Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **1**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

Some stability and boundedness conditions for non-autonomous differential equations with deviating arguments

Cemil Tunç

Abstract

In this article, the author studies the stability and boundedness of solutions for the non-autonomous third order differential equation with a deviating argument, r:

$$x'''(t) + a(t)x''(t) + b(t)g_1(x'(t-r)) + g_2(x'(t)) + h(x(t-r))$$
$$= p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)),$$

where $r > 0$ is a constant. Sufficient conditions are obtained; a stability result in the literature is improved and extended to the preceding equation for the case $p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)) = 0$, and a new boundedness result is also established for the case $p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)) \neq 0$.

1 Introduction

In 1968, Ponzo [10] considered the following nonlinear third order differential equation without a deviating argument:

$$x'''(t) + a(t)x''(t) + b(t)x'(t) + cx(t) = 0.$$

For the preceding equation, he constructed a positive definite Liapunov function with negative semi-definite time derivative. This established the stability of the null solution.

In this paper, instead of the preceding equation, we consider the following non-autonomous third order differential equation with a deviating argument, r :

$$\begin{aligned} & x'''(t)+a(t)x''(t)+b(t)g_1(x'(t-r))+g_2(x'(t))+h(x(t-r)) \\ & = p(t,x(t),x(t-r),x'(t),x'(t-r),x''(t)), \end{aligned} \tag{1}$$

which is equivalent to the system:

$$\begin{aligned} & x'(t)=y(t), \\ & y'(t)=z(t), \\ & z'(t)=-a(t)z(t)-b(t)g_1(y(t))-h(x(t))+b(t)\int_{t-r}^{t} g_1'(y(s))z(s)ds \\ & -g_2(y(t))+\int_{t-r}^{t} h'(x(s))y(s)ds+p(t,x(t),x(t-r),y(t),y(t-r),z(t)), \end{aligned} \tag{2}$$

where r is a positive constant; the functions a, b, g_1, g_2, h and p depend only on the arguments displayed explicitly and the primes in Eq. (1) denote differentiation with respect to $t \in \Re^+ = [0,\infty)$. The functions a, b, g_1, g_2, h and p are assumed to be continuous for their all respective arguments on $\Re^+$, $\Re^+$, $\Re$, $\Re$, $\Re$ and $\Re^+ \times \Re^5$, respectively. Assume also that the derivatives $a'(t) \equiv \frac{d}{dt}a(t)$, $b'(t) \equiv \frac{d}{dt}b(t)$, $h'(x) \equiv \frac{d}{dx}h(x)$ and $g_1'(y) \equiv \frac{d}{dy}g_1(y)$ exist and are continuous; throughout the paper $x(t)$, $y(t)$ and $z(t)$ are abbreviated as x, y and z, respectively. Finally, the existence and uniqueness of solutions of Eq. (1) are assumed and all solutions considered are supposed to be real valued.

The motivation of this paper has come by the result of Ponzo [10, Theorem 2]. Our purpose here is to extend and improve the result established by Ponzo [10, Theorem 2] to the preceding non-autonomous differential equation with the deviating argument r for the asymptotic stability of null solution and the boundedness of all solutions, whenever $p \equiv 0$ and $p \neq 0$ in Eq.(1), respectively.

At the same time, it is worth mentioning that one can recognize that by now many significant theoretical results dealt with the stability and boundedness of solutions of nonlinear differential equations of third order without delay:

$$x'''(t)+a_1x''(t)+a_2x'(t)+a_3x(t)=p(t,x(t),x'(t),x''(t)),$$

in which a_1, a_2 and a_3 are not necessarily constants. In particular, one can refer to the book of Reissig et al. [11] as a survey and the papers of Ezeilo [4,5], Ezeilo and Tejumola [6], Ponzo [10], Swick [14], Tunç [16, 17, 18, 21], Tunç and Ateş [27] and the references cited in these works for some publications performed on the topic. Besides, with respect our observation from the literature, it can be seen some papers on the stability and boundedness of solutions of nonlinear differential equations of third order with delay (see, for example, the papers of Afuwape and Omeike [2], Omeike [9], Sadek [12], Sinha [13], Tejumola and Tchegnani [15], Tunç ([19, 20], [22-26]), Zhu [28]) and the references thereof).

It should be noted that, to the best of our knowledge, we did not find any work based on the result of Ponzo [10, Theorem 2] in the literature. That is to say that, this work is the first attempt carrying the result of Ponzo [10, Theorem 2] to certain non-autonomous differential equations with deviating arguments. The assumptions will be established here are different from that in the papers mentioned above.

2 Main Results

Let $p(t, x, x(t-r), y, y(t-r), z) = 0$. We establish the following theorem

Theorem 1. In addition to the basic assumptions imposed on the functions $a(t)$, $b(t)$, g_1, g_2 and h appearing in Eq. (1), we assume that there are positive constants a, α, β, b_1, b_2, B, c, c_1 and L such that the following conditions hold:

(i) $a(t) \geq 2\alpha + a$, $B \geq b(t) \geq \beta$,

$g_1(0) = g_2(0) = h(0) = 0$,

$0 < c_1 \leq h'(x) \leq c$, $\alpha\beta - c > 0$,

$\frac{g_1(y)}{y} \geq b_1 \geq 1$, $\frac{g_2(y)}{y} \geq b_2$, $(y \neq 0)$ and $|g_1'(y)| \leq L$.

(ii) $[\alpha b(t) - c]\, y^2 \geq 2^{-1}\alpha a'(t) y^2 + b'(t) \int\limits_0^y g_1(\eta) d\eta$.

Then the null solution of Eq. (1) is stable, provided

$$r < \min\left\{ \frac{\alpha b_2}{\alpha(BL + 2c) + c}, \ \frac{2\alpha}{BL(2+\alpha) + c} \right\}.$$

Proof. To prove Theorem 1, we define a Lyapunov functional $V(t, x_t, y_t, z_t)$:

$$\begin{aligned} 2V(t, x_t, y_t, z_t) &= z^2 + 2\alpha yz + 2b(t)\int_0^y g_1(\eta)d\eta + 2\int_0^y g_2(\eta)d\eta + \alpha a(t)y^2 + 2h(x)y \\ &+ 2\alpha\int_0^x h(\xi)d\xi + \lambda_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \lambda_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds, \end{aligned} \tag{3}$$

where λ_1 and λ_2 are some positive constants which will be specified later in the proof.

Now, from the assumptions $\frac{g_1(y)}{y} \geq b_1 \geq 1$, $\frac{g_2(y)}{y} \geq b_2$, $(y \neq 0)$, and $0 < c_1 \leq h'(x) \leq c$, it follows that

$$\begin{aligned} &2b(t)\int_0^y g_1(\eta)d\eta = 2b(t)\int_0^y \frac{g_1(\eta)}{\eta}\eta d\eta \geq \beta b_1 y^2 \geq \beta y^2, \\ &2\int_0^y g_2(\eta)d\eta = 2\int_0^y \frac{g_2(\eta)}{\eta}\eta d\eta \geq b_2 y^2, \\ &h^2(x) = 2\int_0^x h(\xi)h'(\xi)d\xi \leq 2c\int_0^x h(\xi)d\xi. \end{aligned}$$

The preceding inequalities lead to the following:

$$\begin{aligned} 2V(t, x_t, y_t, z_t) &\geq (z+\alpha y)^2 + \beta[y+\beta^{-1}h(x)]^2 + 2\alpha\int_0^x h(\xi)d\xi - \frac{1}{\beta}h^2(x) \\ &+ b_2y^2 + \lambda_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \lambda_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds \\ &\geq (z+\alpha y)^2 + \beta[y+\beta^{-1}h(x)]^2 + 2\alpha\int_0^x h(\xi)d\xi - \frac{2c}{\beta}\int_0^x h(\xi)d\xi \\ &+ b_2y^2 + \lambda_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \lambda_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds. \end{aligned}$$

Now, it is clear

$$\begin{aligned} 2\alpha\int_0^x h(\xi)d\xi - \frac{2c}{\beta}\int_0^x h(\xi)d\xi &= 2\beta^{-1}(\alpha\beta - c)\int_0^x h(\xi)d\xi \\ &\geq c_1\beta^{-1}(\alpha\beta - c)x^2. \end{aligned}$$

Hence

$$\begin{aligned} 2V(t, x_t, y_t, z_t) &\geq (z+\alpha y)^2 + \beta[y+\beta^{-1}h(x)]^2 + 2^{-1}c_1\beta^{-1}(\alpha\beta - c)x^2 + b_2y^2 \\ &+ \lambda_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \lambda_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds. \end{aligned}$$

The preceding inequality allows the existence of some positive constants D_i , $(i = 1,\ 2,\ 3)$, such that

$$V(t, x_t, y_t, z_t) \geq D_1x^2 + D_2y^2 + D_3z^2 \geq D_4(x^2 + y^2 + z^2), \tag{4}$$

where $D_4 = \min\{D_1, D_2, D_3\}$, since $\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds \geq 0$ and $\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds \geq 0$.

Now, along a trajectory of (2) we find

$$
\begin{aligned}
\tfrac{d}{dt}V(t,x_t,y_t,z_t) \;&= -\left[\alpha b(t)g_1(y)y^{-1}+\alpha g_2(y)y^{-1}-h'(x)-2^{-1}\alpha a'(t)\right]y^2+b'(t)\int\limits_0^y g_1(\eta)d\eta\\
&-[a(t)-\alpha]\,z^2+zb(t)\int\limits_{t-r}^{t} g_1'(y(s))z(s)ds+z\int\limits_{t-r}^{t} h'(x(s))y(s)ds\\
&+\alpha yb(t)\int\limits_{t-r}^{t} g_1'(y(s))z(s)ds+\alpha y\int\limits_{t-r}^{t} h'(x(s))y(s)ds\\
&+\lambda_1 y^2 r-\lambda_1\int\limits_{t-r}^{t} y^2(s)ds+\lambda_2 z^2 r-\lambda_2\int\limits_{t-r}^{t} z^2(s)ds.
\end{aligned}
\tag{5}
$$

In view of the assumptions of Theorem 1 and the inequality $2\,|mn| \le m^2+n^2$, we find the following inequalities:

$$
\begin{aligned}
&\left[\alpha b(t)g_1(y)y^{-1}+\alpha g_2(y)y^{-1}-h'(x)-2^{-1}\alpha a'(t)\right]y^2-b'(t)\int\limits_0^y g_1(\eta)d\eta\\
&\ge\left[\alpha b_1 b(t)+\alpha b_2-c-2^{-1}\alpha a'(t)\right]y^2-b'(t)\int\limits_0^y g_1(\eta)d\eta\\
&\ge[\alpha b(t)-c]\,y^2-2^{-1}\alpha a'(t)y^2-b'(t)\int\limits_0^y g_1(\eta)d\eta+\alpha b_2 y^2\\
&\ge \alpha b_2 y^2,
\end{aligned}
$$

$$[a(t)-\alpha]\,z^2 \ge (\alpha+a)z^2,$$

$$zb(t)\int\limits_{t-r}^{t} g_1'(y(s))z(s)ds \le \frac{BL}{2}rz^2+\frac{BL}{2}\int\limits_{t-r}^{t} z^2(s)ds,$$

$$\alpha yb(t)\int\limits_{t-r}^{t} g_1'(y(s))z(s)ds \le \frac{\alpha BL}{2}ry^2+\frac{\alpha BL}{2}\int\limits_{t-r}^{t} z^2(s)ds,$$

$$z\int\limits_{t-r}^{t} h'(x(s))y(s)ds \le \frac{c}{2}rz^2+\frac{c}{2}\int\limits_{t-r}^{t} y^2(s)ds,$$

$$\alpha y\int\limits_{t-r}^{t} h'(x(s))y(s)ds \le \frac{\alpha c}{2}ry^2+\frac{\alpha c}{2}\int\limits_{t-r}^{t} y^2(s)ds.$$

The substituting of the preceding inequalities into (5) gives

$$\begin{aligned}\frac{d}{dt}V(t,x_t,y_t,z_t) &\le -\tfrac{1}{2}\left[\alpha b_2-(\alpha BL+\alpha c+2\lambda_1)r\right]y^2-\tfrac{1}{2}\alpha b_2 y^2\\ &-az^2-\tfrac{1}{2}\left[2\alpha-(BL+c+2\lambda_2)r\right]z^2\\ &+\left[2^{-1}(1+\alpha)c-\lambda_1\right]\int\limits_{t-r}^{t} y^2(s)ds\\ &+\left[2^{-1}(1+\alpha)BL-\lambda_2\right]\int\limits_{t-r}^{t} z^2(s)ds.\end{aligned}$$

Let $\lambda_1=\frac{(1+\alpha)c}{2}$ and $\lambda_2=\frac{(1+\alpha)BL}{2}$. Hence we can write

$$\begin{aligned}\frac{d}{dt}V(t,x_t,y_t,z_t) &\le -\tfrac{1}{2}\left[\alpha b_2-(\alpha BL+\alpha c+2\lambda_1)r\right]y^2-\tfrac{1}{2}\alpha b_2 y^2\\ &-az^2-\tfrac{1}{2}\left[2\alpha-(BL+c+2\lambda_2)r\right]z^2.\end{aligned}$$

Now, the last inequality implies

$$\frac{d}{dt}V(t,x_t,y_t,z_t)\le-\lambda_3\, y^2-\lambda_4\, z^2,$$

for some positive constants λ_3 and λ_4, provided

$$r<\min\left\{\frac{\alpha b_2}{\alpha(BL+2c)+c},\ \ \frac{2\alpha}{BL(2+\alpha)+c}\right\}.$$

This completes the proof of Theorem 1 (see also Burton [3], Hale [7], Krasovskii [8]).

For the case $p(t,x,x(t-r),y,y(t-r),z)\neq 0$, we establish the following theorem.

Theorem 2. Suppose that assumptions (i)-(ii) of Theorem 1 and the following condition hold:

$$|p(t,x,x(t-r),y,y(t-r),z)|\le q(t),$$

where $q\in L^1(0,\infty)$. Then, there exists a finite positive constant K such that the solution $x(t)$ of Eq. (1) defined by the initial functions

$$x(t)=\phi(t),x'(t)=\phi'(t),x''(t)=\phi''(t)$$

satisfies

$$|x(t)|\le K,\left|x'(t)\right|\le K,\left|x''(t)\right|\le K$$

for all $t \geq t_0$, where $\phi \in C^2([t_0 - r, t_0],\ \Re)$, provided

$$r < \min\left\{\frac{\alpha b_2}{\alpha(L+2c)+c},\ \frac{2\alpha}{BL(2+\alpha)+c}\right\}.$$

Proof. It is clear that under the assumptions of Theorem 2, the time derivative of functional $V(t, x_t, y_t, z_t)$ satisfies the following:

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \leq -\lambda_3\, y^2 - \lambda_4\, z^2 + (\alpha y + z)p(t, x, x(t-r), y, y(t-r), z).$$

Hence

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \leq D_5(|y| + |z|)q(t), \tag{6}$$

where $D_5 = \max\{1,\ \ \alpha\}$.

In view of the inequalitiy $|m| < 1 + m^2$, it follows from (6) that

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \leq D_5(2 + y^2 + z^2)\ q(t). \tag{7}$$

By (4) and (7), we get that

$$\begin{aligned}\tfrac{d}{dt}V(t, x_t, y_t, z_t)\ &\leq D_5(2 + D_4^{-1}V(t, x_t, y_t, z_t))q(t)\\ &= 2D_5 q(t) + D_5 D_4^{-1}V(t, x_t, y_t, z_t)q(t).\end{aligned}$$

Integrating the preceding inequality from 0 to t, using the assumption $q \in L^1(0, \infty)$ and the Gronwall-Reid-Bellman inequality, (see Ahmad and Rama Mohana Rao [1]), it follows that

$$\begin{aligned}V(t, x_t, y_t, z_t)\ &\leq V(0, x_0, y_0, z_0) + 2D_5 A + D_5 D_4^{-1}\int_0^t V(s, x_s, y_s, z_s)q(s)ds\\ &\leq \{V(0, x_0, y_0, z_0) + 2D_5 A\}\exp\left(D_5 D_4^{-1}\int_0^t q(s)ds\right)\\ &= \{V(0, x_0, y_0, z_0) + 2D_5 A\}\exp(D_5 D_4^{-1}A) = K_1 < \infty,\end{aligned} \tag{8}$$

where $K_1 > 0$ is a constant, $K_1 = \{V(0, x_0, y_0, z_0) + 2D_5 A\}\ \exp(D_5 D_4^{-1}A)$, and $A = \int_0^\infty q(s)ds$.

Thus, we have from (4) and (8) that

$$x^2 + y^2 + z^2 \leq D_4^{-1}V(t, x_t, y_t, z_t) \leq K,$$

where $K = K_1 D_4^{-1}$.

This fact completes the proof of Theorem 2.

Example. Consider nonlinear delay differential equation of third order:

$$\begin{aligned} &x'''(t) + \{11 + (1+t^2)^{-1}\}x''(t) + 2(1+e^{-t})x'(t-r) + 4x'(t) + x(t-r) \\ &= \tfrac{4}{1+t^2+x^2(t)+x^2(t-r)+x'^2(t)+x'^2(t-r)+x''^2(t)}. \end{aligned} \tag{9}$$

Delay differential Eq. (9) may be expressed as the following system:

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -\{11+(1+t^2)^{-1}\}z - 2(1+e^{-t})y - 4y - x \\ &+2(1+e^{-t})\int_{t-r}^{t} z(s)ds + \int_{t-r}^{t} y(s)ds \\ &+\tfrac{4}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)+z^2}. \end{aligned}$$

Clearly, Eq. (9) is special case of Eq. (1), and we have the following:

$$a(t) = 11 + \frac{1}{1+t^2} \geq 11 = 2\times 5 + 1,$$

$$\alpha = 5, a = 1,$$

$$b(t) = 1 + \frac{1}{e^t},$$

$$1 \leq 1 + \frac{1}{e^t} \leq 2,$$

$$\beta = 1, B = 2,$$

$$g_1(y) = 2y, g_1(0) = 0,$$

$$\frac{g_1(y)}{y} = 2 = b_1 > 1, (y \neq 0),$$

$$g_1'(y) = 2 = L,$$

$$\int_0^y g_1(\eta)d\eta = \int_0^y 2\eta d\eta = y^2,$$

$$g_2(y) = 4y, g_2(0) = 0,$$

$$\frac{g_2(y)}{y} = 4 = b_2, (y \neq 0),$$

$$h(x) = x, h(0) = 0, h'(x) = 1,$$

$$0 < 2^{-1} < h'(x) \le 1,$$

$$c_1 = 2^{-1}, c = 1,$$

$$a'(t) = -\frac{2t}{(1+t^2)^2}, (t \ge 0),$$

$$b'(t) = -\frac{1}{e^t}, (t \ge 0),$$

$$p(t, x, x(t-r), y, y(t-r), z)$$
$$= \frac{4}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)+z^2} \le \frac{4}{1+t^2} = q(t).$$

In view of the above discussion, it follows that

$$\alpha\beta - c = 4 > 0,$$

$$[\alpha b(t) - c]y^2 = [4 + 5e^{-t}]y^2, (t \ge 0),$$

$$\frac{\alpha}{2}a'(t)y^2 + b'(t)\int_0^y g(\eta)d\eta = -\left[\frac{5t}{(1+t^2)^2}\right]y^2 - e^{-t}y^2, (t \ge 0),$$

$$\begin{aligned}[\alpha b(t) - c]y^2 = [4 + 5e^{-t}]y^2 \quad &\ge -\left[\tfrac{5t}{(1+t^2)^2}\right]y^2 - e^{-t}y^2 \\ &= \tfrac{\alpha}{2}a'(t)y^2 + b'(t)\int_0^y g(\eta)d\eta,\end{aligned}$$

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{4}{1+s^2}ds = 2\pi < \infty,$$

that is, $q \in L^1(0, \infty)$ and

$$r < \min\left\{\frac{\alpha b_2}{\alpha(BL+2c)+c}, \ \frac{2\alpha}{BL(2+\alpha)+c}\right\} = \min\left\{\frac{4}{31}, \frac{10}{29}\right\} = \frac{4}{31}.$$

Thus all the assumptions of Theorems 1 and 2 hold. This shows that the null solution of Eq. (9) is stable and all solutions of the same equation are bounded, when $p(t, x, x(t-r)y, y(t-r), z) = 0$ and $\ne 0$, respectively.

References

[1] S. Ahmad; M. Rama Mohana Rao, Theory of ordinary differential equations. With applications in biology and engineering. Affiliated East-West Press Pvt. Ltd., New Delhi, 1999.

[2] A. U. Afuwape; M. O. Omeike, On the stability and boundedness of solutions of a kind of third order delay differential equations. *Appl. Math. Comput.* 200 (2008), no. 1, 444-451.

[3] T. A.Burton, Stability and periodic solutions of ordinary and functional-differential equations. Mathematics in Science and Engineering, 178. Academic Press, Inc., Orlando, FL, 1985.

[4] J. O. C. Ezeilo, On the stability of solutions of certain differential equations of the third order. *Quart. J. Math. Oxford Ser.* (2) 11 (1960), 64-69.

[5] J. O. C. Ezeilo, On the stability of the solutions of some third order differential equations. *J. London Math. Soc.* 43 (1968) 161-167.

[6] J. O. C. Ezeilo; H. O. Tejumola, Boundedness theorems for certain third order differential equations. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) 55 (1973), 194-201 (1974).

[7] J. Hale, Sufficient conditions for stability and instability of autonomous functional-differential equations. *J. Differential Equations* 1 (1965), 452-482.

[8] N. N. Krasovskii, Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay. Translated by J. L. Brenner Stanford University Press, Stanford, Calif. 1963.

[9] M. O. Omeike, Stability and boundedness of solutions of some non-autonomous delay differential equation of the third order. *An. Ştiint. Univ. Al. I. Cuza Iaşi. Mat.* (*N.S.*) 55 (2009), suppl. 1, 49-58.

[10] P. Ponzo, Some stability conditions for linear differential equations. *IEEE Transactions on Automatic Control*, 13 (6), (1968), 721-722.

[11] R. Reissig; G. Sansone; R. Conti, Non-linear Differential Equations of Higher Order, Translated from German. Noordhoff International Publishing, Leyden, 1974.

[12] A. I. Sadek, On the stability of solutions of some non-autonomous delay differential equations of the third order. *Asymptot. Anal.* 43 (2005), no. 1-2, 1-7.

[13] A. S. C. Sinha, On stability of solutions of some third and fourth order delay-differential equations.*Information and Control* 23 (1973), 165-172.

[14] K. E. Swick, Boundedness and stability for a nonlinear third order differential equation. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) 56 (1974), no. 6, 859-865.

[15] H. O. Tejumola; B. Tchegnani, Stability, boundedness and existence of periodic solutions of some third and fourth order nonlinear delay differential equations. *J. Nigerian Math. Soc.* 19, (2000), 9-19.

[16] C. Tunç, Uniform ultimate boundedness of the solutions of third-order nonlinear differential equations. *Kuwait J. Sci. Engrg.* 32 (2005), no. 1, 39-48.

[17] C. Tunç, Boundedness of solutions of a third-order nonlinear differential equation. *JIPAM. J. Inequal. Pure Appl. Math.* 6 (2005), no. 1, Article 3, 6 pp.

[18] C. Tunç, On the asymptotic behavior of solutions of certain third-order nonlinear differential equations. *J. Appl. Math. Stoch. Anal.* 2005, no. 1, 29-35.

[19] C. Tunç, New results about stability and boundedness of solutions of certain non-linear third-order delay differential equations. *Arab. J.* Sci. *Eng.* Sect. *A Sci.* 31 (2006), no. 2, 185-196.

[20] C. Tunç, On the boundedness of solutions of third-order differential equations with delay. (Russian) *Differ. Uravn.* 44 (2008), no. 4, 446-454, 574; translation in *Differ. Equ.* 44 (2008), no. 4, 464-472.

[21] C. Tunç, On the stability and boundedness of solutions of nonlinear vector differential equations of third order. *Nonlinear Anal.* 70 (2009), no. 6, 2232-2236.

[22] C. Tunç, On the boundedness of solutions of delay differential equations of third order. *Arab. J. Sci. Eng. Sect. A Sci.* 34 (2009), no. 1, 227-237.

[23] C. Tunç, Stability criteria for certain third order nonlinear delay differential equations. *Port. Math.* 66 (2009), no. 1, 71-80.

[24] C. Tunç, A new boundedness result to nonlinear differential equations of third order with finite lag. *Commun. Appl.Anal.* 13 (2009), no. 1, 1-10.

[25] C. Tunç, On the stability and boundedness of solutions to third order nonlinear differential equations with retarded argument. *Nonlinear Dynam.* 57 (2009), no.1-2, 97-106.

[26] Tunç, C., On the qualitative behaviors of solutions to a kind of nonlinear third order differential equations with a retarded argument. *An. Ştiint. Univ.* "*Ovidius*" *Constanta Ser. Mat.* 17 (2), (2009), 215-230.

[27] C. Tunç; M. Ateş, Stability and boundedness results for solutions of certain third order nonlinear vector differential equations. *Nonlinear Dynam.* 45 (2006), no. 3-4, 273-281.

[28] Y. F. Zhu, On stability, boundedness and existence of periodic solution of a kind of third order nonlinear delay differential system. *Ann. Differential Equations* 8(2), (1992), 249-259.

Faculty of Arts and Sciences, Department of Mathematics

Yüzüncü Yıl University, 65080, Van, TURKEY.

E-mail address: cemtunc@yahoo.com

(Received May 11, 2009)