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SOME RESULTS OF NONTRIVIAL SOLUTIONS FOR A NONLINEAR PDE IN SOBOLEV SPACE

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Abstract. In this study, we investigate the question of nonexistence of nontrivial solutions of the Robin problem

$$(P)\qquad \left| \begin{array}{l} -\dfrac{\partial^2 u}{\partial x^2} - \sum\limits_{s=1}^{n} \dfrac{\partial}{\partial y_s} a_s(y, \frac{\partial u}{\partial y_s}) + f(y,u) = 0 \text{ in } \Omega = \mathbb{R} \times D, \\ \\ u + \varepsilon \dfrac{\partial u}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial D. \end{array} \right.$$

where $a_s : D \times \mathbb{R} \to \mathbb{R}$ are H^1- functions with constant sign such that

$$(H_1)\qquad 2\int_0^{\xi_s} a_s(y,t_s)dt_s - \xi_s a_s(y,\xi_s) \le 0, s = 1, ..., n$$

and $f : D \times \mathbb{R} \to \mathbb{R}$ is a real continuous locally Liptschitz function such that

$$(H_2)\qquad 2F(y,u) - uf(y,u) \le 0,$$

We show that the function

$$E(x) = \int_D |u(x,y)|^2 \, dy$$

is convex on $\mathbb{R}$. Our proof is based on energy (integral) identities. $\left(D = \prod\limits_{k=1}^{n}]\alpha_k, \beta_k[, \varepsilon > 0 \text{ and } F(y,u) = \int\limits_0^u f(y,\tau)d\tau \right)$.

1. Introduction

The problem of existence and nonexistence of nontrivial solutions of problems of the form

$$\left| \begin{array}{l} -\Delta u + f(u) = 0 \text{ in } \Omega, \\ \\ u = 0 \text{ on } \partial\Omega, \end{array} \right.$$

has been investigated by many authors under various situations. Previous works have been reported by Berestycky, Gallouet & Kavian [1] , M. J. Esteban & P. L. Lions [2], Pucci & J. Serrin [9] and Pohozaev [10]. To illustrate some

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of the typical known results, let us consider Dirichlet problem

$$\left| \begin{array}{l} -\Delta u + f(u) = 0, u \in C^2(\overline{\Omega}), \\ \\ u = 0 \text{ on } \partial\Omega, \end{array} \right.$$

Under hypothesis

$$\begin{cases} \nabla u \in L^2(\Omega), \\ f(0) = 0, \\ F(u) = \int\limits_0^u f(s)ds \in L^1(\Omega), \end{cases}$$

where Ω is a connected unbounded domain of $\mathbb{R}^N$ such as

$$\exists \Lambda \in \mathbb{R}^N, \|\Lambda\| = 1, \langle n(x), \Lambda \rangle \geq 0 \text{ on } \partial\Omega, \langle n(x), \Lambda \rangle \neq 0,$$

($n(x)$ is the outward normal to $\partial\Omega$ at the point x) Esteban & Lions [2] established that the Dirichlet problem does not have nontrivial solutions.

Berestycky, Gallouet & Kavian [1] established that the problem

$$\left| \begin{array}{c} -\Delta u - u^3 + u = 0, \\ u \in H^2(\mathbb{R}^2) \end{array} \right.$$

admits a radial solution

This same solution satisfies

$$\left| \begin{array}{l} -\Delta u - u^3 + u = 0, \\ u \in H^2(]0, +\infty[\times \mathbb{R}) \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \{0\} \times \mathbb{R}, \end{array} \right.$$

this shows that analogous Esteban-Lions result for Neumann problems is not valid.

The Pohozaev identity published in 1965 for solutions of the Dirichlet problem proved absence of nontrivial solutions for some elliptic equations when Ω is a star shaped bounded domain in $\mathbb{R}^n$ and f a continuous function on $\mathbb{R}$ satisfying:

$$(n-2)F(u) - 2nuf(u) > 0,$$

where, $n = \dim \mathbb{R}^n$.

When

$$\Omega = J \times \omega,$$

where $J \subset \mathbb{R}$ is unbounded interval and $\omega \subset \mathbb{R}^n$ domain , Haraux & Khodja [3] established under the assumption

$$\begin{cases} f(0) = 0, \\ 2F(u) - uf(u) \leq 0, \end{cases}$$

if we assume that $u \in H^2(J \times \omega) \cap L^\infty(J \times \omega)$ is a solution of the problems

$$\left| \begin{array}{l} -\Delta u + f(u) = 0 \text{ in } \Omega, \\ \\ \left(u \text{ or } \frac{\partial u}{\partial n}\right) = 0 \text{ on } \partial\left(J \times \omega\right). \end{array} \right.$$

Then these two problems (Dirichlet and Neumann) do have only trivial solution.

When

$$f(u) = u\,(u+1)\,(u+2)\,,$$

and

$$\Omega = \mathbb{R} \times \,]0, a[\,(a < \pi)\,,$$

Neumann problem

$$\left| \begin{array}{l} -\Delta u + u\,(u+1)\,(u+2) = 0 \text{ in } \Omega, \\ \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{array} \right.$$

is still open.

In this work, let $a_i, i = 1, ..., n$ be a sequence in $H^1(D \times \mathbb{R})$ verifying

$$a_i(y, 0) = 0 \text{ in } D = \prod_{k=1}^{n}]\alpha_k, \beta_k[,$$

and $f : D \times \mathbb{R} \to \mathbb{R}$ a locally Lipschitz continuous function such that $f(y, 0) = 0$ in D, so that $u = 0$ is a solution of the equation

$$-\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^{n} \frac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right) + f(y, u) = 0 \text{ in } \ \Omega = \mathbb{R} \times D. \tag{1.1}$$

We assume that

$$u \in H^2(\Omega) \cap L^\infty(\Omega),$$

and satisfies

$$u(x, s) = 0, (x, s) \in \mathbb{R} \times \partial D \tag{1.2}$$

or

$$\frac{\partial u}{\partial n}(x, s) = 0, (x, s) \in \mathbb{R} \times \partial D \tag{1.3}$$

or

$$\left(u + \varepsilon \frac{\partial u}{\partial n}\right)(x, s) = 0, (x, s) \in \mathbb{R} \times \partial D \tag{1.4}$$

Let us denote by:

$$\Gamma = \mathbb{R} \times \partial D = \Gamma_{\alpha_1} \cup \Gamma_{\beta_1} \cup ... \cup \Gamma_{\alpha_n} \cup \Gamma_{\beta_n}\ ,$$

$$\left(\Gamma_{\mu_i} = \left\{\left(x, y_1, ...y_{i-1}, \mu_i, y_{i+1}, ..., y_n\right), x \in \mathbb{R}, 1 \leq i \leq n\right\}\right)$$

the boundary of Ω,

$n(x,s) = (0, n_1(x,s), ..., n_n(x,s))$, the outward normal to $\partial\Omega$ at the point (x,s) and

$\left(\dfrac{\partial^2 u(x,y)}{\partial y_i^2}\right)_{i=1,...,n}$ the second derivative of u with respect to y_i at point (x,y).

If $z \in \Omega$, $k = 1, 2, ...n$ and $\tau \in \{\alpha_1, \beta_1, \alpha_2, \beta_2, ..., \alpha_n, \beta_n\}$ one writes:

$$z := (x,y) = (x, y_1, ..., y_n)$$

$$z_k^\tau := (y_1, ..., y_{k-1}, \tau, y_{k+1}, ..., y_n),$$

$$dz_k^* := dy_1 ... dy_{k-1} dy_{k+1} ... dy_n,$$

$$\int_{\alpha_1}^{\beta_1} ... \int_{\alpha i_{-1}}^{\beta_{i-1}} \int_{\alpha_{i+1}}^{\beta_{i+1}} ... \int_{\alpha_n}^{\beta_n} f(x,y) dy_1 ... dy_{i-1} dy_{i+1} ... dy_n := \int_{D_i^*} f(x,y) dz_i^*.$$

The objective of this paper is to extend the results of [3], [5] to problems (1.1) – (1.2), (1.1) – (1.3) and (1.1) – (1.4).

2. Integral identities

We begin this section by giving an integral identity useful in the sequel.

Lemma 1. *Let*

$$a_i \in H^1(D \times \mathbb{R}), i = 1, ..., n$$

satisfy

$$a_i(., \xi_i) : \overline{D} \to \mathbb{R}, > 0 \text{ or } < 0, \forall \xi_i, i \in \{1, ..., n\},$$

and assume $f : D \times \mathbb{R} \to \mathbb{R}$ *a locally Lipschitz continuous function. Then any solution* $u \in H^2(\mathbb{R} \times D) \cap L^\infty(\mathbb{R} \times D)$ *of* (1.1) *satisfying* (1.4), *verifies for each* $x \in \mathbb{R}$ *and* $\varepsilon \neq 0$ *the integral identity*

$$\begin{aligned} &\int_D \left(-\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \sum_{i=1}^n A_i(y, \tfrac{\partial u}{\partial y_i}) + F(y,u)\right)(x,y)dy \\ &+\varepsilon \sum_{i=1}^n \int_{D_i^*} \left(A_i\left(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i})\right) + A_i(z_i^{\beta_i}, \left(-\varepsilon^{-1}\right) u(x, z_i^{\beta_i}))\right) dz_i^* = 0 \end{aligned} \tag{2.1}$$

Proof. Let

$$H : \mathbb{R} \to \mathbb{R}$$

the function defined by

$$H(x) = \int_D \left(-\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \sum_{i=1}^n A_i(y, \frac{\partial u}{\partial y_i}) + F(y,u)\right)(x,y)dy.$$

The hypotheses on u, a_i, $i = 1,...,n$ and f imply that H is absolutely continuous and thus differentiable almost everywhere on $\mathbb{R}$, we have
(2.2)

$$\frac{d}{dx}H(x) = \int_D \left(-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \sum_{i=1}^n a_i(y, \tfrac{\partial u}{\partial y_i})\frac{\partial^2 u}{\partial y_i \partial x} + f(y,u)\frac{\partial u}{\partial x}\right)(x,y)dy$$
$$= \int_D \left(-\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i}\left(a_i(y, \tfrac{\partial u}{\partial y_i})\right) + f(y,u)\right)\left(\frac{\partial u}{\partial x}\right)(x,y)dy$$
$$+\sum_{i=1}^n \int_{D_i^*} \left(a_i(z_i^{\beta_i}, \tfrac{\partial u}{\partial y_i}(x, z_i^{\beta_i}))\frac{\partial u}{\partial x}(x, z_i^{\beta_i}) - a_i(z_i^{a_i}, \tfrac{\partial u}{\partial y_i}(x, z_i^{\alpha_i}))\frac{\partial u}{\partial x}(x, z_i^{\alpha_i})\right) dz_i^*.$$

Indeed a simple use of Fubini's theorem and an integration by parts yields

$$\int_D a_i(y, \frac{\partial u}{\partial y_i})\left(\frac{\partial^2 u}{\partial y_i \partial x}\right)(x,y)dy = \int_{D_i^*}\left(\int_{\alpha_i}^{\beta_i} a_i(y, \frac{\partial u}{\partial y_i})\left(\frac{\partial^2 u}{\partial y_i \partial x}\right)dy_i\right)dz_i^*$$

$$= \int_{D_i^*}\left(\int_{\alpha_i}^{\beta_i} -\frac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right)\frac{\partial u}{\partial x}(x,y)\,dy_i\right)dz_i^*$$

$$+\int_{D_i^*}\left(a_i(z_i^{\beta_i}, \frac{\partial u}{\partial y_i})\left(\frac{\partial u}{\partial x}\right)(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u}{\partial y_i})\left(\frac{\partial u}{\partial x}\right)(x, z_i^{\alpha_i})\right)dz_i^*$$

$$= \int_D -\frac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right)\left(\frac{\partial u}{\partial x}\right)(x,y)\,dy$$

$$+\int_{D_i^*}\left(a_i(z_i^{\beta_i}, \frac{\partial u(x, z_i^{\beta_i})}{\partial y_i})\left(\frac{\partial u}{\partial x}\right)(x, z_i^{\beta_i}) - a_i(z_i^{a_i}, \frac{\partial u(x, z_i^{\alpha_i})}{\partial y_i})\left(\frac{\partial u}{\partial x}\right)(x, z_i^{\alpha_i})\right)dz_i^*.$$

By summing up these formulas with respect to i and substituting them in (2.2), one obtains

$$\frac{d}{dx}H(x) = \int_D \left(-\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right) + f(y,u)\right)\left(\frac{\partial u}{\partial x}\right)(x,y)\,dy$$

$$+\sum_{i=1}^n \int_{D_i^*}\left(a_i(z_i^{\beta_i}, \frac{\partial u(x, z_i^{\beta_i})}{\partial y_i})\frac{\partial u}{\partial x}(x, z_i^{\beta_i}) - a_i(z_i^{a_i}, \frac{\partial u(x, z_i^{\alpha_i})}{\partial y_i})\frac{\partial u}{\partial x}(x, z_i^{\alpha_i})\right)dz_i^*.$$

As u satisfies equation (1.1), the above expression reduces to
(2.3)

$$\frac{d}{dx}H(x) = \sum_{i=1}^n \int_{D_i^*}\left(a_i(z_i^{\beta_i}, \frac{\partial u(x, z_i^{\beta_i})}{\partial y_i})\frac{\partial u}{\partial x}(x, z_i^{\beta_i}) - a_i(z_i^{a_i}, \frac{\partial u(x, z_i^{\alpha_i})}{\partial y_i})\frac{\partial u}{\partial x}(x, z_i^{\alpha_i})\right)dz_i^*.$$

Now observe that $\left(u+\varepsilon\dfrac{\partial u}{\partial n}\right)(x,s)=0$ on $\partial\Omega$, is equivalent to

$$(2.4)\quad \begin{cases} (u-\varepsilon\dfrac{\partial u}{\partial y_i})\,(x,y_1,..,y_{i-1},\alpha_i,y_{i+1},..,y_n)=0 \\ & x\in\mathbb{R},\alpha_i<y_i<\beta_i. \\ (u+\varepsilon\dfrac{\partial u}{\partial y_i})\,(x,y_1,..,y_{i-1},\beta_i,y_{i+1},..,y_n)=0 \end{cases}$$

This allows to write formula (2.3) in the following form

$$\frac{d}{dx}H(x)=\sum_{i=1}^{n}\int_{D_i^*}\left(a_i(z_i^{\beta_i},(-\varepsilon^{-1})\,u(x,z_i^{\beta_i}))\frac{\partial u}{\partial x}(x,z_i^{\beta_i})-a_i(z_i^{a_i},\varepsilon^{-1}u(x,z_i^{\alpha_i}))\frac{\partial u}{\partial x}(x,z_i^{\alpha_i})\right)dz_i^*.$$

$$=-\varepsilon\sum_{i=1}^{n}\int_{D_i^*}\frac{\partial}{\partial x}\left(A_i(z_i^{\beta_i},-\varepsilon^{-1}u(x,z_i^{\beta_i}))+A_i\left(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i})\right)\right)dz_i^*,$$

i.e

$$\frac{d}{dx}\left(H(x)+\varepsilon\sum_{i=1}^{n}\int_{\mathbb{D}_i}\left(A_i\left(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i})\right)+A_i(z_i^{\beta_i},\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}))\right)dz_i^*\right)=0.$$

Integrating this expression,with respect to x one obtains

$$H(x)+\varepsilon\sum_{i=1}^{n}\int_{D_i^*}\left(A_i\left(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i})\right)+A_i(z_i^{\beta_i},\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}))\right)dz_i^*=const.$$

Since

$$u\,(x,y)\in H^2(\mathbb{R}\times D),$$

one must get

$$\int_{-\infty}^{+\infty}\left(H(x)+\varepsilon\sum_{i=1}^{n}\int_{D_i^*}\left(A_i\left(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i})\right)+A_i(z_i^{\beta_i},\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}))\right)dz_i^*\right)dx<\infty.$$

We conclude that the constant is null which is the desired result.

Lemma 2. *Let u be chosen as in Lemma1.1. If one assumes u to be solution of problems* $(1.1)-(1.3)$ *or* $(1.1)-(1.4)$*, then for each $x\in\mathbb{R}$, the solution u verifies*

$$(2.5)\qquad \int_D\left(-\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2+\sum_{i=1}^{n}A_i(y,\frac{\partial u}{\partial y_i})+F(y,u)\right)(x,y)dy=0.$$

Proof. To prove (2.5) it suffices to show that the second term of (2.1) vanishes if u verifies (1.2) or (1.3), i.e

$$\int\limits_{D_i^*} \left(A_i \left(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i}) \right) + A_i(z_i^{\beta_i}, \left(-\varepsilon^{-1}\right) u(x, z_i^{\beta_i})) \right) dz_i^* = 0.$$

If one supposes that $u(x,s) = 0$ for $(x,s) \in \mathbb{R} \times \partial D$, it is immediate, that

$$A_i \left(z_i^{\alpha_i}, 0 \right) = A_i(z_i^{\beta_i}, 0), \forall i = 1, ..., n.$$

Now if the boundary condition is $\dfrac{\partial u}{\partial n}(x,s) = 0$ for $(x,s) \in \mathbb{R} \times \partial D$, then

$$\frac{\partial u}{\partial n}(x,s) = \langle \nabla u.n \rangle (x,s) = 0 \ (x,s) \in \mathbb{R} \times \partial D,$$

i.e

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial x}(x, z_i^{\alpha_i}) = \dfrac{\partial u}{\partial x}(x, z_i^{\beta_i}) = 0 \\ \dfrac{\partial u}{\partial y_1}(x, z_1^{\alpha_1}) = \dfrac{\partial u}{\partial y_1}(x, z_1^{\beta_1}) = 0 \\ \quad\quad\quad\quad . \\ \quad\quad\quad\quad . \\ \quad\quad\quad\quad . \\ \dfrac{\partial u}{\partial y_i}(x, z_i^{\alpha_i}) = \dfrac{\partial u}{\partial y_i}(x, z_i^{\beta_i}) = 0 \\ \quad\quad\quad\quad . \\ \quad\quad\quad\quad . \\ \quad\quad\quad\quad . \\ \dfrac{\partial u}{\partial y_n}(x, z_n^{\alpha_n}) = \dfrac{\partial u}{\partial y_n}(x, z_n^{\beta_n}) = 0 \end{array}\right| , x \in \mathbb{R}, \alpha_i \leq y_i \leq \beta_i,$$

consequently

$$a_i(z_i^{a_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\alpha_i})) = a_i(z_i^{\beta_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\beta_i})) = 0, \forall i = 1, ..., n.$$

because

$$a_i(x, 0) = 0, \forall x \in \overline{D}, \forall i = 1, ..., n..$$

Finally one gets

$$A_i \left(z_i^{\alpha_i}, 0 \right) = A_i(z_i^{\beta_i}, 0) = 0, \forall i = 1, ..., n.$$

3. Main results

The goal of this section is to establish the nonexistence of nontrivial solutions to Robin problem.

Theorem 1. *Let $a_i, i = 1, ..., n$ and f satisfying respectively*

$$\begin{aligned} & a_i(., \xi_i) : \overline{D} \to \mathbb{R}, > 0 \text{ or } < 0, \\ & 2A_i(y, \xi_i) - a_i(y, \xi_i)\xi_i \leq 0, \end{aligned} \qquad \forall \xi_i, \forall i \in \{1, ..., n\} \tag{3.1}$$

$$2F(y, u) - uf(y, u) \leq 0 \, , \tag{3.2}$$

and assume

$$u \in H^2(\Omega) \cap L^\infty(\Omega)$$

to be a solution of $(1.1) - (1.4)$. *Then the function*

$$x \mapsto E(x) = \int_D |u(x, y)|^2 \, dy \text{ is convex on } \mathbb{R}.$$

Proof. To begin the proof, we see that almost everywhere in $\Omega = \mathbb{R} \times D$, we have

$$(u\frac{\partial^2 u}{\partial x^2})(x, y) = (\frac{1}{2}\frac{\partial^2}{\partial x^2}(u^2) - \left|\frac{\partial u}{\partial x}\right|^2)(x, y).$$

In fact by multiplying equation (1.1) by $\frac{u}{2}$ and integrating the new equation over D, we obtain

$$0 = \int_D \left(-\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right) + f(y, u)\right)\frac{u}{2}(x, y)dy$$

(3.3)

$$= \int_D \left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}(u^2) + \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 - \frac{1}{2}\sum_{i=1}^n \frac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right)u + \frac{u}{2}f(y, u)(x, y)\right)dy.$$

A simple use of Fubini's theorem and an integration by parts yields,

$$\int_D \frac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right)(u)(x, y)\, dy = \int_{D_i^*}(\int_{\alpha_i}^{\beta_i} \frac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right)u dy_i)dz_i^* =$$

$$= -\int_D a_i(y, \frac{\partial u}{\partial y_i})\frac{\partial u}{\partial y_i}(x, y)\, dy + \int_{D_i^*}(a_i(z_i^{\beta_i}, \frac{\partial u(x, z_i^{\beta_i})}{\partial y_i})u(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u(x, z_i^{\alpha_i})}{\partial y_i})u(x, z_i^{\alpha_i}))dz_i^*$$

Instead of (3.3), we obtain

$$\int_D \left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}(u^2) + \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{2}\sum_{i=1}^n a_i(y, \frac{\partial u}{\partial y_i})\frac{\partial u}{\partial y_i} + \frac{1}{2}uf(y, u)\right)(x, y)dy$$

$$= \frac{1}{2}\sum_{i=1}^{n}\int_{D_i^*}(a_i(z_i^{\beta_i},\frac{\partial u(x,z_i^{\beta_i})}{\partial y_i})u(x,z_i^{\beta_i}) - a_i(z_i^{\alpha_i},\frac{\partial u(x,z_i^{\alpha_i})}{\partial y_i})u(x,z_i^{\alpha_i}))dz_i^*$$

From (2.4) it follows that

$$\sum_{i=1}^{n}\int_{D_i^*}(a_i(z_i^{\beta_i},\frac{\partial u(x,z_i^{\beta_i})}{\partial y_i})u(x,z_i^{\beta_i}) - a_i(z_i^{\alpha_i},\frac{\partial u(x,z_i^{\alpha_i})}{\partial y_i})u(x,z_i^{\alpha_i}))dz_i^*$$

$$= \sum_{i=1}^{n}\int_{D_i^*}(a_i(z_i^{\beta_i},\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}))u(x,z_i^{\beta_i}) - a_i(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i}))u(x,z_i^{\alpha_i}))dz_i^*$$

$$= \sum_{i=1}^{n}\int_{D_i^*}(\frac{1}{-\varepsilon^{-1}}a_i(z_i^{\beta_i},-\varepsilon^{-1}u(x,z_i^{\beta_i}))\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}) - \frac{1}{\varepsilon^{-1}}a_i(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i}))\varepsilon^{-1}u(x,z_i^{\alpha_i}))dz_i^*$$

i.e

$$\int_D\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\left(u^2\right) + \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{2}\sum_{i=1}^{n}a_i(y,\frac{\partial u}{\partial y_i})\frac{\partial u}{\partial y_i} + \frac{1}{2}uf(y,u)\right)(x,y)dy$$

$$+\frac{\varepsilon}{2}\sum_{i=1}^{n}\int_{D_i^*}(a_i(z_i^{\beta_i},\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}))\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}) + a_i(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i}))\varepsilon^{-1}u(x,z_i^{\alpha_i}))dz_i^* = 0,$$

Combining this formula and (2.1) we obtain

$$\int_D\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\left(u^2\right) + \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{2}\sum_{i=1}^{n}a_i(y,\frac{\partial u}{\partial y_i})\frac{\partial u}{\partial y_i} + \frac{1}{2}uf(y,u)\right)(x,y)dy$$

$$+\frac{\varepsilon}{2}\sum_{i=1}^{n}\int_{D_i^*}(a_i(z_i^{\beta_i},\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}))\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}) + a_i(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i}))\varepsilon^{-1}u(x,z_i^{\alpha_i}))dz_i^*$$

$$= \int_D\left(-\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \sum_{i=1}^{n}A_i(y,\frac{\partial u}{\partial y_i}) + F(y,u)\right)(x,y)dy$$

$$+\varepsilon\sum_{i=1}^{n}\int_{D_i^*}(A_i\left(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i})\right) + A_i(z_i^{\beta_i},\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i})))dz_i^*$$

i.e

$$\int_D\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\left(u^2\right) + \left|\frac{\partial u}{\partial x}\right|^2\right)(x,y)dy$$

$$= \int_D\left(\sum_{i=1}^{n}(A_i(y,\frac{\partial u}{\partial y_i}) - \frac{1}{2}a_i(y,\frac{\partial u}{\partial y_i})\frac{\partial u}{\partial y_i}) + F(y,u) - \frac{1}{2}uf(y,u)\right)(x,y)dy$$

$$+\varepsilon\sum_{i=1}^{n}\int_{D_i^*}\left(A_i\left(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i})\right)-\frac{1}{2}a_i(z_i^{\alpha_i},\varepsilon^{-1}u(x,z_i^{\alpha_i}))\varepsilon^{-1}u(x,z_i^{\alpha_i})\right)dz_i^*$$

$$+\varepsilon\sum_{i=1}^{n}\int_{D_i^*}\left(A_i(z_i^{\beta_i},\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}))-\frac{1}{2}a_i(z_i^{\beta_i},\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i}))\left(-\varepsilon^{-1}\right)u(x,z_i^{\beta_i})\right)dz_i^*.$$

Hypotheses (3.1) and (3.2) imply that

$$\frac{d^2}{dx^2}\left(\int_D|u\left(x,y\right)|^2dy\right)\geq 4\int_D\left|\frac{\partial u}{\partial x}\left(x,y\right)\right|^2dy\geq 0,\forall x\in\mathbb{R}.$$

This completes the proof.

Remark 1. *The convexity of the function $E(x)$ on $\mathbb{R}$ implies the triviality of the solution $u(x,y)$ of the problem* $(1.1)-(1.4)$*.*

Theorem 2. *Let the function $a_i, i=1,...,n$ and f be as described as in Theorem 3.1. We assume $u\in H^2(\Omega)\cap L^\infty(\Omega)$ is a solution of* $(1.1)-(1.2)$ *or* $(1.1)-(1.3)$*, then the function $E(x)$ defined above is convex on $\mathbb{R}$.*

Proof. By similar arguments as in the proof of Theorem 3.1, we obtain

$$\int_D\left(-\frac{1}{4}\frac{\partial}{\partial x^2}^2\left(u^2\right)+\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2+\frac{1}{2}\sum_{i=1}^{n}a_i(y,\frac{\partial u}{\partial y_i})\frac{\partial u}{\partial y_i}+\frac{1}{2}uf(y,u)\right)(x,y)dy$$

$$=\frac{1}{2}\sum_{i=1}^{n}\int_{D_i}(a_i(z_i^{\beta_i},\frac{\partial u(x,z_i^{\beta_i})}{\partial y_i})u(x,z_i^{\beta_i})-a_i(z_i^{\alpha_i},\frac{\partial u(x,z_i^{\alpha_i})}{\partial y_i})u(x,z_i^{\alpha_i}))dz_i^*$$

Now if $u\left(x,s\right)=0$ or $\frac{\partial u}{\partial n}\left(x,s\right)=0$, for $(x,s)\in\mathbb{R}\times\partial D$ this formula reduces to

$$\int_D\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\left(u^2\right)-\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2+\frac{1}{2}\sum_{i=1}^{n}a_i(y,\frac{\partial u}{\partial y_i})\frac{\partial u}{\partial y_i}+\frac{1}{2}uf(y,u)\right)(x,y)dy=0$$

We can now employ (2.5) to transform this identity into the following form

$$\int_D\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\left(u^2\right)+\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2+\frac{1}{2}\sum_{i=1}^{n}a_i(y,\frac{\partial u}{\partial y_i})\frac{\partial u}{\partial y_i}+\frac{1}{2}uf(y,u)\right)(x,y)dy$$

$$=\int_D\left(-\frac{1}{2}\left|\frac{\partial u(x,y)}{\partial x}\right|^2+\sum_{i=1}^{n}A_i(y,\frac{\partial u(x,y)}{\partial y_i})+F(y,u(x,y))\right)dy$$

i.e

$$\int_D\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\left(u^2\right)+\left|\frac{\partial u}{\partial x}\right|^2\right)(x,y)dy$$

$$= \int_D \left(\sum_{i=1}^{n} \left(A_i(y, \frac{\partial u(x,y)}{\partial y_i}) - \frac{1}{2} a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial u}{\partial y_i} \right) + F(y,u) - \frac{1}{2} u f(y,u) \right) (x,y) dy.$$

Our assumptions on a_i, and f imply the desired result.

4. Applications

A practical tool for characterizing the assumption (3.1) or (3.2) of Theorem 3.1 is the following Proposition.

Proposition 1. *Let*

$$f : \mathbb{R} \to \mathbb{R}$$

a Lipschitzian real function such that

$$f(0) = 0.$$

We suppose that f is concave on $]-\infty, 0[$ and convex on $]0, +\infty[$. Then the function f satisfies the assumption (3.1) or (3.2) of Theorem 3.1.

***Application 4.1*:** Taking

$$a_i(y, \frac{\partial u(x,y)}{\partial y_i}) = \frac{\partial u(x,y)}{\partial y_i}$$

then the equation (1.1) becomes

$$-\Delta u + f(y,u) = 0 \text{ in } \Omega \tag{4.1}$$

***Application 4.2*:** We can put

$$a_i(y, \frac{\partial u}{\partial y_i}(x,y)) = c_i \frac{\partial u(x,y)}{\partial y_i}$$

with c_i are reals constants. In this case (1.1) can be rewritten as

$$-\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^{n} c_i \frac{\partial^2 u}{\partial y_i^2} + f(y,u) = 0 \text{ in } \Omega \tag{4.2}$$

***Application 4.3*:** We can also put

$$a_i(y, \frac{\partial u(x,y)}{\partial y_i}) = p_i(y) \frac{\partial u(x,y)}{\partial y_i}$$

with $p_i(y) < 0$ or > 0 in $\overline{D}$, it follows that the equation(1.1) is equivalent to

$$-\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left(p_i(y) \frac{\partial u}{\partial y_i} \right) + f(y,u) = 0 \text{ in } \Omega \tag{4.3}$$

We observe that in this three applications, we have

$$2A_i(y, \xi_i) - a_i(y, \xi_i)\, \xi_i \equiv 0, \forall \xi_i,\ i = 1, ..., n.$$

5. Examples

To conclude this work, let us give a few simple examples illustrating the use of Theorem 3.1.

Example 1. *The problem*

$$(5.1)\quad \left|\begin{array}{l} -\dfrac{\partial^2 u}{\partial x^2} - \sum\limits_{i=1}^{n} \dfrac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right) + \theta\,(y)\,|u|^{p-1}\, u = 0 \ \textit{in}\ \Omega = \mathbb{R} \times D \\ \\ (u + \varepsilon \dfrac{\partial u}{\partial n})(x, s) = 0, (x, s) \in \mathbb{R} \times \partial D \end{array}\right.$$

where

$$\theta : \overline{D} \to \mathbb{R},$$

is a nonnegative continuous real function, $p \geq 1$ does not have nontrivial solutions.

Indeed,

$$2F(y, u) - uf(y, u) = \theta\,(y)\,(\frac{2}{p+1} - 1)\,|u|^{p+1} \leq 0.$$

Theorem 3.1 give the desired result.

Example 2. *Let $\rho : \overline{D} \to \mathbb{R}$, be a continuous function . The problem*

$$(5.2)\quad \left|\begin{array}{l} -\dfrac{\partial^2 u}{\partial x^2} - \sum\limits_{i=1}^{n} \dfrac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right) + \rho\,(y)\, u = 0 \ \textit{in}\ \mathbb{R} \times D \\ \\ u + \varepsilon \frac{\partial u}{\partial n} = 0 \ \textit{on}\ \mathbb{R} \times \partial D \end{array}\right.$$

considered in $H^2(\mathbb{R} \times D) \cap L^\infty(\mathbb{R} \times D)$ does not have nontrivial solutions.

A simple calculation gives

$$2F(y, u) - uf(y, u) \equiv 0.$$

and

$$\frac{1}{4}\frac{d^2}{dx^2}\left(\int\limits_D (|u(x, y)|^2\, dx\right) = \int\limits_D \left(\left|\frac{\partial u}{\partial x}\right|^2 + F(y, u) - \frac{1}{2} uf(y, u)\right) dx$$

$$= \int\limits_D \left|\frac{\partial u}{\partial x}\right|^2 (x, y) dx \geq 0$$

Example 3. *Let*

$$\theta_1, \theta_2 : \overline{D} \to \mathbb{R},$$

be two continuous nonnegative functions, $p, q \geq 1$ and

$$f(y, u) = mu + \theta_1\,(y)\,|u|^{p-1}\, u + \theta_2\,(y)\,|u|^{q-1}\, u), m \in \mathbb{R}.$$

The problem

$$(5.3)\quad \left| \begin{array}{l} -\dfrac{\partial^2 u}{\partial x^2} - \sum\limits_{i=1}^{n} \dfrac{\partial}{\partial y_i}\left(a_i(y, \frac{\partial u}{\partial y_i})\right) + f(y,u) = 0 \ \ in\ \mathbb{R}\times D \\ \\ u + \varepsilon \dfrac{\partial u}{\partial n} = 0 \quad on\ \mathbb{R}\times \partial D \end{array} \right.$$

does not have nontrivial solutions.

It suffices to remark that,

$$2F(y,u) - uf(y,u) =$$

$$\theta_1\,(y)\,(\frac{2}{p+1} - 1)\,|u|^{p+1} + \theta_2\,(y)\,(\frac{2}{q+1} - 1)\,|u|^{q+1} \leq 0$$

and then apply theorem 3.1.

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