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BOUNDEDNESS AND EXPONENTIAL STABILITY FOR PERIODIC TIME DEPENDENT SYSTEMS

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ABSTRACT. The time dependent 2-periodic system

$$\dot{x}(t) = A(t)x(t), \; t \in \mathbb{R}, \quad x(t) \in \mathbb{C}^n \qquad (A(t))$$

is uniformly exponentially stable if and only if for each real number μ and each 2-periodic, $\mathbb{C}^n$-valued function f, the solution of the Cauchy Problem

$$\begin{cases} \dot{y}(t) = A(t)y(t) + e^{i\mu t}f(t), & t \in \mathbb{R}_+, \; y(t) \in \mathbb{C}^n \\ y(0) = 0 \end{cases}$$

is bounded. In this note we prove a result that has the above result as an immediate corollary. Some new characterizations for uniform exponential stability of $(A(t))$ in terms of the Datko type theorems are also obtained as corollaries.

1. INTRODUCTION

The concept of asymptotical stability is fundamental in the theory of ordinary and partial differential equations. In this way the stability theory leads to the real world applications. The recent advances of stability theory interact with spectral theory, harmonic analysis, modern topics of complex functions theory and also with control theory. This note begins with the following simple remark.

Let a be a complex number. The scalar differential equation

$$\dot{x}(t) = ax(t), \quad t \in \mathbb{R}$$

is asymptotically stable, i.e. $\lim_{t\to\infty} |e^{(t-t_0)a}| = 0$ for all $t_0 \in \mathbb{R}$, if and only if the real part of a is negative or if and only if for each real number μ and each complex scalar b, the solution of the Cauchy Problem

$$\begin{cases} \dot{y}(t) = ay(t) + e^{i\mu t}b, & t \in \mathbb{R}_+ \\ y(0) = 0. \end{cases}$$

is bounded.

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In appropriate terms, this nice and elementary result may be extended to the case when a is a quadratic matrix having complex entries and b is a vector in $\mathbb{C}^n$. The result can also be extended to the case when a is a bounded linear operator acting on a Banach space and b is a vector in this space. See e.g. [4].

The stability result for matrices has been recently generalized by the second author of this note for exponential dichotomy. See [5].

The similar problem for discrete time-dependent periodic systems in infinite dimensional Banach spaces has been developed in [1]. In fact, this note can be seen as a continuous finite dimensional version of this latter quoted paper.

2. Notations and Preliminary Results

By $\mathcal{X}$ we denote the Banach algebra of all quadratic matrices with complex entries endowed with the usual operatorial norm. An eigenvalue of a matrix $L \in \mathcal{X}$ is any complex scalar λ having the property that there exists a nonzero vector $v \in \mathbb{C}^n$ such that $Lv = \lambda v$. The spectrum of the matrix L, denoted by $\sigma(L)$, consists by all its eigenvalues. The resolvent set of L, denoted by $\rho(L)$, is the complement in $\mathbb{C}$ of $\sigma(L)$.

We begin with few lemmas which will be useful later.

Let $h_1, h_2 : [0,2] \to \mathbb{C}$ given by

$$h_1(u) = \begin{cases} u, \ \text{u} \in [0,1) \\ 2-u, \ \text{u} \in [1,2] \end{cases} \tag{1}$$

and

$$h_2(u) = u(2-u).$$

Lemma 1. *For each real number μ we have that*

$$I_1(\mu) := \int_0^2 h_1(u) e^{i\mu u} du = \frac{1}{\mu^2}[2e^{i\mu} - e^{2i\mu} - 1] \tag{2}$$

and

$$I_2(\mu) := \int_0^2 h_2(u) e^{i\mu u} du = e^{2i\mu}(2 - i\mu) - (2 + i\mu). \tag{3}$$

Moreover, $I_1(\mu) \neq 0$ if and only if μ is in the set $\mathbb{C} \setminus \{2k\pi : k \in \mathbb{Z}\}$ and $I_2(\mu) \neq 0$ for all $\mu \in \{2k\pi : k \in \mathbb{Z}\}$.

Proof. After an obvious calculation we can see that the equality (2) is fulfilled and thus $I_1(\mu) = 0$ if and only if

$$\begin{cases} 2\sin\mu - \sin 2\mu = 0 \\ 2\cos\mu - \cos 2\mu - 1 = 0. \end{cases}$$

This happens if and only if $\mu \in \{2k\pi : k \in \mathbb{Z}\}$. Using (3) we get $I_2(\mu) \neq 0$ for all $\mu \in \{2k\pi : k \in \mathbb{Z}\}$. □

Lemma 2. *Let L be a quadratic matrix of order $n \geq 1$ having complex entries. If*

$$\sup_{m\in\{1,2,3\dots\}} \|L^m\| = M < \infty$$

then each absolute value of the eigenvalue λ of L is less than or equal to 1.

Proof. Let λ be an eigenvalue of L. Suppose on contrary that $|\lambda| > 1$. Then there exists a non zero vector $x \in \mathbb{C}^n$ such that $Lx = \lambda x$. Therefore $L^m x = \lambda^m x$ for all $m = 1, 2, \dots$ and then

$$M \geq \|L^m\| \geq \frac{\|L^m x\|}{\|x\|} = |\lambda|^m \to \infty \quad when \quad m \to \infty.$$

This is a contradiction and the proof of Lemma 2 is finished. □

Lemma 3. *Let L be as above. If*

$$\sup_{N\in\{1,2,3\dots\}} \|I + L + \cdots + L^N\| = K < \infty \tag{4}$$

then 1 is not an eigenvalue of L.

Proof. Suppose $1 \in \sigma(L)$. Then $Lx = x$ for some non zero vector x in $\mathbb{C}^n$ and $L^k x = x,$ for all $k = 1, 2, \dots N$. Therefore

$$\begin{aligned} \sup_{N\in\{1,2,3\dots\}} \|I + L + \cdots + L^N\| &= \sup_{N\in\{1,2,3\dots\}} \sup_{\xi\neq 0} \frac{\|(I + L + \cdots + L^N)(\xi)\|}{\|\xi\|} \\ &\geq \sup_{N\in\{1,2,3\dots\}} \frac{N\|x\|}{\|x\|} = \infty, \end{aligned}$$

which is a contradiction . This completes the proof. □

Corollary 1. *Let T be a quadratic matrix of order $n \geq 1$ having complex entries. If for a real number μ, have that*

$$\sup_{N\in\{1,2,3\dots\}} \|I + e^{i\mu}T + \cdots + (e^{i\mu}T)^N\| = K(\mu) < \infty \tag{5}$$

then $e^{-i\mu}$ is not an eigenvalue of T.

Proof. We apply Lemma 3 for $L = e^{i\mu}T$. This yields that $1 \in \rho(e^{i\mu}T)$ and then $I - e^{i\mu}T$ is an invertible matrix. Equivalently $e^{i\mu}(e^{-i\mu}I - T)$ is an invertible matrix i.e. $e^{-i\mu} \in \rho(T)$. □

Corollary 2. *Let T be as above. If for each real number μ the inequality (4) is fulfilled then the spectrum of T lies in the interior of the circle of radius one.*

Proof. We use the identity

$$(I - e^{i\mu}T)(I + e^{i\mu}T + \cdots + (e^{i\mu}T)^{N-1}) = I - (e^{i\mu}T)^N.$$

Passing to the norm we get :

$$\begin{aligned} \|(e^{i\mu}T)^N\| &\leq 1 + \|(I - e^{i\mu}T)\|\|(I + e^{i\mu}T + \cdots + (e^{i\mu}T)^{N-1})\| \\ &\leq 1 + (1 + \|T\|)K(\mu). \end{aligned}$$

From Lemma 2 follows that the absolute value of each eigenvalue λ of $e^{i\mu}T$ is less than or equal to one and from Lemma 3, $e^{-i\mu}$ is in the resolvent set of T. □

The infinite dimensional version of Corollary 2 has been stated in [3].

3. Boundedness and exponential stability

Consider the homogenous time-dependent differential system

$$\dot{x} = A(t)x, \tag{$A(t)$}$$

where $A(t)$ is a 2-periodic continuous function, i.e. $A(t+2) = A(t)$ for all $t \in \mathbb{R}$. The choice of 2 as period is due to the method of the proof but the result may be preserved with arbitrary period $T > 0$ instead of 2. It is well-known that the system $(A(t))$ is *uniformly exponentially stable*, i.e. there exist two positive constants N and ν such that

$$\|\Phi(t)\Phi^{-1}(s)\| \leq Ne^{-\nu(t-s)} \quad \text{for all} \quad t \geq s,$$

if and only if the spectrum of the matrix $V := \Phi(2)$ lies inside of the circle of radius one. See e.g. [2], where even the infinite dimensional version of this result is stated.

It is natural to ask if the negativeness of all eigenvalues of $A(t)$ yields the exponential stability of the system $(A(t))$. We give here a counterexample, adapted from [6], in order to justify that the answer of the previous question is NO. Let us denote by $\Phi(t)$ the fundamental matrix of $(A(t))$ i.e. the unique solution of the operatorial Cauchy Problem

$$\begin{cases} \dot{X}(t) = A(t)X(t) \\ X(0) = I. \end{cases} \qquad (A(t), 0, I)$$

Let us consider the matrices:

$$D(t) := \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, A = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & \pi - 5 \\ -\pi & -1 \end{pmatrix}.$$

Define $A(t) = D(-\pi t)AD(\pi t)$ and $\Phi(t) = D(-\pi t)e^{tB}$. Then

$$\Phi(0) = I, \Phi'(t) = A(t)\Phi(t), \sigma(A(t)) = \{-1\} \text{ for all } t \in \mathbf{R}$$

and $\sigma(\Phi(2)) = \{e^{2\lambda_1}, e^{2\lambda_2}\}$, where $\lambda_1 = \rho - 1, \lambda_2 = -\rho - 1$ and $\rho^2 = \pi(5 - \pi)$. This shows that the system $(A(t))$ is not uniformly exponentially stable because $e^{2\lambda_1}$ is a real number greater than one. As a consequence of the uniqueness of the solution of the Cauchy Problem $(A(t), 0, I)$, have that $\Phi(2 + \tau) = \Phi(\tau)\Phi(2)$ for all $\tau \in \mathbb{R}$.

Let us consider also the vectorial non-homogenous Cauchy Problem

$$\begin{cases} \dot{y}(t) = A(t)y(t) + e^{i\mu t} f(t), \quad t \in \mathbb{R}_+ \\ y(0) = 0, \end{cases} \qquad (A(t), \mu, f(t), 0, 0)$$

where f is some continuous function. With $P_{2,0}(\mathbb{R}_+, \mathbb{C}^n)$ we shall denote the space consisting of all continuous and 2-periodic functions g with the property that $g(0) = 0$. We endow this space with the norm "sup". For each $k \in \{1, 2\}$ let us consider the set $\mathcal{A}_k$ consisting by all functions $f \in P_{2,0}(\mathbf{R}_+, \mathbb{C}^n)$ given for $t \in [0, 2]$ by $f(t) = \Phi(t)h_k(t)$.

Theorem 1. *The following two statements hold true.*

(i) *If the system $(A(t))$ is uniformly exponentially stable then for each continuous and bounded function f and each real number μ the solution of $(A(t), \mu, f, 0, 0)$ is bounded.*

(ii) *Let $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$. If for each $f \in \mathcal{A}$ and for each real number μ the solution of the Cauchy Problem $(A, \mu, f, 0, 0)$ is bounded then the system $(A(t))$ is uniformly exponentially stable.*

Proof. The solution ψ_f of $(A(t), \mu, f, 0, 0)$ is given by

$$\psi_f(t) = \int_0^t \Phi(t)\Phi^{-1}(s)e^{i\mu s} f(s)ds.$$

The assertion **(i)** is now an easy consequence of the following estimates.

$$\begin{aligned}\|\psi_f(t)\| &\leq \int_0^t \|\Phi(t)\Phi^{-1}(s)\|\|f(s)\|ds \\ &\leq \int_0^t Ne^{-\nu(t-s)}\|f(s)\|ds \\ &\leq Ne^{-\nu t}\int_0^t e^{\nu s}\|f(s)\|ds.\end{aligned}$$

Let $\sup_{\tau\in[0,2]} \|f(\tau)\| = M_f$. Then

$$\begin{aligned}\|\psi_f(t)\| &\leq Ne^{-\nu t}\int_0^t e^{\nu s}M_f ds \\ &= M_f\frac{N}{\nu}(1-e^{-\nu t}) \\ &\leq \frac{N}{\nu}M_f.\end{aligned}$$

Thus ψ_f is bounded.

The argument for the second statement is a bit more difficult.

Let $b \in \mathbb{C}^n$ and $f_1 \in P_{2,0}(\mathbb{R}_+, \mathbb{C}^n)$ given on $[0,2]$ by

$$f_1(\tau) = \begin{cases} \Phi(\tau)(\tau b), & \text{if } \tau \in [0,1) \\ \Phi(\tau)(2-\tau)b, & \text{if } \tau \in [1,2]. \end{cases}$$

and h_1 defined in (1). Then for each $\tau \in \mathbb{R}$ have that $f_1(\tau) = \Phi(\tau)h_1(\tau)b$. For each natural number n, one has

$$\begin{aligned}\psi_{f_1}(2n) &= \int_0^{2n} \Phi(2n)\Phi^{-1}(s)e^{i\mu s}f_1(s)ds \\ &= \sum_{k=0}^{n-1}\int_{2k}^{2k+2} \Phi(2n)\Phi^{-1}(s)e^{i\mu s}f_1(s)ds.\end{aligned}$$

Put $s = 2k+\tau$, and using the fact that $\Phi^{-1}(2k+\tau) = \Phi^{-1}(2k)\Phi^{-1}(\tau)$, we get

$$\begin{aligned}\psi_{f_1}(2n) &= \sum_{k=0}^{n-1}\int_0^1 \Phi(2n)\Phi^{-1}(2k+\tau)e^{2i\mu k}e^{i\mu\tau}f_1(\tau)d\tau \\ &= \sum_{k=0}^{n-1} e^{2i\mu k}\Phi(2n-2k)b\int_0^2 e^{i\mu\tau}h_1(\tau)d\tau.\end{aligned}$$

Let us denote

$$A_1 = \mathbb{C}\backslash\{2k\pi : k \in \mathbb{Z}\} \quad \text{and} \quad M_1(\mu) = \int_0^2 e^{i\mu\tau} h_1(\tau) d\tau.$$

We know that $M_1(\mu) \neq 0$ for every $\mu \in A_1$ and thus

$$\psi_{f_1}(2n)(M_1(\mu))^{-1} = \sum_{k=0}^{n-1} e^{2i\mu k}\Phi(2n-2k)b, \quad \text{for all} \quad \mu \in A_1. \tag{6}$$

Consider also the function h_2 defined by

$$h_2(\tau) = \tau(1-\tau), \quad \tau \in [0,2]$$

and the function $f_2 \in P_{2,0}(\mathbb{R}_+, \mathbb{C}^n)$ given on $[0,2]$ by the formula

$$f_2(\tau) = \Phi(\tau)h_2(\tau)b.$$

By the same procedure, we obtain

$$\psi_{f_2}(2n)(M_2(\mu))^{-1} = \sum_{k=0}^{n-1} e^{2i\mu k}\Phi(2n-2k)b, \mu \in \{2k\pi : k \in \mathbb{Z}\}, \tag{7}$$

where

$$M_2(\mu) = \int_0^2 e^{i\mu\tau} h_2(\tau) d\tau.$$

We know that ψ_{f_1} and ψ_{f_2} are bounded functions. Then there are two positive constants $K_1(\mu, f_1), K_2(\mu, f_2)$ such that

$$\|\psi_{f_1}(2n)\| \leq K_1(\mu, f_1) \;\text{ and }\; \|\psi_{f_2}(2n)\| \leq K_2(\mu, f_2) \;\text{ for all }\; n = 1, 2 \ldots.$$

From (6) follows that if $\mu \in A_1$ then

$$\left\| \sum_{k=0}^{n-1} e^{2i\mu k}\Phi(2n-2k)b \right\| \leq \frac{K_1(\mu, f_1)}{|M_1(\mu)|} = r_1(\mu, f_1)$$

and analogously using (7), if $\mu \in \{2k\pi : k \in \mathbb{Z}\}$ then we get

$$\left\| \sum_{k=0}^{n-1} e^{2i\mu k}\Phi(2n-2k)b \right\| \leq \frac{K_2(\mu, f_2)}{|M_2(\mu)|} = r_2(\mu, f_2).$$

Now for each real number μ and each $b \in \mathbb{C}^n$, the above inequalities yield

$$\left\| \sum_{k=0}^{n-1} e^{2i\mu k}\Phi(2n-2k)b \right\| \leq r_1(\mu, f_1) + r_2(\mu, f_2). \tag{8}$$

On the other hand replacing j by $n-k$, we get

$$\sum_{k=0}^{n-1} e^{2i\mu k}\Phi(2n-2k)b = e^{2i\mu n}\sum_{j=1}^{n} e^{-2i\mu j}\Phi(2j)b. \tag{9}$$

Following Uniform Boundedness Principle and using the relations (8) and (9) we can find a positive constant $L(\mu)$ such that

$$\left\|\sum_{j=1}^{n} e^{-2i\mu s}(\Phi(2))^j\right\| \le L(\mu) < \infty.$$

Now we can apply Corollary 2 for $T = \Phi(2)$ and can say that the spectrum of $\Phi(1)$ lies in the interior of the circle of radius one, i.e. the system $(A(t))$ is uniformly exponentially stable. This completes the proof. $\square$

Corollary 3. *The system $(A(t))$ is uniformly exponentially stable if and only if for each real number μ and each function f belonging to $P_{2,0}(\mathbb{R}_+, \mathbb{C}^n)$ the solution of $(A(t), \mu, f, 0, 0)$ is bounded.*

Using the periodicity of Φ and of f it is easy to see that the solution ψ_f is bounded if the sequence $(\psi_f(n))$ is bounded as well. If we return to (6) and (7) we should be able to recapture the inequality (8) under the assumption that for each vector b the series $(\sum_{j\ge 0} ||\Phi(2j)b||)$ is convergent. Then the following Corollary of Datko type may be stated as well.

Corollary 4. *With the above notations we have that the system $(A(t))$ is uniformly exponentially stable if and only if for each vector b the following inequality holds true.*

$$\sum_{j=1}^{\infty} ||\Phi(2j)b|| < \infty. \tag{10}$$

It is not difficult to see that the requirement (10) may be replaced by an apparently weaker one, namely with the inequality

$$\sum_{j=1}^{\infty} |\langle\Phi(2j)b, b\rangle| < \infty, \quad \forall b \in \mathbb{C}^n.$$

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