *such that either* $x(c) = 0$ *or* $x(c)x^{\sigma}(c) < 0$.

Theorem 3 (Chain Rule)**.** *(*[7, Theorem 1.93]*) Assume* $f : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\tilde{\mathbb{T}} := f(\mathbb{T})$ *is a time scale. Let* $g : \tilde{\mathbb{T}} \to \mathbb{R}$*. If* $f^{\Delta}(t)$ *and* $g^{\tilde{\Delta}}(f(t))$ *exist for* $t \in \mathbb{T}^{\kappa}$*, then*

$$(g \circ f)^{\Delta} = (g^{\tilde{\Delta}} \circ f) f^{\Delta}.$$

Lemma 1. [8, Corollaries 1.15-16, p.5] *Let* f *be a continuous function on* $[u, v]_{\mathbb{T}}$ *that is* Δ *differentiable on* $[u, v)_{\mathbb{T}}$.

i. *If* $f^{\Delta}(t) = 0$ *for all* $t \in [u, v)_{\mathbb{T}}$*, then* f *is a constant function on* $[u, v]_{\mathbb{T}}$.
ii. f *is increasing, decreasing, nondecreasing, and nonincreasing on* $[u, v]_{\mathbb{T}}$ *if* $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$*, and* $f^{\Delta}(t) \leq 0$ *for all* $t \in [u, v)_{\mathbb{T}}$*, respectively.*

Definition 1. [7, Definitions 2.25, 2.45] *A function* $p : \mathbb{T} \to \mathbb{R}$ *is said to be* regressive *provided that* $1 + \mu(t)p(t) \neq 0$ *for all* $t \in \mathbb{T}^{\kappa}$*. The set of all regressive rd-continuous functions* $p : \mathbb{T} \to \mathbb{R}$ *is denoted by* $\mathcal{R}$*. We also denote by* $\mathcal{R}^{+}$ *the set* $\mathcal{R}^{+} = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$ *of positively regressive functions.*

Let $p \in \mathcal{R}$ and $\mu(t) > 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \mathrm{Log}(1 + \mu(z)p(z))\, \Delta z\right). \tag{1.6}$$

The exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\Delta} = p(t)y$, $y(s) = 1$. In particular, if $\mathbb{T} = \mathbb{Z}$, then

$$e_p(t, t_0) = \prod_{s=t_0}^{t-1} (1 + p(s)), \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{1.7}$$

Other properties of the exponential function are given by the following.

Lemma 2. [6, Lemma 2.7.] *If* $p, q \in \mathcal{R}$*, then*

$$e_p(t, t) = 1, \quad e_p(t, s) = 1/e_p(s, t), \quad e_p(t, u)e_p(u, s) = e_p(t, s),$$
$$e_p(\sigma(t), s)(1 + \mu(t)p(t))e_p(t, s), \quad e_p(s, \sigma(t)) = \frac{e_p(s, t)}{1 + \mu(t)p(t)},$$
$$e_p^{\Delta}(., s) = p e_p(., s), \quad e_p^{\Delta}(s, .) = (\ominus p)e_p(s, .),$$
$$e_{p \oplus q} = e_p e_q, \quad e_{p \ominus q} = \frac{e_p}{e_q}.$$

Theorem 4. [7, Theorem 6.1] *Let* $\tau \in \mathbb{T}$*,* $y, f \in C_{rd}$*, and* $p \in \mathrm{R}^{+}$*. Then*

$$y^{\Delta}(t) \leq p(t)y(t) + f(t) \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq y(\tau)e_p(t, \tau) + \int_{\tau}^{t} e_p(t, \sigma(s))f(s)\Delta s \text{ for all } t \in [\tau, \infty)_{\mathbb{T}}.$$

2. Gronwall's inequality

There is no doubt that Gronwall's inequality [14, p.293] plays a substantial role in the investigation of stability and convergence properties of solutions of Volterra integral equations. The purpose of this section is to extend the scope of time scale analogue of Gronwall's inequality, which will be used to obtain function bounds for the solutions of Volterra integro-dynamic equations on time scales. A variant of Gronwall's inequality on time scales is given as follows:

Theorem 5. [7, Corollary 6.7, p.257] *Let $y \in C_{rd}$ and $\omega \in \mathcal{R}^+$, $\omega \geq 0$, and $\alpha \in \mathbb{R}$. Then*

$$y(t) \leq \alpha + \int_{t_0}^{t} y(s)\omega(s)\Delta s \text{ for all } t \in [t_0, \infty)_{\mathbb{T}} \tag{2.1}$$

implies

$$y(t) \leq \alpha e_{\omega}(t, t_0) \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}.$$

For more on Gronwall's inequalities on time scales we refer to [7, p.256], [18], and [20].

One may easily see by setting $\mathbb{T} = [0, 1] \cup [2, \infty)$, $\alpha = 3/2$, $t_0 = 0$, $\omega(t) = -t$, and $y(t) = 1$ for $t \in [0, 1]$ and $y(t) = 0$ for $t \in [2, \infty)$ that nonnegativity condition on the function ω in Theorem 5 cannot be omitted. However, in the next theorem, we keep positive regressivity condition $\omega \in \mathcal{R}^+$ and rule out nonnegativity condition on ω by making more stringent assumption than (2.1). Therefore, we obtain important relaxations for the particular cases. For instance, if $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, i.e., all functions $\omega : \mathbb{R} \to \mathbb{R}$ are positively regressive, if $\mathbb{T} = \mathbb{Z}$, the functions $\omega : \mathbb{Z} \to \mathbb{R}$ satisfying $\omega(t) > -1$ for all $t \in \mathbb{Z}$ are positively regressive. That is, the following result is valid for all functions $\omega : \mathbb{R} \to \mathbb{R}$ and for all functions $\widetilde{\omega} : \mathbb{Z} \to \mathbb{R}$ satisfying $\widetilde{\omega}(t) > -1$.

Theorem 6 (An extension of Gronwall's inequality)**.** *Let f and γ be continuous functions on $[t_0, T)_{\mathbb{T}}$ where $T \leq \infty$. Suppose that $-\gamma \in \mathcal{R}^+$.*

i. *If*

$$f(t) + \int_{\tau}^{t} \gamma(s)f(s)\Delta s \leq f(\tau) \tag{2.2}$$

for all $\tau, t \in [t_0, T)_{\mathbb{T}}$ with $\tau \leq t$, then

$$f(t) \leq f(t_0)e_{-\gamma}(t, t_0) \text{ for } t \in [t_0, T)_{\mathbb{T}}. \tag{2.3}$$

ii. *If*

$$f(t) + \int_{\tau}^{t} \gamma(s)f(s)\Delta s \geq f(\tau)$$

for all $\tau, t \in [t_0, T)_{\mathbb{T}}$ with $\tau \leq t$, then

$$f(t) \geq f(t_0)e_{-\gamma}(t, t_0) \text{ for } t \in [t_0, T)_{\mathbb{T}}.$$

Hereafter, we present some results which are essential for the proof of Theorem 6.

Lemma 3. *Let f, γ, and ξ be rd-continuous functions on $[t_0, T)_{\mathbb{T}}$ where $T \leq \infty$. Let $\xi \in \mathcal{R}^+$.*

i. *If* $\gamma(t) > 0$ *for* $t \in (t_0, T)_{\mathbb{T}}$ *and*

$$\int_{\tau}^{t} \gamma(s)[f(s) - f(\tau)e_{\xi}(s,\tau)]\Delta s \leq 0 \tag{2.4}$$

for all $\tau, t \in [t_0, T)_{\mathbb{T}}$ *with* $\tau \leq t$, *then*

$$f(t) \leq f(t_0)e_{\xi}(t, t_0) \text{ for } t \in [t_0, T)_{\mathbb{T}}. \tag{2.5}$$

ii. *If* $\gamma(t) < 0$ *for* $t \in (t_0, T)_{\mathbb{T}}$ *and*

$$\int_{\tau}^{t} \gamma(s)[f(s) - f(\tau)e_{\xi}(s,\tau)]\Delta s \geq 0 \tag{2.6}$$

for all $\tau, t \in [t_0, T)_{\mathbb{T}}$ *with* $\tau \leq t$, *then*

$$f(t) \leq f(t_0)e_{\xi}(t, t_0) \text{ for } t \in [t_0, T)_{\mathbb{T}}. \tag{2.7}$$

Proof. Let $\gamma(t) > 0$ for $t \in (t_0, T)_{\mathbb{T}}$ and (2.4) be satisfied. Suppose contrary that there is a nonempty interval $(t_1, t_2)_{\mathbb{T}} \subset [t_0, T)_{\mathbb{T}}$, with $t_0 \leq t_1 < t_2 < T$ such that

$$f(t_1) \leq f(t_0)e_{\xi}(t_1, t_0)$$

and

$$f(t) > f(t_0)e_{\xi}(t, t_0) \text{ for all } t \in (t_1, t_2)_{\mathbb{T}}, \tag{2.8}$$

i.e. (2.5) does not hold. Then using $e_{\xi}(t, t_1)e_{\xi}(t_1, t_0) = e_{\xi}(t, t_0)$ we have

$$f(t) > f(t_0)e_{\xi}(t, t_1)e_{\xi}(t_1, t_0) \geq f(t_1)e_{\xi}(t, t_1), \tag{2.9}$$

which, along with $\gamma > 0$, yields

$$\int_{t_1}^{t_2} \gamma(t)[f(t) - f(t_1)e_{\xi}(t, t_1)]\Delta t > 0, \tag{2.10}$$

contradicting our assumption (2.4). The statement (ii.) can be verified by applying similar arguments. The proof is complete. □

One can similarly prove the next result by reversing the directions of the inequalities (2.8-2.10).

Corollary 1. *Let* f, γ, *and* ξ *be rd-continuous functions on* $[t_0, T)_{\mathbb{T}}$ *where* $T \leq \infty$. *Suppose that* $\xi \in \mathcal{R}^+$.

i. *If* $\gamma(t) > 0$ *for* $t \in (t_0, T)_{\mathbb{T}}$ *and*

$$\int_{\tau}^{t} \gamma(s)[f(s) - f(\tau)e_{\xi}(s,\tau)]\Delta s \geq 0$$

for all $\tau, t \in [t_0, T)_{\mathbb{T}}$ *with* $\tau \leq t$, *then*

$$f(t) \geq f(t_0)e_{\xi}(t, t_0) \text{ for } t \in [t_0, T)_{\mathbb{T}}.$$

ii. *If* $\gamma(t) < 0$ *for* $t \in (t_0, T)_{\mathbb{T}}$ *and*

$$\int_{\tau}^{t} \gamma(s)[f(s) - f(\tau)e_{\xi}(s,\tau)]\Delta s \leq 0$$

for all $\tau, t \in [t_0, T)_{\mathbb{T}}$ *with* $\tau \leq t$*, then*

$$f(t) \geq f(t_0)e_{\xi}(t, t_0) \text{ for } t \in [t_0, T)_{\mathbb{T}}.$$

Now, we are ready to prove Theorem 6.

Proof of Theorem 6. We proceed by considering two cases: First, we consider the case in which γ is equivalently zero, strictly positive or strictly negative. Second, we handle the proof for the case when γ changes sign.

Case I. If $\gamma \equiv 0$, then (2.3) follows from (2.2). Suppose $\gamma(t) > 0$ for all $t \in (t_0, T)_{\mathbb{T}}$. For an arbitrary $\tau \in [t_0, T)_{\mathbb{T}}$ define

$$x_{\tau}(t) = \int_{\tau}^{t} \gamma(s)f(s)\Delta s \text{ for } t \in [\tau, T)_{\mathbb{T}}.$$

Invoking the differentiation rule ([7, Theorem 1.117]) we have

$$x_{\tau}^{\Delta}(t) = \gamma(t)f(t).$$

Then multiplying both sides of (2.2) by $\gamma(t)$, we obtain

$$x_{\tau}^{\Delta}(t) + \gamma(t)x_{\tau}(t) \leq f(\tau)\gamma(t), \tag{2.11}$$

Let us denote by p the function

$$p(t) = \ominus\left(-\gamma(t)\right).$$

Evidently, $\gamma(t) > 0$ and $-\gamma \in \mathcal{R}^{+}$ imply that $p = \frac{\gamma}{1-\mu\gamma} > 0$ and that

$$e_p^{\sigma}(t,\tau) = (1 + \mu p)e_p(t,\tau) > 0$$

for all $t \in (t_0, T)_{\mathbb{T}}$. Using the equality

$$\begin{aligned} e_p^{\sigma}(t,\tau)\gamma(t) &= (1 + \mu(t)p(t))\,\gamma(t)e_p(t,\tau) \\ &= \left(1 + \frac{\mu(t)\gamma(t)}{1 - \mu(t)\gamma(t)}\right)\gamma(t)e_p(t,\tau) \\ &= \frac{\gamma(t)}{1 - \mu(t)\gamma(t)}e_p(t,\tau) \\ &= \ominus\left(-\gamma(t)\right)e_p(t,\tau) \\ &= e_p^{\Delta}(t,\tau), \end{aligned}$$

and a multiplying both sides of (2.11) by $e_p^{\sigma}(t,\tau)$ we find

$$e_p^{\sigma}(t,\tau)x_{\tau}^{\Delta}(t) + e_p^{\sigma}(t,\tau)\gamma(t)x_{\tau}(t) \leq f(\tau)\gamma(t)e_p^{\sigma}(t,\tau),$$

which yields

$$e_p^{\sigma}(t,\tau)x_{\tau}^{\Delta}(t) + e_p^{\Delta}(t,\tau)x_{\tau}(t) \leq f(\tau)e_p^{\Delta}(t,\tau),$$

and hence,

$$\left[e_p(t,\tau)\left(x_{\tau}(t) - f(\tau)\right)\right]^{\Delta} \leq 0 \text{ for all } t \in [\tau, T)_{\mathbb{T}}.$$

Lemma 1 implies that the function $e_p(t,\tau)\,(x_\tau(t)-f(\tau))$ is nonincreasing on $[\tau,T)_{\mathbb{T}}$. That is,

$$e_p(t,\tau)\,(x_\tau(t)-f(\tau)) \le e_p(\tau,\tau)\,(x_\tau(\tau)-f(\tau))$$

for all $t\in[\tau,T)_{\mathbb{T}}$. Utilizing $e_p(\tau,\tau)=1$ and $x_\tau(\tau)=0$, we obtain

$$e_p(t,\tau)x_\tau(t) \le f(\tau)(e_p(t,\tau)-1).$$

Multiplying this inequality by $e_{-\gamma}(t,\tau)$, we get

$$x_\tau(t) \le f(\tau)\,(1-e_{-\gamma}(t,\tau)) = f(\tau)\int_\tau^t \gamma(s)e_{-\gamma}(s,\tau)\Delta s,$$

and therefore,

$$\int_\tau^t \gamma(s)[f(s)-f(\tau)e_{-\gamma}(t,\tau)]\Delta s \le 0 \text{ for all } t\in[\tau,T)_{\mathbb{T}}. \tag{2.12}$$

Since $\tau\in[t_0,T)_{\mathbb{T}}$ was arbitrary (2.12) holds for all $t,\tau\in[t_0,T)_{\mathbb{T}}$ satisfying $\tau\le t$. This is (2.4) with $\xi=-\gamma$. Consequently, we obtain (2.3) by making use of Lemma 3. To get (2.3) in the case when $\gamma(t)<0$ for all $t\in(t_0,T)$, it suffices to reverse direction of all the inequalities above and use the fact that (2.6) implies (2.7).

Case II. Now, suppose γ changes sign. Hereafter, we will use continuity of the function γ on $[t_0,T)_{\mathbb{T}}$ to show that the interval $[t_0,T)_{\mathbb{T}}$ can be partitioned into disjoint subintervals of the form $[t_{n-1},t_n)_{\mathbb{T}}$ so that γ is strictly negative, strictly positive, or identically zero on each of the open intervals $(t_{n-1},t_n)_{\mathbb{T}}$.

Let us define the set $S\subset[t_0,T)_{\mathbb{T}}$ as follows

$$S=\{t\in[t_0,T)_{\mathbb{T}}:\gamma(t)\gamma(\sigma(t))<0\}.$$

It is obvious that the set S consists only of right scattered points of $[t_0,T)_{\mathbb{T}}$ and can be expressed as follows

$$S=\cup_{t\in S}[t,\sigma(t)).$$

Let us separate this set from $[t_0,T)_{\mathbb{T}}$ and define

$$K=[t_0,T)_{\mathbb{T}}-S.$$

Denote the set of zeros of γ in K by A, i.e.,

$$A=\{t\in K:\gamma(t)=0\}.$$

Since the single point set $\{0\}$ is closed in $\mathbb{R}$, we get by continuity of γ that the set

$$A=\gamma^{-1}\{\{0\}\}\cap K=\{t\in K:\gamma(t)=0\}$$

is closed in K (here, we are considering $\mathbb{R}$ with its standard topology and the subset K with the subspace topology inherited from the topology on $\mathbb{R}$). Thus, the complement $\widetilde{A}=K-A$ of A in K is open in K. Consequently, the set $\widetilde{A}$ is composed of disjoint open intervals in K, each of which have one of the following forms: $(a,b)\cap\widetilde{A}$ or $[t_0,b)\cap\widetilde{A}$, where $a,b\in[t_0,T]_{\mathbb{T}}$ and $a<b$. We conclude from Theorem 2 that on each of these open intervals, the function γ is either strictly positive or strictly negative. Because, if there exist two points $t_1,t_2\in(a,b)\cap\widetilde{A}$ such that $\gamma(t_1)\gamma(t_2)<0$ then, Theorem 2 implies the existence of a point $c\in[t_1,t_2)\cap\widetilde{A}$ such that either $\gamma(c) = 0$ or $\gamma(c)\gamma(\sigma(c)) < 0$. This is not possible since $\widetilde{A} \cap (A \cup S) = \varnothing$. So, there is an increasing sequence $(t_n)_{n\in J}$ of distinct points such that

$$\cup_{n\in J}[t_{n-1}, t_n)_{\mathbb{T}} = [t_0, T)_{\mathbb{T}}$$

and the values of γ on (t_{n-1}, t_n) are always positive or always negative or always zero. Quantitative properties of the index set J depends on the function γ and the interval $[t_0, T)_{\mathbb{T}}$, i.e., the set J can be either a finite set $\{1, 2, ..., N\}$ or the set of natural numbers $\mathbb{N}$.
From Case I we know that the inequality

$$f(t) \le f(t_{n-1})e_{-\gamma}(t, t_{n-1}) \text{ for } t \in [t_{n-1}, t_n)_{\mathbb{T}} \tag{2.13}$$

is satisfied, where $[t_{n-1}, t_n)_{\mathbb{T}}$ is any subinterval of the above mentioned partition of $[t_0, T)_{\mathbb{T}}$. The rest of the proof proceeds by induction. Suppose that (2.3) holds on $[t_0, t_{n-1})_{\mathbb{T}} = \cup_{k=1}^{n-1}[t_{k-1}, t_k)$, i.e.,

$$f(t) \le f(t_0)e_{-\gamma}(t, t_0) \text{ for } t \in [t_0, t_{n-1})_{\mathbb{T}}. \tag{2.14}$$

If t_{n-1} is a left dense point, then continuity of both sides of (2.14) implies

$$f(t_{n-1}) \le f(t_0)e_{-\gamma}(t_{n-1}, t_0). \tag{2.15}$$

To see that (2.15) holds in the case when t_{n-1} is left scattered, assume that $t_{n-2} \in [t_0, t_{n-1})_{\mathbb{T}}$ is a point such that $\sigma(t_{n-2}) = t_{n-1}$ and define the function

$$F(t) := f(t_0)e_{-\gamma}(t, t_0). \tag{2.16}$$

By (2.2) we have

$$\begin{aligned} f(t_{n-2}) &\ge f(t_{n-1}) + \int_{t_{n-2}}^{t_{n-1}} \gamma(s)f(s)\Delta s \\ &= f(t_{n-1}) + \mu(t_{n-2})\gamma(t_{n-2})f(t_{n-2}), \end{aligned}$$

and therefore,

$$(1 - \mu(t_{n-2})\gamma(t_{n-2}))f(t_{n-2}) \ge f(t_{n-1}). \tag{2.17}$$

It follows from (2.14), (2.16), and (2.17) that

$$\begin{aligned} F(t_{n-1}) &= F(t_{n-2}) + \mu(t_{n-2})F^{\Delta}(t_{n-2}) \\ &= f(t_0)e_{-\gamma}(t_{n-2}, t_0)\{1 - \mu(t_{n-2})\gamma(t_{n-2})\} \\ &\ge f(t_{n-2})\{1 - \mu(t_{n-2})\gamma(t_{n-2})\} \ge f(t_{n-1}). \end{aligned}$$

Hence, (2.15) holds in any case. Thus, by (2.13) and (2.15) we get that

$$\begin{aligned} f(t) &\le f(t_{n-1})e_{-\gamma}(t, t_{n-1}) \\ &\le f(t_0)e_{-\gamma}(t_{n-1}, t_0)e_{-\gamma}(t, t_{n-1}) \\ &\le f(t_0)e_{-\gamma}(t, t_0) \end{aligned}$$

for $t \in [t_{n-1}, t_n)_{\mathbb{T}}$. This shows that (2.3) holds on the interval $[t_0, t_n)_{\mathbb{T}}$. By induction we conclude that (2.3) holds on the entire interval $[t_0, T)_{\mathbb{T}}$.
For the proof of second statement of theorem we reverse the directions of inequalities (2.2-2.3) and invoke Corollary 1 to modify the proof of the first statement accordingly. The proof is complete. □

3. Shift operators on time scales

In this section, we introduce the shift operators $\delta_{\pm} : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ to construct the integro-dynamic equation

$$x^{\Delta}(t) = -a(t)x(t) + \int_{t_0}^{t} b(\delta_-(s,t))x(s)\Delta s, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.1}$$

An arbitrary time scale (e.g., $\mathbb{T} = q^{\mathbb{N}}$) does not have to include $t-s$ and 0. Therefore, different than the kernel $b(t-s)$ and the lower limit 0 of the integral in (1.2), we use $b(\delta_-(s,t))$ and t_0 in (3.1), respectively. An intuition for the determination of the shift operator δ_- can be developed by understanding the idea behind the use of $b(t-s)$ in (1.2). Informally, the expression $b(t-s)$ in (1.2) can be regarded as a shift (or delay) of the function b. However, since $t-s \notin q^{\mathbb{N}}$ for the time scale $\mathbb{T} = q^{\mathbb{N}}$, the expression $b(t-s)$ cannot be used as the shift of b. On the other hand, $t/s \in q^{\mathbb{N}}$ for all $t, s \in q^{\mathbb{N}}$ satisfying $t \geq s \geq 1$. Inspired by these examples and common properties of the operations $t-s$ and t/s, we can construct backward shift operator δ_- on time scales. Similarly, we can describe properties of the forward shift operator δ_+ considering the properties of the operations $t+s$ and ts.

Definition 2. *Suppose we are given an initial point $t_0 \in \mathbb{T}$ so that there exists operators $\delta_{\pm} : [t_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}} \to [t_0, \infty)_{\mathbb{T}}$ satisfying the following properties:*

P.1 *$\delta_+(s,t) \in [t_0, \infty)_{\mathbb{T}}$ for all $s, t \in [t_0, \infty)_{\mathbb{T}}$ and $\delta_-(s,t) \in [t_0, \infty)_{\mathbb{T}}$ for all $s, t \in [t_0, \infty)_{\mathbb{T}}$ satisfying $t_0 \leq s \leq t$,*

P.2 *Given a fixed element $T_0 \in [t_0, \infty)_{\mathbb{T}}$, the functions $\delta\pm$ are strictly increasing with respect to their second arguments, i.e.,*

$$T_0 \leq t < u \text{ implies } \delta_{\pm}(T_0, t) < \delta_{\pm}(T_0, u),$$

P.3 *If $T_1 < T_2$ for $T_1, T_2 \in [t_0, \infty)_{\mathbb{T}}$, then*

$$\delta_-(T_1, u) > \delta_-(T_2, u) \text{ for all } u \in [T_2, \infty)_{\mathbb{T}}$$

and

$$\delta_+(T_1, u) < \delta_+(T_2, u) \text{ for all } u \in [t_0, \infty)_{\mathbb{T}},$$

P.4 *$\delta_-(t_0, u) = \delta_+(t_0, u) = u$ for all $u \in [t_0, \infty)_{\mathbb{T}}$,*

P.5 *$\delta_+(t, s) = \delta_+(s, t)$ for all $t, s \in [t_0, \infty)_{\mathbb{T}}$,*

P.6 *$\delta_-(\delta_+(s, u), \delta_+(u, v)) = \delta_-(s, v)$ for all $u \in [t_0, \infty)_{\mathbb{T}}$ and $s, v \in [t_0, \infty)_{\mathbb{T}}$ with $s \leq v$,*

P.7 *$\delta_+(\delta_-(s, u), \delta_-(u, v)) = \delta_-(s, v)$ for all $s, u, v \in [t_0, \infty)_{\mathbb{T}}$ satisfying $s \leq u \leq v$.*

Then the operators δ_- and δ_+ associated with the initial point t_0 are called backward and forward shift operators on $[t_0, \infty)_{\mathbb{T}}$, respectively.

Generalized shifts and the associated geometry on a general time scale were first dealt with in [11]. Also, generalized convolution on time scales was treated by [13]. Afterwards, in [10, Definition 2.1] shift operators was defined to propose convolution on time scales. Note that the shift operators $\delta_{\pm}$ defined here are different than the ones in the above mentioned literature.

Example 1. *The operators $\delta_{\pm}$ described in (1.2-1.5) satisfy P.1-7 on the given time scales.*

This example shows that we can define different type shift operators on the same time scale. For instance, on $\mathbb{T} = \mathbb{R}$, we have the shift operators $\delta_{\pm}(s,t) = t \pm s$ and $\widehat{\delta}_{\pm}(s,t) = ts^{\pm 1}$ with the initial points 0 and 1, respectively.

Making use of properties P.1-7, we obtain the following result.

Lemma 4. i. *For a fixed* $T \in (t_0, \infty)_{\mathbb{T}}$ *we have*

$$\delta_+(T,t) > \delta_+(t_0,t) = t \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}$$

and

$$\delta_-(T,t) < \delta_+(t_0,t) = t \text{ for all } t \in [T,\infty)_{\mathbb{T}}.$$

ii. $u = \delta_-(s,t)$ *implies* $t = \delta_+(u,s)$ *for all* $t \in [t_0,\infty)_{\mathbb{T}}$ *and* $s,u \in [t_0,t]_{\mathbb{T}}$,

iii. $u = \delta_-(s,t)$ *implies* $s = \delta_-(u,t)$ *for all* $t \in [t_0,\infty)_{\mathbb{T}}$ *and* $s,u \in [t_0,t]_{\mathbb{T}}$,

iv. $\delta_-(\delta_-(s,t),t) = s$ *for all* $s,t \in [t_0,\infty)_{\mathbb{T}}$ *satisfying* $s \le t$,

v. $\delta_+(\delta_-(s,t),s) = t$ *for all* $s,t \in [t_0,\infty)_{\mathbb{T}}$ *satisfying* $s \le t$,

vi. $\delta_-(s,\delta_+(s,u)) = u$ *for all* $s,u \in [t_0,\infty)_{\mathbb{T}}$,

vii. $\delta_-(u,u) = t_0$ *for all* $u \in [t_0,\infty)_{\mathbb{T}}$.

Proof. (i) follows from P.2 and P.3. To show (ii), use P.4-5 and P.7 to obtain

$$\begin{aligned}\delta_+(u,s) &= \delta_+(\delta_-(s,t),\delta_-(t_0,s))\\ &= \delta_+(\delta_-(t_0,s),\delta_-(s,t))\\ &= \delta_-(t_0,t) = t.\end{aligned}$$

For the proof of (iii) we use (ii) to get

$$u = \delta_-(s,t) \Rightarrow t = \delta_+(u,s).$$

This, along with P. 4 and P.6, yields

$$s = \delta_-(t_0,s) = \delta_-(\delta_+(t_0,u),\delta_+(u,s)) = \delta_-(u,t).$$

(iv) is direct implication of (iii) since $u = \delta_-(s,t)$ implies $s = \delta_-(u,t)$ and

$$s = \delta_-(u,t) = \delta_-(\delta_-(s,t),t).$$

(v) is obtained by assuming $u = \delta_-(s,t)$, using (ii), i.e., $t = \delta_+(u,s)$, and

$$t = \delta_+(u,s) = \delta_+(\delta_-(s,t),s).$$

To verify (vi) take $u = \delta_-(s,t)$ use (ii) to get $t = \delta_+(s,u)$ and

$$u = \delta_-(s,t) = \delta_-(s,\delta_+(s,u)).$$

(vii) is proven by substituting $s = v = t_0$ in P.6. The proof is complete. □

Notice that shift operators $\delta_\pm$ are defined once the initial point $t_0 \in \mathbb{T}$ is known. For instance, we choose the initial point $t_0 = 0$ to define shift operators $\delta_\pm(s,t) = t \pm s$ on $\mathbb{T} = \mathbb{R}$. However, if we take the initial point $\lambda \in (0,\infty)$ then we can define new shift operators by $\widetilde{\delta}_\pm(s,t) = t \mp \lambda \pm s$ and in terms of $\delta_\pm$ as

$$\widetilde{\delta}_\pm(s,t) = \delta_\mp(\lambda,\delta_\pm(s,t)).$$

Example 2. *In the following table, we give several particular time scales to show the change in the formula of shift operators as the initial points change.*

	$\mathbb{T} = \mathbb{N}^{1/2}$		$\mathbb{T} = h\mathbb{Z}$		$\mathbb{T} = 2^{\mathbb{N}}$	
t_0	0	λ	0	$h\lambda$	1	2^λ
$\delta_-(s,t)$	$\sqrt{t^2 - s^2}$	$\sqrt{t^2 + \lambda^2 - s^2}$	$t - s$	$t + h\lambda - s$	t/s	$2^\lambda t s^{-1}$
$\delta_+(s,t)$	$\sqrt{t^2 + s^2}$	$\sqrt{t^2 - \lambda^2 + s^2}$	$t + s$	$t - h\lambda + s$	ts	$2^{-\lambda} t s$

where $\lambda \in \mathbb{Z}_+$, $\mathbb{N}^{1/2} = \{\sqrt{n} : n \in \mathbb{N}\}$, $2^{\mathbb{N}} = \{2^n : n \in \mathbb{N}\}$, *and* $h\mathbb{Z} = \{hn : n \in \mathbb{Z}\}$.

In general, let $\mathbb{T}$ be a time scale with shift operators $\delta_{\pm}$ associated with initial point t_0. Choosing a new initial point $\lambda \in [t_0, \infty)_{\mathbb{T}}$, we can define the new shift operators $\widetilde{\delta}_{\pm}$ associated with λ by

$$\widetilde{\delta}_{-}(s,t) = \delta_{+}(\lambda, \delta_{-}(s,t)), \tag{3.2}$$

$$\widetilde{\delta}_{+}(s,t) = \delta_{-}(\lambda, \delta_{+}(s,t)). \tag{3.3}$$

Using P.1-7, one may easily verify that the new shift operators $\widetilde{\delta}_{\pm}$ satisfy the following properties:

- $\widetilde{P}.1$ $\widetilde{\delta}_{+}(s,t) \in [\lambda, \infty)_{\mathbb{T}}$ for all $s,t \in [\lambda, \infty)_{\mathbb{T}}$ and $\widetilde{\delta}_{-}(s,t) \in [\lambda, \infty)_{\mathbb{T}}$ for all $s,t \in [\lambda, \infty)_{\mathbb{T}}$ satisfying $\lambda \leq s \leq t$,
- $\widetilde{P}.2$ Given a fixed element $T_0 \in [\lambda, \infty)_{\mathbb{T}}$, the functions $\widetilde{\delta}\pm$ are strictly increasing with respect to their second arguments, i.e.,
$$T_0 \leq t < u \text{ implies } \widetilde{\delta}_{\pm}(T_0, t) < \widetilde{\delta}_{\pm}(T_0, u),$$
- $\widetilde{P}.3$ If $T_1 < T_2$ for T_1,$T_2 \in [\lambda, \infty)_{\mathbb{T}}$, then
$$\widetilde{\delta}_{-}(T_1, u) > \widetilde{\delta}_{-}(T_2, u) \text{ for all } u \in [T_2, \infty)_{\mathbb{T}}$$
and
$$\widetilde{\delta}_{+}(T_1, u) < \widetilde{\delta}_{+}(T_2, u) \text{ for all } u \in [\lambda, \infty)_{\mathbb{T}},$$
- $\widetilde{P}.4$ $\widetilde{\delta}_{-}(\lambda, u) = \widetilde{\delta}_{+}(\lambda, u) = u$ for all $u \in [\lambda, \infty)_{\mathbb{T}}$,
- $\widetilde{P}.5$ $\widetilde{\delta}_{+}(t,s) = \widetilde{\delta}_{+}(s,t)$ for all $t,s \in [\lambda, \infty)_{\mathbb{T}}$,
- $\widetilde{P}.6$ $\widetilde{\delta}_{-}(\widetilde{\delta}_{+}(s,u), \widetilde{\delta}_{+}(u,v)) = \widetilde{\delta}_{-}(s,v)$ for all $u \in [t_0, \infty)_{\mathbb{T}}$ and $s,v \in [t_0, \infty)_{\mathbb{T}}$ with $s \leq v$,
- $\widetilde{P}.7$ $\widetilde{\delta}_{+}(\widetilde{\delta}_{-}(s,u), \widetilde{\delta}_{-}(u,v)) = \widetilde{\delta}_{-}(s,v)$ for all $s,u,v \in [t_0, \infty)_{\mathbb{T}}$ satisfying $s \leq u \leq v$.

Moreover, the properties given in Lemma 4 are also valid for the operators $\widetilde{\delta}_{\pm}$.

4. Function Bounds for Solutions of Volterra Equations

Hereafter, we suppose that $\mathbb{T}$ is a time scale including an initial point t_0 so that there exists shift operators $\delta_{\pm}$ satisfying properties P.1-7 in Definition 2. Let $a, b : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ be two continuous functions with $b(t) \geq 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and $-a \in \mathcal{R}^{+}$.Hereafter, we denote by $x(t) := x(t, t_0, x_0)$ the unique differentiable solution of

$$x^{\Delta}(t) = -a(t)x(t) + \int_{t_0}^{t} b(\delta_{-}(s,t))x(s)\Delta s \tag{4.1}$$

satisfying $x(t_0) = x_0$. For the existence and boundedness of such a solution we refer the reader to [1], [3], and [12]. For brevity, we shall use the notation $x(t)$ instead of $x(t, t_0, x_0)$.

Theorem 7. *Let γ be defined by*

$$\gamma(u) = a(u) - \int_{t_0}^{u} b(\delta_{-}(s,u))e_p(u,s)\Delta s, \tag{4.2}$$

where p is given by

$$p(t) = \ominus(-a(t)). \tag{4.3}$$

i. *If* $x_0 \geq 0$, *then*

$$x_0 e_{-a}(t,t_0) \leq x(t) \leq x_0 e_{-\gamma}(t,t_0) \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}. \tag{4.4}$$

ii. *If* $x_0 < 0$, *then*

$$x_0 e_{-\gamma}(t,t_0) \leq x(t) \leq x_0 e_{-a}(t,t_0) \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}. \tag{4.5}$$

Proof. First we show that $x(t)$ is nonnegative for all $t \in [t_0,\infty)_{\mathbb{T}}$. If $x_0 = 0$, then $x(t) \equiv 0$ is the unique differentiable solution of (4.1) and (4.4) holds for such x. Hereafter, we assume that $x_0 > 0$. Let the set Ω be defined by

$$\Omega := \{t \in \mathbb{T} : t > t_0 \text{ and } b(s) = 0 \text{ for all } s \in [t_0,t)_{\mathbb{T}}\}.$$

If Ω is unbounded above, then $b(t) = 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. Hence, (4.1) turns into

$$x^{\Delta}(t) = -a(t)x(t) \text{ and } x(t_0) = x_0 \tag{4.6}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$, which has the unique solution

$$x(t) = x_0 e_{-a}(t,t_0). \tag{4.7}$$

The solution (4.7) can also be derived from the fact that (4.6) implies

$$x(t) + \int_{\tau}^{t} a(u)x(u)\Delta u = x(\tau)$$

for all τ, t satisfying $t_0 \leq \tau \leq t$. This, along with Theorem 6, yields (4.7). On the other hand, by (P.1-P.4) we get that

$$u = \delta_-(t_0,u) \geq \delta_-(s,u) \geq \delta_-(u,u) = t_0 \text{ for } t_0 \leq s \leq u, \tag{4.8}$$

i.e., $\delta_-(s,u) \in [t_0,u)_{\mathbb{T}}$ for $t_0 \leq s \leq u$. That is, if Ω is unbounded abaove, then $b(\delta_-(s,u)) = 0$ for all $s \in [t_0,u)_{\mathbb{T}}$ and

$$\gamma(u) = a(u) \text{ for all } u \in [t_0,\infty)_{\mathbb{T}}, \tag{4.9}$$

by (4.2). Thus, (4.4) follows from (4.7) and (4.9).
It remains to show that (4.4) is satisfied whenever Ω is empty or bounded above. To do so, we first need to show that $x(t) > 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. Define the non-negative number $T \geq t_0$ by

$$T = \begin{cases} t_0 & \text{if } \Omega = \varnothing \\ \sup\Omega & \text{otherwise} \end{cases}.$$

If $T = t_0$, then $x(T) = x_0 > 0$. Let $T > t_0$ and $b(T) > 0$. If T is left scattered, then there exists a right scattered point $\widehat{t} \in [t_0,T)_{\mathbb{T}}$ such that $\sigma(\widehat{t}) = T$. Since $b(t) = 0$ for all $t \in [t_0,\widehat{t}]_{\mathbb{T}}$ we have $x(\widehat{t}) > 0$. Using (4.1) and the formula

$$x^{\sigma}(t) = x(t) + \mu(t)x^{\Delta}(t), \tag{4.10}$$

(see [7, Theorem 1.16 (iv)]), we find

$$x(T) = x(\widehat{t}) + \mu(\widehat{t})x^{\Delta}(\widehat{t}) = \big(1 - a(\widehat{t})\mu(\widehat{t})\big)\, x(\widehat{t}) > 0.$$

If $T \in (t_0,\infty)_{\mathbb{T}}$ is a left dense point, then the inequality $x(t) = x_0 e_{-a}(t,t_0) > 0$ on $[t_0,T)_{\mathbb{T}}$ and continuity of x imply $x(t) > 0$ for all $[t_0,T]_{\mathbb{T}}$.
To see nonnegativity of $x(t)$ on the interval $(T,\infty)_{\mathbb{T}}$ it suffices to prove that the set M_- given by

$$M_- := \{t \in (T,\infty)_{\mathbb{T}} : x(t) < 0\}$$

is empty. Suppose contrary that $M_- \neq \varnothing$ and denote by t_1 the real number

$$t_1 := \inf M_-.$$

Henceforth, we show that $t_1 \in M_-$, i.e., $x(t_1) < 0$. It follows from continuity of x that

$$x(t_1) \leq 0,$$

in which the case $x(t_1) = 0$ leads to a contradiction in the sign of $x^{\Delta}(t_1)$. To see this, let $x(t_1) = 0$. Thus, t_1 is right dense, and hence, M_- includes a continuous interval (t_1, a) on which x is nonincreasing, i.e., $x^{\Delta}(t_1) \leq 0$. On the other hand, from (4.1) and Theorem 1 we arrive at

$$\begin{aligned} x^{\Delta}(t_1) &= -a(t_1)x(t_1) + \int_{t_0}^{t_1} b(\delta_-(s,t_1))x(s)\Delta s \\ &\geq \int_{t_0}^{T} b(\delta_-(s,t_1))x(s)\Delta s \\ &= \Lambda_1 \int_{t_0}^{T} b(\delta_-(s,t_1))\Delta s \end{aligned}$$

where Λ_1 is a real number satisfying

$$m_1 \leq \Lambda_1 \leq M_1 \tag{4.11}$$

in which m_1 and M_1 are given by

$$m_1 = \min\{x(t) : t \in [t_0, T]_{\mathbb{T}}\} \text{ and } M_1 = \max\{x(t) : t \in [t_0, T]_{\mathbb{T}}\}$$

Evidently,

$$m_1 > 0. \tag{4.12}$$

On the other hand, similar to (4.8) we get that $t_0 \leq s \leq T$ implies

$$t_1 = \delta_-(t_0, t_1) \geq \delta_-(s, t_1) \geq \delta_-(T, t_1) > \delta_-(t_1, t_1) = t_0.$$

This shows that $b(\delta_-(s,t_1))$ is not equally zero on the interval $[t_0, T)_{\mathbb{T}}$. From (4.11-4.12) we find

$$\Lambda_1 \int_{t_0}^{T} b(\delta_-(s,t_1))\Delta s > 0.$$

Therefore, we have $t_1 \in M_-$, i.e., $x(t_1) < 0$. Since $x(T)x(t_1) < 0$, Theorem 2 guarantees the existence of a $c \in [T, t_1)$ such that

$$x(c) = 0 \text{ or } x(c)x(\sigma(c)) < 0,$$

where $x(c)x(\sigma(c)) < 0$ is not possible. To see this, we show that the set given by

$$D := \{t \in [T, t_1)_{\mathbb{T}} : x(t)x^{\sigma}(t) < 0\}$$

is empty. If there exists a $t^* \in D$, then $x(t^*) > 0$ and $x(\sigma(t^*)) < 0$. Since $t_1 = \inf M_- \in M_-$ we have

$$\sigma(t^*) = t_1 \notin D.$$

This along with $-a \in \mathcal{R}^+$, i.e., $1 - \mu(t)a(t) > 0$, implies

$$x(t_1) = x(\sigma(t^*)) = \{1 - \mu(t^*)a(t^*)\}\, x(t^*) + \mu(t^*) \int_{t_0}^{t^*} b(\delta_-(s,t^*))x(s)\Delta s \geq 0,$$

where we also used (4.1) and (4.10). This leads to a contradiction. Hence, we have $x(c) = 0$ for a $c \in (T, t_1)_{\mathbb{T}}$. This shows that the set

$$M_0 = \{t \in (T, t_1)_{\mathbb{T}} : x(t) = 0\}$$

is non-empty. Let

$$\eta = \sup M_0.$$

It follows from continuity of x that $\eta \in M_0$. Since $D = \varnothing$, as a consequence of Theorem 2, there cannot be any element $t \in (\eta, t_1)_{\mathbb{T}}$ such that $x(t) > 0$. Thus, $(\eta, t_1)_{\mathbb{T}} = \varnothing$ and then x is strictly decreasing on $[\eta, t_1)_{\mathbb{T}}$, i.e., $x^{\Delta}(\eta) < 0$ by Lemma 1. However, we get from Theorem 1 that

$$\begin{aligned} x^{\Delta}(\eta) &= -a(c)x(\eta) + \int_{t_0}^{\eta} b(\delta_-(s,\eta))x(s)\Delta s \\ &= \Lambda \int_{t_0}^{\eta} b(\delta_-(s,\eta))\Delta s \geq 0, \end{aligned}$$

where Λ is a real number satisfying $0 \leq \Lambda \leq \sup\{x(t) : t \in (t_0, \eta)_{\mathbb{T}}\}$. We obtain this contradiction by assuming that $M \neq \varnothing$. Consequently, $M_- = \varnothing$, i.e., $x(t) \geq 0$ for all $t \in [t_0, T)_{\mathbb{T}}$. Taking the integral in (4.1) from τ to t, we arrive at

$$x(t) - x(\tau) = -\int_{\tau}^{t} a(u)x(u)\Delta u + \int_{\tau}^{t}\int_{t_0}^{u} b(\delta_-(s,u))x(s)\Delta s\Delta u. \tag{4.13}$$

Having both $b(\delta_-(s,u))$ and $x(s)$ are nonnegative we obtain

$$x(t) + \int_{\tau}^{t} a(u)x(u)\Delta u \geq x(\tau), \tag{4.14}$$

for all $t, \tau \in [t_0, \infty)_{\mathbb{T}}$ satisfying $\tau \leq t$. By Theorem 6, we find the lower bound

$$x(t) \geq x_0 e_{-a}(t, t_0) \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.15}$$

Using this lower bound we can write

$$x(s) \leq e_p(u, s)x(u) \text{ for all } u \geq s \geq t_0, \tag{4.16}$$

where p is as in (4.3). Combining (4.13) and (4.16), for all $\tau, t \in [t_0, \infty)_{\mathbb{T}}$ with $\tau \leq t$

$$\begin{aligned} x(t) &\leq x(\tau) - \int_{\tau}^{t} a(u)x(u)\Delta u + \int_{\tau}^{t}\int_{t_0}^{u} b(\delta_-(s,u))e_p(u,s)x(u)\Delta s\Delta u \\ &= x(\tau) - \int_{\tau}^{t}\left[a(u) - \int_{t_0}^{u} b(\delta_-(s,u))e_p(u,s)\Delta s\right]x(u)\Delta u \\ &= x(\tau) - \int_{\tau}^{t} \gamma(u)x(u)\Delta u, \end{aligned} \tag{4.17}$$

where γ is defined as in (4.2). Note that $-\gamma \in \mathcal{R}^+$ since $-a \in \mathcal{R}^+$ and $b(\delta_-(s,u)) \geq 0$ for all $s \in [t_0, u)_{\mathbb{T}}$. Theorem 6 yields

$$x(t) \leq x_0 e_{-\gamma}(t, t_0). \tag{4.18}$$

Consequently, the result (4.4) follows from inequalities (4.15) and (4.18).
To see that (4.5) holds for $x_0 = x(t_0) < 0$, it is enough to employ (4.4) by taking into account that $-x(t)$ is the unique solution of Eq. (4.1) satisfying the initial condition $-x(t_0) = -x_0 > 0$. The proof is complete. □

Let $\lambda \in [t_0, \infty)$ be any fixed element. Consider the integro-dynamic equation

$$x^{\Delta}(t) = -a(t)x(t) + \int_{\lambda}^{t} b(\delta_-(s,t))x(s)\Delta s. \tag{4.19}$$

Henceforth, we denote by $X(t)$ the unique differentiable solution $X(t, \lambda, X_0)$ of Eq. (4.19) satisfying the initial condition $X(\lambda) = X_0$.

In the next corollary, we shall provide lower and upper bounds for $X(t)$ by using the results of Theorem 7. To be able to employ Theorem 7 in the analysis, first of all the kernel of the integral term in (4.19) should include the shift operator associated with λ. As we have mentioned in Section 3 we may move the initial point t_0 to λ, and define the new shift operators $\widetilde{\delta}_{\pm}$ associated with λ as in (3.2) and (3.3), respectively. Let us define the function $\widetilde{b} : [\lambda, \infty)_{\mathbb{T}} \to [0, \infty)$ by

$$\widetilde{b}(u) = b(\delta_-(\lambda, u)). \tag{4.20}$$

It is obvious that $\widetilde{b}(u) \geq 0$ for all $[\lambda, \infty)_{\mathbb{T}}$. Also we get by Lemma 4 (vi) and (3.2) that

$$\begin{aligned}
\widetilde{b}(\widetilde{\delta}_-(s,t)) &= b(\delta_-(\lambda, \widetilde{\delta}_-(s,t))) \\
&= b(\delta_-(\lambda, \delta_+(\lambda, \delta_-(s,t)))) \\
&= b(\delta_-(s,t)).
\end{aligned}$$

Hence, we can rewrite Eq. (4.19) as follows

$$x^{\Delta}(t) = -a(t)x(t) + \int_{\lambda}^{t} \widetilde{b}(\widetilde{\delta}_-(s,t))x(s)\Delta s. \tag{4.21}$$

Define the functions

$$\widetilde{\gamma}(u) = a(u) - \int_{\lambda}^{u} \widetilde{b}(\widetilde{\delta}_-(s,u)) e_p(u,s)\Delta s, \tag{4.22}$$

and

$$q(u) = a(u) - \int_{\lambda}^{u} \widetilde{b}(\widetilde{\delta}_-(s,u))\Delta s,$$

respectively. Note that $\widetilde{b} = b$, and hence, $\widetilde{\gamma} = \gamma$ whenever $\lambda = t_0$.

Corollary 2. *Let $a, b : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ be two continuous functions with $b(t) \geq 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and let $-a \in \mathcal{R}^+$. Let the function $\widetilde{b}$ be given by (4.20). Suppose*

$$a(u) \geq \int_{\lambda}^{u} \widetilde{b}(\widetilde{\delta}_{-}(s,u)) e_p(u,s) \Delta s \tag{4.23}$$

for all $u \in [\lambda, \infty)_{\mathbb{T}}$.

i. *If $X_0 \geq 0$, then $X(t)$ is nonincreasing on $[\lambda, \infty)_{\mathbb{T}}$ and*

$$X_0 e_{-q}(t,\lambda) \leq X(t) \leq X_0 e_{-\widetilde{\gamma}}(t,\lambda) \tag{4.24}$$

for all $t \in [\lambda, \infty)_{\mathbb{T}}$,

ii. *If $X_0 < 0$, then $X(t)$ is nondecreasing on $[\lambda, \infty)_{\mathbb{T}}$ and*

$$X_0 e_{-\widetilde{\gamma}}(t,\lambda) \leq X(t) \leq X_0 e_{-q}(t,\lambda) \tag{4.25}$$

for all $t \in [\lambda, \infty)_{\mathbb{T}}$.

Proof. Considering Eq. (4.21), we will proceed in a way similar to proof of Theorem 7. First we will show that (4.24) holds. By (4.23), we know that $\widetilde{\gamma}(t) \geq 0$ for all $t \in [\lambda, \infty)_{\mathbb{T}}$. As in Theorem 7, we can show that if $X_0 \geq 0$, then $X(t) \geq 0$ for all $t \in [\lambda, \infty)_{\mathbb{T}}$. Similar to (4.17) we have

$$\begin{aligned} X(t) &\leq X(\tau) - \int_{\tau}^{t} a(u)X(u)\Delta u + \int_{\tau}^{t}\int_{\lambda}^{u} \widetilde{b}(\widetilde{\delta}_{-}(s,u)) e_p(u,s) X(u) \Delta s \Delta u \\ &= X(\tau) - \int_{\tau}^{t} \left[a(u) - \int_{\lambda}^{u} \widetilde{b}(\widetilde{\delta}_{-}(s,u)) e_p(u,s)\Delta s \right] X(u)\Delta u \\ &= X(\tau) - \int_{\tau}^{t} \widetilde{\gamma}(u) X(u) \Delta u, \end{aligned}$$

for all $t, \tau \in [\lambda, \infty)_{\mathbb{T}}$ with $\tau \leq t$. Hence,

$$X(u) \leq X(s) \text{ for all } u, s \in [\tau, \infty)_{\mathbb{T}} \text{ with } s \leq u.$$

This shows monotonicity of solutions. Substituting $X(u)$ for $X(s)$ in

$$X(t) - X(\tau) = -\int_{\tau}^{t} a(u)X(u)\Delta u + \int_{\tau}^{t}\int_{\lambda}^{u} \widetilde{b}(\widetilde{\delta}_{-}(s,u)) X(s) \Delta s \Delta u$$

we arrive at

$$X(\tau) \leq X(t) + \int_{\tau}^{t} \left[a(u) - \int_{\lambda}^{u} \widetilde{b}(\widetilde{\delta}_{-}(s,u))\Delta s \right] X(u)\Delta u = X(t) + \int_{\tau}^{t} q(u)X(u)\Delta u \tag{4.26}$$

for all $t, \tau \in [\lambda, \infty)_{\mathbb{T}}$ with $\tau \leq t$. Since $-a \in \mathcal{R}^+$ and

$$\int_{\lambda}^{u} \widetilde{b}(\widetilde{\delta}_{-}(s,u)) \geq 0 \text{ for all } u \in [\lambda, \infty)_{\mathbb{T}},$$

we have $-q \in \mathcal{R}^+$ on $[\lambda,\infty)_{\mathbb{T}}$. Applying Theorem 6 to (4.26), we get the lower bound

$$X_0 e_{-q}(t,\lambda) \le X(t).$$

The upper bound in (4.24) can easily be obtained by using the similar arguments to the ones in (4.16-4.18). In the case $X_0 < 0$, (4.25) can be proved as it is done at the end of Theorem 7. The proof is complete. □

We need the condition (4.23) to guarantee the monotonicity of the solutions of (4.19). In the following corollary, we rule out this condition to obtain weaker conditions leading to lower and upper bounds for the solutions of (4.19).

Corollary 3. *Let $a, b : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ be two continuous functions with $b(t) \ge 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ and let $-a \in \mathcal{R}^+$.*

i. *If $X_0 \ge 0$, then*

$$X_0 e_{-a}(t,\lambda) \le X(t) \le X_0 e_{-\widetilde{\gamma}}(t,\lambda) \tag{4.27}$$

for all $t \in [\lambda,\infty)_{\mathbb{T}}$,

ii. *If $X_0 < 0$, then*

$$X_0 e_{-\widetilde{\gamma}}(t,\lambda) \le X(t) \le X_0 e_{-a}(t,\lambda) \tag{4.28}$$

for all $t \in [\lambda,\infty)_{\mathbb{T}}$.

Proof. Notice that this corollary is equivalent to Theorem 7 when $\lambda = t_0$. To verify (4.27) for $\lambda > t_0$, we proceed in a way similar to proof of Corollary 2. The upper bound in (4.27) can be obtained as it is done in the proof of Corollary 2. So, we only need to find the lower bound. From (4.21) we have

$$X(t) - X(\tau) = -\int_{\tau}^{t} a(u)X(u)\Delta u + \int_{\tau}^{t}\int_{\lambda}^{u} \widetilde{b}(\widetilde{\delta}_-(s,u))X(s)\Delta s\Delta u,$$

in which $X(t) \ge 0$ whenever $X_0 \ge 0$ by Theorem 7. Since the function $\widetilde{b}$ is also nonnegative, we find

$$X(t) + \int_{\tau}^{t} a(u)X(u)\Delta u \ge X(\tau), \text{ for all } t,\tau \in [\lambda,\infty)_{\mathbb{T}} \text{ with } \tau \le t,$$

and therefore, by Theorem 6

$$X_0 e_{-a}(t,\lambda) \le X(t).$$

The proof is completed as we did in the proof of Theorem 7. □

Considering [4, Examples 3.2, 3.7] one may see the consistency of the results with the special case $\mathbb{T} = \mathbb{R}$ (see also [5]). For the case $\mathbb{T} = \mathbb{Z}$ the following example numerically illustrates the bounds obtained in Corollary 3.

Example 3. *Let us take $\mathbb{T} = \mathbb{Z}$, $a(t) = 1/2$, $\delta_-(k,t) = t-k$, and*

$$b(t-k) = \frac{1}{2^{t-k}(t-k+6)(t-k+5)}$$

to construct the convolution type Volterra integro difference equation

$$x(t+1) = \frac{x(t)}{2} + \sum_{k=1}^{t-1} \frac{x(k)}{2^{t-k}(t-k+6)(t-k+5)}, \quad t \in [1,\infty)_{\mathbb{Z}}. \tag{4.29}$$

It is obvious from (1.7) that

$$e_{-a}(t,1) = (0.5)^{t-1}$$

and

$$\begin{aligned}\widetilde{\gamma}(t) &= \frac{1}{2} - \sum_{k=1}^{t-1} \frac{1}{(t-k+6)(t-k+5)} \\ &= \frac{1}{3} + \frac{1}{t+5}.\end{aligned}$$

Hence,

$$e_{-\widetilde{\gamma}}(t,1) = \prod_{k=1}^{t-1} \left(\frac{2}{3} - \frac{1}{k+5} \right).$$

Let $X(t)$ *denote the solution of (4.29) satisfying* $X(1) = 1$*. Using MATLAB 6.12, the numerical values of the functions* $L(t) := e_{-a}(t,1) = (0.5)^{t-1}$*,* $X(t)$*,* $U(t) := e_{-\widetilde{\gamma}}(t,1)$*, and* $E(t) := \exp(-\frac{1}{3}(t-1))$ *for* $t \in [1,10]_{\mathbb{Z}}$ *are computed as follows:*

t	$L(t)$	$X(t)$	$U(t)$	$E(t)$
1	1	1	1	1
2	0,5	0,5	0,5	0,7165
3	0,25	0,2619	0,2619	0,5134
4	0,125	0,1414	0,1419	0,3679
5	0,0625	0,0778	0,0788	0,2636
6	0,0313	0,0433	0,0447	0,1889
7	0,0156	0,0243	0,0257	0,1353
8	0,0078	0,0137	0,015	0,097
9	0,0039	0,0077	0,0088	0,0695
10	0,002	0,0044	0,0053	0,0498

As depicted in Figure 1, graphs of the functions L *and* U *lie alongside that of* $X(t)$*.*

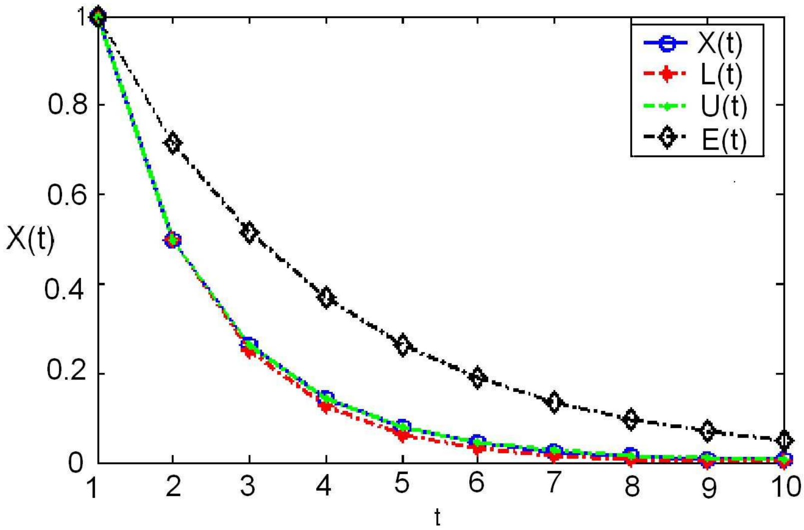

Fig. 1. Upper and lower bounds for a solution $X(t)$ of (4.29)

We continue this section by listing some remarks. The following remark can be found in [9].

Remark 1. (1) *For a nonnegative φ with $-\varphi \in \mathcal{R}^+$, we have the inequalities*

$$1 - \int_s^t \varphi(u) \leq e_{-\varphi}(t,s) \leq \exp\left\{-\int_s^t \varphi(u)\right\} \text{ for all } t \geq s$$

(2) *If φ is rd-continuous and nonnegative, then*

$$1 + \int_s^t \varphi(u) \leq e_{\varphi}(t,s) \leq \exp\left\{\int_s^t \varphi(u)\right\} \text{ for all } t \geq s$$

It follows from (1.6) that $e_\varphi(t,s) > 0$ for $\varphi \in \mathcal{R}^+$ and $t \geq s$. One may derive the next result using Remark 1.

Remark 2. (1) *If $\varphi \in \mathcal{R}^+$ and $\varphi(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_\varphi(t,s) \leq \exp\left(\int_s^t \varphi(r)\Delta r\right) < 1.$$

(2) *If $\varphi \in \mathcal{R}^+$, then*

$$0 < e_\varphi(t,s) \leq \exp\left(\int_s^t \varphi(r)\Delta r\right) \tag{4.30}$$

for all $t \in [s,\infty)_{\mathbb{T}}$.

For more on inequalities regarding the exponential function on time scales see also [7, Theorem 2.44, p.66].

Corollary 4. *In addition to all assumptions of Corollary 3 suppose also that*

$$\lim_{t\to\infty} \int_\lambda^t \widetilde{\gamma}(s)\Delta s = \infty, \tag{4.31}$$

then for any $X_0 \in \mathbb{R}$, $X(t) = X(t, \lambda, X_0)$ *tends to zero as* $t \to \infty$.

Proof. Since $\widetilde{b}$ takes only nonnegative values, we get by (4.22) that

$$a(u) \geq \widetilde{\gamma}(u) \text{ for all } u \in [\lambda, \infty)_{\mathbb{T}}, \tag{4.32}$$

and hence,

$$1 - \mu(u)a(u) \leq 1 - \mu(u)\widetilde{\gamma}(u) \text{ for all } u \in [\lambda, \infty)_{\mathbb{T}},$$

i.e., $-\widetilde{\gamma} \in \mathcal{R}^+$. On the other hand, by (4.27), (4.28), and (4.30) we obtain

$$0 \leq X(t) \leq X_0 \exp\left(-\int_\lambda^t \widetilde{\gamma}(s)\Delta s\right) \text{ for } X_0 \geq 0$$

and

$$X_0 \exp\left(-\int_\lambda^t \widetilde{\gamma}(s)\Delta s\right) \leq X(t) < \exp\left(-\int_\lambda^t \widetilde{\gamma}(s)\Delta s\right) \text{ for } X_0 < 0$$

for all $t \in [\lambda, \infty)_{\mathbb{T}}$. The proof follows from (4.31). □

Remark 3. *In the particular case* $\mathbb{T} = \mathbb{Z}$, *Corollary 4 provides alternative conditions implying asymptotic stability of zero solution of convolution type Volterra difference equations*

$$x_{n+1} = ax_n + \sum_{s=0}^{n-1} b_{n-s}x_s, \quad n \geq 0,$$

handled in [15, Theorem 1.1] *(see also* [16] *and* [17]*).*

Corollary 5. *In addition to assumptions of Corollary 2 suppose also that there exists an* $\varepsilon > 0$ *such that*

$$a(u) - \int_\lambda^u \widetilde{b}(\widetilde{\delta}_-(s,u))e_p(u,s)\Delta s \geq \varepsilon \tag{4.33}$$

holds for all $u \in [\lambda, \infty)_{\mathbb{T}}$.

i. *If* $X_0 > 0$, *then the solution* $X(t) = X(t, \lambda, X_0)$ *of (4.19) is strictly decreasing on* $[\lambda, \infty)_{\mathbb{T}}$ *and*

$$X_0 e_{-q}(t, \lambda) \leq X(t) \leq X_0 \exp\left(-\varepsilon(t - \lambda)\right) \tag{4.34}$$

for all $t \in [\lambda, \infty)_{\mathbb{T}}$,

ii. *If* $X_0 < 0$, *then the solution* $X(t) = X(t, \lambda, X_0)$ *of (4.19) is strictly increasing on* $[\lambda, \infty)_{\mathbb{T}}$ *and*

$$X_0 \exp\left(-\varepsilon(t - \lambda)\right) \leq X(t) \leq X_0 e_{-q}(t, \lambda) \tag{4.35}$$

for all $t \in [\lambda, \infty)_{\mathbb{T}}$.

Proof. Upper and lower bounds in (4.34) follow from (4.24), (4.30), and (4.33). The upper bound in (4.35) is obtained from (4.25). On the other hand, for $X_0 < 0$, (4.30) yields

$$e_{-\widetilde{\gamma}}(t, \lambda) \leq \exp\left(-\int_\lambda^t \widetilde{\gamma}(s)\Delta s\right) \leq \exp\left(-\varepsilon(t - \lambda)\right),$$

and

$$X_0 e_{-\widetilde{\gamma}}(t, \lambda) \geq X_0 \exp\left(-\int_\lambda^t \widetilde{\gamma}(s)\Delta s\right) \geq X_0 \exp\left(-\varepsilon(t - \lambda)\right).$$

Hence, the lower bound in (4.35) can be found by using (4.25). Note that (4.33) implies that $\widetilde{\gamma}$ is strictly positive. Thus, by (4.17) the solution $X(t) = X(t, \lambda, X_0)$ of (4.19) is strictly decreasing on $[\lambda, \infty)_{\mathbb{T}}$ provided $X_0 > 0$. For $X_0 < 0$, monotonicity of $X(t) = X(t, \lambda, X_0)$ is obtained by applying the same type of argument that ends the proof of Theorem 7. The proof is complete. $\square$

It is obvious that the function $\widetilde{\gamma}$ in Example 3 satisfies the inequality $\widetilde{\gamma}(t) \geq \frac{1}{3}$. This is (4.33) with $\varepsilon = \frac{1}{3}$. Hence, considering Figure 1 and (4.34) with $\varepsilon = \frac{1}{3}$ one may see the validity of the upper bound $E(t) = \exp(-\frac{1}{3}(t-1))$ for the solutions of (4.29).

Corollary 6. *Suppose all assumptions of Corollary 5. Then*

i. *If $X_0 > 0$, then the solution $X(t) = X(t, \lambda, X_0)$ of (4.19) is strictly decreasing on $[\lambda, \infty)_{\mathbb{T}}$ and*

$$X_0 e_{-q}(t, \lambda) \leq X(t) \leq X_0 e_{\ominus\varepsilon}(t, \lambda) \tag{4.36}$$

for all $t \in [\lambda, \infty)_{\mathbb{T}}$,

ii. *If $X_0 < 0$, then the solution $X(t) = X(t, \lambda, X_0)$ of (4.19) is strictly increasing on $[\lambda, \infty)_{\mathbb{T}}$ and*

$$X_0 e_{\ominus\varepsilon}(t, \lambda) \leq X(t) \leq X_0 e_{-q}(t, \lambda) \tag{4.37}$$

for all $t \in [\lambda, \infty)_{\mathbb{T}}$.

Proof. Monotonicity of the solutions $X(t, \lambda, X_0)$ can be obtained similar to that in Corollary 5. Since $\widetilde{\gamma} > \varepsilon$ by (4.33), the inequality

$$-\widetilde{\gamma} < -\varepsilon < \frac{-\varepsilon}{1 + \mu(t)\varepsilon} = \ominus\varepsilon$$

yields

$$e^{\Delta}_{-\widetilde{\gamma}}(t, \lambda) = -\widetilde{\gamma}(t) e_{-\widetilde{\gamma}}(t, \lambda) \leq \ominus\varepsilon e_{-\widetilde{\gamma}}(t, \lambda).$$

This, along with Theorem 4, implies

$$e_{-\widetilde{\gamma}}(t, \lambda) \leq e_{\ominus\varepsilon}(t, \lambda) \text{ for all } t \in [\lambda, \infty)_{\mathbb{T}}.$$

Thus, the bounds in (4.36) and (4.37) follow from (4.24) and (4.25). The proof is complete. $\square$

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