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Some existence results for boundary value problems of fractional semilinear evolution equations

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Abstract

In this paper, we study the existence of solutions for a two-point boundary value problem of fractional semilinear evolution equations in a Banach space. Our results are based on the contraction mapping principle and Krasnoselskii's fixed point theorem.

Keywords: Evolution equations of fractional order, boundary conditions, existence of solutions, fixed point theorem.
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1 Introduction

In some real world problems, fractional-order models are found to be more adequate than integer-order models. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For examples and details, see [1-13] and the references therein.
In this paper, we consider a two-point boundary value problem involving fractional semilinear evolution equations and prove some existence results in a Banach space. Precisely, we study the following boundary value problem:

$$\begin{cases} {}^cD^q x(t) = A(t)x(t) + f(t, x(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ \alpha x(0) + \beta x'(0) = \gamma_1, \qquad \alpha x(1) + \beta x'(1) = \gamma_2, \end{cases} \tag{1.1}$$

where ${}^cD$ is the Caputo fractional derivative, $A(t)$ is a bounded linear operator on X for each $t \in [0, 1]$ (the function $t \to A(t)$ is continuous in the uniform operator topology), $f : [0, 1] \times X \to X$ and $\alpha > 0$, $\beta \geq 0$, $\gamma_{1,2}$ are real numbers. Here, $(X, \|.\|)$ is a Banach space and $\mathcal{C} = C([0, 1], X)$ denotes the Banach space of all continuous functions from $[0, 1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|.\|_{\mathcal{C}}$.

2 Preliminaries

Let us recall some basic definitions [8, 11, 13] on fractional calculus.

Definition 2.1. For a function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^{c}D^{q}g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^{q}g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a function $g(t)$ is defined by

$$D^{q}g(t) = \frac{1}{\Gamma(n-q)} \Big(\frac{d}{dt}\Big)^{n} \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q]+1,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Now, we state a known result due to Krasnoselskii [14] which is needed to prove the existence of at least one solution of (1.1).

Theorem 2.1. Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that: (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

As argued in reference [1], the solution of the boundary value problem (1.1) can be written as

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(f(s, x(s)) + A(s)x(s)\Big) ds \\ &+ \int_0^1 \Big[\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big] \Big(f(s, x(s)) + A(s)x(s)\Big) ds \\ &+ \frac{1}{\alpha^2}[(\alpha(1-t) + \beta)\gamma_1 + (\beta + \alpha t)\gamma_2]. \end{aligned}$$

3 Main results

Theorem 3.1. Let $f : [0,1] \times X \to X$ be a jointly continuous function mapping bounded subsets of $[0,1] \times X$ into relatively compact subsets of X, and

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \ \forall t \in [0,1], \ x,y \in X.$$

Then the boundary value problem (1.1) has a unique solution provided

$$(L + A_1) \leq \frac{1}{2}\Big[\frac{\beta+2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big]^{-1},$$

where $A_1 = \max_{t\in[0,1]} \|A(t)\|$.

Proof. Define $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{F}x)(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(f(s,x(s)) + A(s)x(s)\Big)ds \\ &+ \int_0^1 \Big[\frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big]\Big(f(s,x(s)) + A(s)x(s)\Big)ds \\ &+ \frac{1}{\alpha^2}[(\alpha(1-t)+\beta)\gamma_1 + (\beta+\alpha t)\gamma_2], \ t \in [0,1].\end{aligned}$$

Setting $\sup_{t\in[0,1]} \|f(t,0)\| = M$ and choosing

$$r \geq 2\Big[M\Big(\frac{\beta+2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big) + \frac{\alpha+\beta}{\alpha^2}\Big(\gamma_1+\gamma_2\Big)\Big],$$

we show that $\mathcal{F}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned}\|(\mathcal{F}x)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(\|f(s,x(s))\| + \|A(s)\|\|x(s)\|\Big)ds \\ &+ \int_0^1 \Big|\beta-\alpha t\Big|\Big[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big]\Big(\|f(s,x(s))\| + \|A(s)\|\|x(s)\|\Big)ds \\ &+ \frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1|+|\gamma_2|\Big) \\ &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| + \|A(s)\|\|x(s)\|\Big)ds \\ &+ \int_0^1 \Big|\beta-\alpha t\Big|\Big[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big] \\ &\times \Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| + \|A(s)\|\|x(s)\|\Big)ds + \frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1|+|\gamma_2|\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq ((L+A_1)r+M)\Big[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&\quad+\int_0^1 \Big|\beta-\alpha t\Big|\Big(\frac{(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big)ds\Big]+\frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1|+|\gamma_2|\Big) \\
&= ((L+A_1)r+M)\Big[\frac{t^q}{\Gamma(q+1)}+|\beta-\alpha t|\Big(\frac{1}{\alpha\Gamma(q+1)}+\frac{\beta}{\alpha^2\Gamma(q)}\Big)\Big] \\
&+ \frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1|+|\gamma_2|\Big) \\
&\leq (L+A_1)[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}]r+M\Big[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big] \\
&+ \frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1|+|\gamma_2|\Big)\leq r.
\end{aligned}
$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
&\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\| \\
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|\Big(\|f(s,x(s))-f(s,y(s))\|+\|A(s)(x(s)-y(s)\|\Big)ds \\
&\quad+\int_0^1 \Big|\beta-\alpha t\Big|\Big[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big] \\
&\times \Big(\|f(s,x(s))-f(s,y(s))\|+\|A(s)(x(s)-y(s)\|\Big)ds \\
&\leq (L+A_1)\|x-y\|_{\mathcal{C}}\Big[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&\quad+\int_0^1 |\beta-\alpha t|\Big(\frac{(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big)ds\Big] \\
&\leq (L+A_1)\|x-y\|_{\mathcal{C}}\Big[\frac{t^q}{\Gamma(q+1)}+|\beta-\alpha t|\Big(\frac{1}{\alpha\Gamma(q+1)}+\frac{\beta}{\alpha^2\Gamma(q)}\Big)\Big] \\
&\leq (L+A_1)\Big[\frac{1}{\alpha\Gamma(q+1)}(2\alpha+\beta)+\frac{\beta^2+\alpha\beta)}{\alpha^2\Gamma(q)}\Big]\|x-y\|_{\mathcal{C}} \\
&\leq \Lambda_{\alpha,\beta,q,L,A_1}\|x-y\|_{\mathcal{C}},
\end{aligned}
$$

where $\Lambda_{\alpha,\beta,q,L,A_1} = (L+A_1)\Big[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big]$, which depends only on the parameters involved in the problem. As $\Lambda_{\alpha,\beta,q,L,A_1} < 1$, therefore $\mathcal{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle.

Theorem 3.2. Assume that $f : [0,1]\times X \to X$ is a jointly continuous function and maps bounded subsets of $[0,1]\times X$ into relatively compact subsets of X. Furthermore, assume that

($\mathbf{H_1}$) $\|f(t,x)-f(t,y)\| \leq L\|x-y\|,\ \forall t\in[0,1],\ x,y\in X;$

($\mathbf{H_2}$) $\|f(t,x)\| \leq \mu(t),\ \ \forall (t,x)\in[0,1]\times X,$ and $\mu \in L^1([0,1],R^+)$.

If $(L+A_1)(\frac{\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)})<1$, then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Let us fix

$$r \geq \frac{\|\mu\|_{L^1}[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}]+\frac{\alpha+\beta}{\alpha^2}(|\gamma_1|+|\gamma_2|)}{1-A_1[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}]},$$

and consider $B_r=\{x\in\mathcal{C}:\|x\|\leq r\}$. We define the operators Φ and Ψ on B_r as

$$(\Phi x)(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(f(s,x(s))+A(s)x(s)\Big)ds,$$

$$\begin{aligned}(\Psi x)(t) &= \int_0^1 \Big[\frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big]\Big(f(s,x(s))+A(s)x(s)\Big)ds \\ &+ \frac{1}{\alpha^2}[(\alpha(1-t)+\beta)\gamma_1+(\beta+\alpha t)\gamma_2].\end{aligned}$$

For $x,y\in B_r$, we find that

$$\|\Phi x+\Psi y\|\leq(\|\mu\|_{L^1}+A_1 r)\Big[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big]+\frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1|+|\gamma_2|\Big)\leq r.$$

Thus, $\Phi x+\Psi y\in B_r$. It follows from the assumption (H_1) that Ψ is a contraction mapping for

$$(L+A_1)\Big(\frac{\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big)<1.$$

The continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$\|\Phi x\|\leq\frac{(\|\mu\|_{L^1}+A_1 r)}{\Gamma(q+1)}.$$

Now we prove the compactness of the operator Φ. Setting $\Omega=[0,1]\times B_r$, we define $\sup_{(t,x)\in\Omega}\|f(t,x)\|=\overline{f}$, and consequently we have

$$\begin{aligned}&\|(\Phi x)(t_1)-(\Phi x)(t_2)\| \\ &=\Big\|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}](f(s,x(s))+A(s)x(s))ds \\ &+\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\Big\| \\ &\leq\frac{(\overline{f}+A_1 r)}{\Gamma(q+1)}|2(t_2-t_1)^q+t_1^q-t_2^q|,\end{aligned}$$

which is independent of x. So Φ is relatively compact on B_r. Hence, by the Arzela Ascoli Theorem, Φ is compact on B_r. Thus all the assumptions of Theorem 2.1 are satisfied and the conclusion of Theorem 2.1 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$.

Example. Consider the following boundary value problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = \frac{t}{20}x + \frac{1}{(t+5)^2}\frac{|x|}{1+|x|}, & t \in [0,1], \\ x(0) + x'(0) = 0, & x(1) + x'(1) = 0. \end{cases} \tag{3.1}$$

Here, $f(t,x(t)) = \frac{1}{(t+5)^2}\frac{|x|}{1+|x|}$, $A(t) = \frac{t}{20}$, $\alpha = 1$, $\beta = 1$, $\gamma_1 = 0 = \gamma_2$. Clearly $\|f(t,x) - f(t,y)\| \le L\|x-y\|$ with $L = \frac{1}{25}$ and $A_1 = \frac{1}{20}$. Further,

$$2(L + A_1)\Big(\frac{\beta + 2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2\Gamma(q)}\Big) = \frac{36}{25\sqrt{\pi}} < 1.$$

Thus, all the assumptions of Theorem 3.1 are satisfied. So, the conclusion of Theorem 3.1 applies and the boundary value problem (3.1) has a unique solution on $[0,1]$.

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