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PERIODIC BOUNDARY VALUE PROBLEMS OF SECOND ORDER RANDOM DIFFERENTIAL EQUATIONS

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Abstract

In this paper, an existence and the existence of extremal random solutions are proved for a periodic boundary value problem of second order ordinary random differential equations. Our investigations have been placed in the space of real-valued functions defined and continuous on closed and bounded intervals of real line together with the applications of the random version of a nonlinear alternative of Leray-Schauder type and an algebraic random fixed point theorem of Dhage [5]. An example is also indicated for demonstrating the realizations of the abstract theory developed in this paper.

Key words and phrases: Random differential equation; Periodic boundary conditions; Random solution; Existence theorem; Extremal solutions.
AMS (MOS) Subject Classifications: 47H40; 47N20; 60H25

1 Statement of the Problem

Let $\mathbb{R}$ denote the real line and let $J = [0, 2\pi]$ be a closed and bounded interval in $\mathbb{R}$. Let $C^1(J, \mathbb{R})$ denote the class of real-valued functions defined and continuously differentiable on J. Given a measurable space $(\Omega, \mathcal{A})$ and for a given measurable function $x : \Omega \to C^1(J, \mathbb{R})$, consider a second order periodic boundary value problem of ordinary random differential equations (in short PBVP)

$$\left.\begin{aligned} -x''(t,\omega) &= f(t, x(t,\omega), \omega) \;\text{ a.e. } t \in J \\ x(0,\omega) &= x(2\pi,\omega),\; x'(0,\omega) = x'(2\pi,\omega) \end{aligned}\right\} \tag{1}$$

for all $\omega \in \Omega$, where $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$.

By a *random solution* of the random PBVP (1) we mean a measurable function $x : \Omega \to AC^1(J, \mathbb{R})$ that satisfies the equations in (1), where $AC^1(J, \mathbb{R})$ is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on J.

The random PBVP (1) is new to the theory of periodic boundary value problems of ordinary differential equations. When the random parameter ω is absent, the random PBVP (1) reduces to the classical PBVP of second order ordinary differential equations,

$$\left.\begin{aligned} -x''(t) &= f(t, x(t)) \ \text{ a.e. } t \in J \\ x(0) &= x(2\pi), \ x'(0) = x'(2\pi) \end{aligned}\right\} \tag{2}$$

where, $f : J \times \mathbb{R} \to \mathbb{R}$.

The study of PBVP (2) has been discussed in several papers by many authors for different aspects of the solutions. See for example, Lakshmikantham and Leela [12], Leela [13], Nieto [14, 15], Yao [16], and the references therein. In this paper, we discuss the random PBVP (1) for existence as well as for existence of extremal solutions under suitable conditions of the nonlinearity f which thereby generalize several existence results of the PBVP (2) proved in the above mentioned paper. Our analysis rely on the random versions of nonlinear alternative of Leray-Schauder type (see Dhage [5, 6]) and an algebraic random fixed point theorem of Dhage [5].

The paper is organized as follows: In Section 2 we give some preliminaries and definitions needed in the sequel. The main existence result is given in Section 3, while the result on extremal solutions is given in Section 4. Finally, in Section 5, an example is presented to illustrate the abstract result developed in Section 3.

2 Auxiliary Results

Let E denote a Banach space with the norm $\|\cdot\|$ and let $Q : E \to E$. Then Q is called **compact** if $Q(E)$ is a relatively compact subset of E. Q is called **totally bounded** if $Q(B)$ is totally bounded subset of E for any bounded subset B of E. Q is called **completely continuous** if is continuous and totally bounded on E. Note that every compact operator is totally bounded, but the converse may not be true. However, both the notions coincide on bounded sets in the Banach space E.

We further assume that the Banach space E is separable, i.e., E has a countable dense subset and let β_E be the σ-algebra of Borel subsets of E. We say a mapping $x : \Omega \to E$ is measurable if for any $B \in \beta_E$,

$$x^{-1}(B) = \{\omega \in \Omega \mid x(\omega) \in B\} \in \mathcal{A}.$$

Similarly, a mapping $x : \Omega \times E \to E$ is called jointly measurable if for any $B \in \beta_E$, one has

$$x^{-1}(B) = \{(\omega, , x) \in \Omega \times E \mid x(\omega, x) \in B\} \in \mathcal{A} \times \beta_E,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σalgebras $\mathcal{A}$ and β_E those defined in Ω and E respectively. The details of the measurablity of the functions appears in Himmelberg

[9]. Note that a continuous map f from a Banach space E into itself is measurable, but the converse may not be true.

Let $Q : \Omega \times E \to E$ be a mapping. Then Q is called a random operator if $Q(\omega, x)$ is measurable in ω for all $x \in E$ and it is expressed as $Q(\omega)x = Q(\omega, x)$. In this case we also say that $Q(\omega)$ is a random operator on E. A random operator $Q(\omega)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $Q(\omega, x)$ is continuous (resp. compact, totally bounded and completely continuous) in x for all $\omega \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [10]. The study of random equations and their solutions have been discussed in Bharucha-Reid [1] and Hans [7] which is further applied to different types of random equations such as random differential and random integral equations etc. See Itoh [10], Bharucha-Reid [2] and the references therein. In this paper, we employ the following random nonlinear alternative in proving the main result of this paper.

Theorem 2.1 (Dhage [5, 6]) *Let U be a non-empty, open and bounded subset of the separable Banach space E such that $0 \in U$ and let $Q : \Omega \times \overline{U} \to E$ be a compact and continuous random operator. Further suppose that there does not exists an $u \in \partial U$ such that $Q(\omega)x = \alpha x$ for all $\omega \in \Omega$, where $\alpha > 1$ and ∂U is the boundary of U in E.. Then the random equation $Q(\omega)x = x$ has a random solution, i.e., there is a measurable function $\xi : \Omega \to E$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$.*

An immediate corollary to above theorem in applicable form is

Corollary 2.1 *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}}_r(0)$ be the open and closed balls centered at origin of radius r in the separable Banach space E and let $Q : \Omega \times \overline{\mathcal{B}}_r(0) \to E$ be a compact and continuous random operator. Further suppose that there does not exists an $u \in E$ with $\|u\| = r$ such that $Q(\omega)u = \alpha u$ for all $\omega \in \Omega$, where $\alpha > 1$. Then the random equation $Q(\omega)x = x$ has a random solution, i.e., there is a measurable function $\xi : \Omega \to \overline{\mathcal{B}}_r(0)$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$.*

The following theorem is often used in the study of nonlinear discontinuous random differential equations. We also need this result in the subsequent part of this paper.

Theorem 2.2 (Carathéodory) *Let $Q : \Omega \times E \to E$ be a mapping such that $Q(\cdot, x)$ is measurable for all $x \in E$ and $Q(\omega, \cdot)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \mapsto Q(\omega, x)$ is jointly measurable.*

The following lemma appears in Nieto [15] and is useful in the study of second order periodic boundary value problems of ordinary differential equations.

Lemma 2.1 *For any real number $m > 0$ and $\sigma \in L^1(J, \mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{array}{ll} -x''(t) + m^2 x(t) & = \sigma(t) \ \textit{a.e.}\ t \in J \\ x(0) = x(2\pi), & x'(0) = x'(2\pi) \end{array}\right\} \tag{3}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^{2\pi} G_m(t,s)\sigma(s)\, ds \tag{4}$$

where,

$$G_m(t,s) = \begin{cases} \dfrac{1}{2m(e^{2m\pi}-1)}\left[e^{m(t-s)} + e^{m(2\pi-t+s)}\right], & 0 \le s \le t \le 2\pi, \\ \\ \dfrac{1}{2m(e^{2m\pi}-1)}\left[e^{m(s-t)} + e^{m(2\pi-s+t)}\right], & 0 \le t < s \le 2\pi. \end{cases} \tag{5}$$

Notice that the Green's function G_m is continuous and nonnegative on $J \times J$ and the numbers

$$\alpha = \min\{\, |G_m(t,s)| \,:\, t,s \in [0, 2\pi]\,\} = \frac{e^{m\pi}}{m(e^{2m\pi}-1)}$$

and

$$\beta = \max\{\, |G_m(t,s)| \,:\, t,s \in [0, 2\pi]\,\} = \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}$$

exist for all positive real number m.

We need the following definitions in the sequel.

Definition 2.1 *A function $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ is called random Carathéodory if*

(i) the map $(t,\omega) \to f(t,x,\omega)$ is jointly measurable for all $x \in \mathbb{R}$, and

(ii) the map $x \to f(t,x,\omega)$ is continuous for all $t \in J$ and $\omega \in \Omega$.

Definition 2.2 *A function $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ is called random L^1-Carathéodory if*

(iii) for each real number $r > 0$ there is a measurable and bounded function $q_r : \Omega \to L^1(J, \mathbb{R})$ such that

$$|f(t,x,\omega)| \le q_r(t,\omega) a.\ e.t \in J$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}$ with $|x| \le r$.

Remark 2.1 If f is random L^1-Carathéodory on $J \times \mathbb{R} \times \Omega$, then the function $t \mapsto f(t,x,\omega)$ is Lebesgue integrable on J for all $\omega \in \Omega$ and $x \in \mathbb{R}$ with $|x| \le r$.

3 Existence Result

We seek the random solutions of PBVP (1) in the Banach space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We equip the space $C(J,\mathbb{R})$ with the supremum norm $\|\cdot\|$ defined by

$$\|x\| = \sup_{t\in J} |x(t)|.$$

It is known that the Banach space $C(J,\mathbb{R})$ is separable. By $L^1(J,\mathbb{R})$ we denote the space of Lebesgue measurable real-valued functions defined on $\mathbb{R}_+$. By $\|\cdot\|_{L^1}$ we denote the usual norm in $L^1(J,\mathbb{R})$ defined by

$$\|x\|_{L^1} = \int_0^{2\pi} |x(t)|\, dt.$$

For a given real number $m > 0$, consider the random PBVP,

$$\left.\begin{array}{c} -x''(t,\omega) + m^2 x(t,\omega) = f_m(t, x(t,\omega),\omega) \ \text{ a.e. } t \in J \\ x(0,\omega) = x(2\pi,\omega),\ x'(0,\omega) = x'(2\pi,\omega) \end{array}\right\} \tag{6}$$

for all $\omega \in \Omega$, where the function $f_m : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ is defined by

$$f_m(t,x,\omega) = f(t,x,\omega) + m^2 x.$$

Remark 3.1 We remark that the random PBVP (1) is equivalent to the random PBVP (6) and therefore, a random solution to the PBVP (6) implies the random solution to the PBVP (1) and vice versa.

The random PBVP (6) is equivalent to the random integral equation,

$$x(t,\omega) = \int_0^{2\pi} G_m(t,s) f_m(s, x(s,\omega),\omega)\, ds \tag{7}$$

for all $t \in J$ and $\omega \in \Omega$, where the function $G_m(t,s)$ is defined by (5).

We consider the following set of hypotheses in what follows:

(H_1) The function f_m is random Carathéodory on $J \times \mathbb{R} \times \Omega$.

(H_2) There exists a measurable and bounded function $\gamma : \Omega \to L^2(J,\mathbb{R})$ and a continuous and nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$ such that

$$|f_m(t,x,\omega)| \le \gamma(t,\omega)\psi(|x|)\text{a.e.}t \in J$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}$.

Our main existence result is

Theorem 3.1 *Assume that the hypotheses* (H_1)*-*(H_2) *hold. Suppose that there exists a real number* $r > 0$ *such that*

$$r > \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\|\gamma(\omega)\|_{L^1}\psi(r) \tag{8}$$

for all $\omega \in \Omega$*. Then the random PBVP (1) has a random solution defined on* J*.*

Proof. Set $E = C(J, \mathbb{R})$ and define a mapping $Q : \Omega \times E \to E$ by

$$Q(\omega)x(t) = \int_0^{2\pi} G_m(t,s) f_m(s, x(s,\omega), \omega)\, ds \tag{9}$$

for all $t \in J$ and $\omega \in \Omega$.

Since the map $t \mapsto G_m(t,s)$ is continuous on J, $Q(\omega)$ defines a mapping $Q : \Omega \times E \to E$. Define a closed ball $\overline{\mathcal{B}}_r(0)$ in E centered at origin 0 of radius r, where the real number r satisfies the inequality (8). We show that Q satisfies all the conditions of Corollary 2.1 on $\overline{\mathcal{B}}_r(0)$.

First we show that Q is a random operator on $\overline{\mathcal{B}}_r(0)$. Since $f_m(t,x,\omega)$ is random Carathéodory, the map $\omega \to f_m(t,x,\omega)$ is measurable in view of Theorem 2.2. Similarly, the product $G_m(t,s)f_m(s,x(s,\omega),\omega)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$\omega \to \int_0^{2\pi} G_m(t,s) f_m(s, x(s,\omega), \omega)\, ds = Q(\omega)x(t)$$

is measurable. As a result, Q is a random operator on $\Omega \times \overline{\mathcal{B}}_r(0)$ into E.

Next we show that the random operator $Q(\omega)$ is continuous on $\overline{\mathcal{B}}_r(0)$. Let $\{x_n\}$ be a sequence of points in $\overline{\mathcal{B}}_r(0)$ converging to the point x in $\overline{\mathcal{B}}_r(0)$. Then it is enough to prove that $\lim\limits_{n\to\infty} Q(\omega)x_n(t) = Q(\omega)x(t)$ for all $t \in J$ and $\omega \in \Omega$. By dominated convergence theorem, we obtain:

$$\begin{aligned}
\lim_{n\to\infty} Q(\omega)x_n(t) &= \lim_{n\to\infty} \textstyle\int_0^{2\pi} G_m(t,s) f_m(s, x_n(s,\omega), \omega)\, ds \\
&= \textstyle\int_0^{2\pi} G_m(t,s) \lim_{n\to\infty}[f_m(s, x_n(s,\omega), \omega)]\, ds \\
&= \textstyle\int_0^{2\pi} G_m(t,s) f_m(s, x(s,\omega), \omega)\, ds \\
&= Q(\omega)x(t)
\end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$. This shows that $Q(\omega)$ is a continuous random operator on $\overline{\mathcal{B}}_r(0)$.

Now, we show that $Q(\omega)$ is a compact random operator on $\overline{\mathcal{B}}_r(0)$. To finish, it is enough to prove that $Q(\omega)(\overline{\mathcal{B}}_r(0))$ is uniformly bounded and equi-continuous set in E

for each $\omega \in \Omega$. Since the map $\omega \mapsto \gamma(t,\omega)$ is bounded and $L^2(J,\mathbb{R}) \subset L^1(J,\mathbb{R})$, by hypothesis (H$_2$), there is a constant c such that $\|\gamma(\omega)\|_{L^1} \leq c$ for all $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed. Then for any $x : \Omega \to \overline{\mathcal{B}}_r(0)$, one has

$$
\begin{aligned}
|Q(\omega)x(t)| &\leq \int_0^{2\pi} G_m(t,s)|f_m(s,x(s,\omega),\omega)|\,ds \\
&\leq \int_0^{2\pi} G_m(t,s)\gamma(s,\omega)\psi(|x(s,\omega)|)\,ds \\
&\leq \int_0^{2\pi} G_m(t,s)\gamma(s,\omega)\psi(\|x(\omega)\|)\,ds \\
&\leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\left(\int_0^{2\pi}\gamma(s,\omega)\,ds\right)\psi(r) \\
&\leq K_1
\end{aligned}
$$

for all $t \in J$, where $K_1 = \dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\,c\,\psi(r)$. This shows that $Q(\omega)(\overline{\mathcal{B}}_r(0))$ is a uniformly bounded subset of E for each $\omega \in \Omega$.

Next, we show that $Q(\omega)(\overline{\mathcal{B}}_r(0))$ is an equi-continuous set in E. Let $x \in \overline{\mathcal{B}}_r(0)$ be arbitrary. Then for any $t_1, t_2 \in J$, one has

$$
\begin{aligned}
&|Q(\omega)x(t_1) - Q(\omega)x(t_1)| \\
&\leq \int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|\,|f_m(s,x(s,\omega),\omega)|\,ds \\
&\leq \int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|\,\gamma(s,\omega)\psi(|x(s,\omega)|)\,ds \\
&\leq \int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|\,\gamma(s,\omega)\psi(r)\,ds \\
&\leq \left(\int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|^2\,ds\right)^{1/2}\left(\int_0^{2\pi} |\gamma(s,\omega)|^2\,ds\right)^{1/2}\psi(r). \qquad (10)
\end{aligned}
$$

Hence for all $t_1, t_2 \in J$,

$$|Q(\omega)x(t_1) - Q(\omega)x(t_1)| \to 0 \text{ as } t_1 \to t_2,$$

uniformly for all $x \in \overline{\mathcal{B}}_r(0)$. Therefore, $Q(\omega)(\overline{\mathcal{B}}_r(0))$ is an equi-continuous set in E. As $Q(\omega)(\overline{\mathcal{B}}_r(0))$ is uniformly bounded and equi-continuous, it is compact by Arzelá-Ascoli theorem for each $\omega \in \Omega$. Consequently, $Q(\omega)$ is a completely continuous random operator on $\overline{\mathcal{B}}_r(0)$.

Finally, we prove that there does not exist an $u \in E$ with $\|u\| = r$ such that $Q(\omega)u(t) = \alpha u(t,\omega)$ for all $t \in J$ and $\omega \in \Omega$, where $\alpha > 1$. Suppose not. Then there exists such an element u in E satisfying $Q(\omega)u(t) = \alpha u(t,\omega)$ for some $\omega \in \Omega$. Let $\lambda = \frac{1}{\alpha}$. Then $\lambda < 1$ and $\lambda Q(\omega)u(t) = u(t,\omega)$ for some $\omega \in \Omega$. Now for this $\omega \in \Omega$, one has

$$\begin{aligned}
|u(t,\omega)| &\leq \lambda |Q(\omega)u(t)| \\
&\leq \int_0^{2\pi} G_m(t,s)|f_m(s,u(s,\omega),\omega)|\,ds \\
&\leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \int_0^{2\pi} \gamma(s,\omega)\psi(|u(s,\omega)|)\,ds \\
&\leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \int_0^{2\pi} \gamma(s,\omega)\psi(\|u(\omega)\|)\,ds \\
&\leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \|\gamma(\omega)\|_{L^1}\psi(\|u(\omega)\|)
\end{aligned}$$

for all $t \in J$. Taking supremum over t in the above inequality yields

$$\|u(\omega)\| \leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \|\gamma(\omega)\|_{L^1}\psi(\|u(\omega)\|)$$

or

$$r \leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \|\gamma(\omega)\|_{L^1}\psi(r)$$

for some $\omega \in \Omega$. This contradicts to the condition (8).

Thus, all the conditions of Corollary 2.1 are satisfied. Hence the random equation $Q(\omega)x(t) = x(t,\omega)$ has a random solution in $\overline{\mathcal{B}}_r(0)$, i.e., there is a measurable function $\xi : \Omega \to \overline{\mathcal{B}}_r(0)$ such that $Q(\omega)\xi(t) = \xi(t,\omega)$ for all $t \in J$ and $\omega \in \Omega$. As a result, the random PBVP (1) has a random solution defined on J. This completes the proof. □

4 Extremal Random Solutions

A closed set K of the Banach space E is called a cone if

(i) $K + K \subseteq K$,

(ii) $\lambda K \subset K$ for all $\lambda \in \mathbb{R}_+$, and

(iii) $\{-K\} \cap K = \{\theta\}$,

where θ is the zero element of E. We introduce an order relation $\leq$ in E with the help of the cone K in E as follows. Let $x, y \in E$, then we define

$$x \leq y \iff y - x \in K. \tag{11}$$

A cone K in the Banach space E is called normal, if the norm $\|\cdot\|$ is semi-monotone on K i.e., if $x, y \in K$, then $\|x+y\| \leq \|x\| + \|y\|$. Again a cone K is called regular, if every monotone order bounded sequence in E converges in norm. Similarly, a cone K is called fully regular, if every monotone norm bounded sequence converges in E. The details of different types of cones and their properties appear in Deimling [3] and Heikkilä and Lakshmikantham.

We introduce an order relation $\leq$ in $C(J,\mathbb{R})$ with the help of a cone K in it defined by

$$K = \{x \in C(J,\mathbb{R}) \mid x(t) \geq 0 \text{ for all}\, t \in J\}.$$

Thus, we have

$$x \leq y \Longleftrightarrow x(t) \leq y(t) \text{ for all}\, t \in J.$$

It is known that the cone K is normal in $C(J,\mathbb{R})$. For any function $a, b : \Omega \to C(J,\mathbb{R})$ we define a random interval $[a, b]$ in $C(J,\mathbb{R})$ by

$$\begin{aligned}[a,b] &= \{x \in C(J,\mathbb{R}) \mid a(\omega) \leq x \leq b(\omega)\ \forall\, \omega \in \Omega\} \\ &= \bigcap_{\omega \in \Omega} [a(\omega), b(\omega)].\end{aligned}$$

Definition 4.1 *An operator $Q : \Omega \times E \to E$ is called nondecreasing if $Q(\omega)x \leq Q(\omega)y$ for all $\omega \in \Omega$ and for all $x, y \in E$ for which $x \leq y$.*

We use the following random fixed point theorem of Dhage [4, 5] in what follows.

Theorem 4.1 (Dhage [4]) *Let $(\Omega, \mathcal{A})$ be a measurable space and let $[a, b]$ be a random order interval in the separable Banach space E. Let $Q : \Omega \times [a, b] \to [a, b]$ be a completely continuous and nondecreasing random operator. Then Q has a least fixed point x_* and a greatest random fixed point y^* in $[a, b]$. Moreover, the sequences $\{Q(\omega)x_n\}$ with $x_0 = a$ and $\{Q(\omega)y_n\}$ with $y_0 = a$ converge to x_* and y^* respectively.*

We need the following definitions in the sequel.

Definition 4.2 *A measurable function $a : \Omega \to C(J,\mathbb{R})$ is called a lower random solution for the PBVP (1) if*

$$\left.\begin{aligned} -a''(t,\omega) \leq f(t, a(t,\omega), \omega)\, a.e. t \in J \\ a(0,\omega) \leq a(2\pi,\omega),\ a'(0,\omega) = a'(2\pi,\omega) \end{aligned}\right\}$$

for all $\omega \in \Omega$. Similarly, a measurable function $b : \Omega \to C(J,\mathbb{R})$ is called an upper random solution for the PBVP (1) if

$$\left.\begin{aligned} -b''(t,\omega) \geq f(t, b(t,\omega), \omega)\, a.e. t \in J \\ b(0,\omega) \geq b(2\pi,\omega),\ b'(0,\omega) = b'(2\pi,\omega) \end{aligned}\right\}$$

for all $\omega \in \Omega$.

Note that a random solution for the random PBVP (1) is lower as well as upper random solution for the random PBVP (1) defined on J.

Remark 4.1 *We remark that lower and upper random solutions to the PBVP (1) implies respectively the lower and upper random solutions to the PBVP (6) on J and vice versa.*

Definition 4.3 *A random solution r_M for the random PBVP (1) is called maximal if for all random solutions of the random PBVP (1), one has $x(t,\omega) \leq r_M(t,\omega)$ for all $t \in J$ and $\omega \in \Omega$. Similarly, a minimal random solution to the PBVP (1) on J is defined.*

Remark 4.2 *We remark that maximal and minimal random solutions to the PBVP (1) implies respectively the maximal and minimal random solutions to the PBVP (6) on J and vice versa.*

Definition 4.4 *A function $f : J \times \mathbb{R} \times \Omega$ is called random* **Chandrabhan** *if*

(i) the map $(t,\omega) \mapsto f(t,x,\omega)$ is jointly measurable,

(ii) the map $x \mapsto f(t,x,\omega)$ is continuous and nondecreasing for all $t \in J$ and $\omega \in \Omega$.

Definition 4.5 *A function $f(t,x,\omega)$ is called random L^1-***Chandrabhan** *if for each real number $r > 0$ there exists a measurable function $q_r : \Omega \to L^1(J,\mathbb{R})$ such that for all $\omega \in \Omega$*

$$|f(t,x,\omega)| \leq q_r(t,\omega) a.e. t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

We consider the following set of assumptions:

(H_3) The function f_m is random Chandrabhan on $J \times \mathbb{R} \times \Omega$.

(H_4) The PBVP (1) has a lower random solution a and upper random solution b with $a \leq b$ on J.

(H_5) The function $q : J \times \Omega \to \mathbb{R}_+$ defined by

$$q(t,\omega) = |f_m(t,a(t,\omega),\omega)| + |f_m(t,b(t,\omega),\omega)|$$

is Lebesgue integrable in t for all $\omega \in \Omega$.

Remark 4.3 *If the hypotheses (H_3) and (H_5) hold, then for each $\omega \in \Omega$,*

$$|f_m(t,x(t,\omega),\omega)| \leq q(t,\omega)$$

for all $t \in J$ and $x \in [a,b]$ and the map $\omega \to q(t,\omega)$ is measurable on Ω.

Remark 4.4 Hypothesis (H_3) is natural and used in several research papers on random differential and integral equations (see Dhage [4, 5] and the references given therein). Hypothesis (H_4) holds, in particular, if there exist measurable functions $u, v : \Omega \to C(J, \mathbb{R})$ such that for each $\omega \in \Omega$,

$$u(t, \omega) \leq f(t, x, \omega) \leq v(t, \omega)$$

for all $t \in J$ and $x \in \mathbb{R}$. In this case, the lower and upper random solutions to the random PBVP (1) are given by

$$a(t, \omega) = \int_0^{2\pi} G_m(t, s) u(s, \omega)\, ds$$

and

$$b(t, \omega) = \int_0^{2\pi} G_m(t, s) v(s, \omega)\, ds$$

respectively, where $G_m(t, s)$ is associated with the PBVP (3) on J. The details about the lower and upper random solutions for different types of random differential equations may be found in Ladde and Lakshmikantham [11]. Finally, hypothesis (H_5) remains valid if the function f_m is L^1-Carathéodory on $J \times \mathbb{R} \times \Omega$.

Theorem 4.2 *Assume that the hypotheses (H_1), (H_3)-(H_5) hold. Then the PBVP (1) has a minimal random solution $x_*(\omega)$ and a maximal random solution $y^*(\omega)$ defined on J. Moreover,*

$$x_*(t, \omega) = \lim_{n \to \infty} x_n(t, \omega) \quad \text{and} \quad y^*(t, \omega) = \lim_{n \to \infty} y_n(t, \omega)$$

for all $t \in J$ and $\omega \in \Omega$, where the random sequences $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are given by

$$x_{n+1}(t, \omega) = \int_0^{2\pi} G_m(t, \omega) f_m(s, x_n(s, \omega), \omega)\, ds, \quad n \geq 0 \quad \textit{with} \quad x_0 = a,$$

and

$$y_{n+1}(t, \omega) = \int_0^{2\pi} G_m(t, \omega) f_m(s, y_n(s, \omega), \omega)\, ds, \quad n \geq 0 \quad \textit{with} \quad y_0 = b$$

for all $t \in J$ and $\omega \in \Omega$.

Proof. Set $E = C(J, \mathbb{R})$ and define an operator $Q : \Omega \times [a, b] \to E$ by (9). We show that Q satisfies all the conditions of Theorem 4.1 on $[a, b]$.

It can be shown as in the proof of Theorem 3.1 that Q is a random operator on $\Omega \times [a, b]$. We show that it is L^1-Chandrabhan. First we show that $Q(\omega)$ is nondecreasing on $[a, b]$. Let $x, y : \Omega \to [a, b]$ be arbitrary such that $x \leq y$ on Ω. Then,

$$\begin{aligned} Q(\omega)x(t) &\leq \int_0^{2\pi} G_m(t, s) f_m(s, x(s, \omega), \omega)\, ds \\ &\leq \int_0^{2\pi} G_m(t, s) f_m(s, y(s, \omega), \omega)\, ds \\ &= Q(\omega)y(t) \end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$. As a result, $Q(\omega)x \leq Q(\omega)y$ for all $\omega \in \Omega$ and that Q is nondecreasing random operator on $[a, b]$.

Secondly, by hypothesis (H_4),

$$\begin{aligned}
a(t,\omega) &\leq Q(\omega)a(t) \\
&= \int_0^{2\pi} G_m(t,s) f_m(s, a(s,\omega), \omega)\, ds \\
&= \int_0^{2\pi} G_m(t,s) f_m(s, x(s,\omega), \omega)\, ds \\
&= Q(\omega)x(t) \\
&\leq Q(\omega)b(t) \\
&= \int_0^{2\pi} G_m(t,s) f_m(s, b(s,\omega), \omega)\, ds \\
&\leq b(t,\omega)
\end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$. As a result Q defines a random operator $Q : \Omega \times [a, b] \to [a, b]$.

Next, since (H_5) holds, the hypothesis (H_2) is satisfied with $\gamma(t,\omega) = q(t,\omega)$ for all $(t,\omega) \in J \times \Omega$ and $\psi(r) = 1$ for all real number $r \geq 0$. Now it can be shown as in the proof of Theorem 3.1 that the random operator $Q(\omega)$ is completely continuous on $[a, b]$ into itself. Thus, the random operator $Q(\omega)$ satisfies all the conditions of Theorem 4.1 and so the random operator equation $Q(\omega)x = x(\omega)$ has a least and a greatest random solution in $[a, b]$. Consequently, the PBVP (1) has a minimal and a maximal random solution defined on J. This completes the proof. □

Remark 4.5 The conclusion of the Theorem 4.2 also remains true if we replace the hypotheses (H_3) and (H_5) with the following one.

(H_6) The function f_m is random L^1-Chandrabhan on $J \times \mathbb{R} \times \Omega$.

To see this, let hypothesis (H_6) hold. Since the cone K in $C(J, \mathbb{R})$ is normal, the random order interval $[a, b]$ is norm-bounded. Hence there is a real number $r > 0$ such that $\|x\| \leq r$ for all $x \in [a, b]$. Now f_m is L^1-Chandrabhan, so there is a measurable function $q_r : \Omega \to C(J, \mathbb{R})$ such that

$$|f_m(t, x, \omega)| \leq q_r(t,\omega) \text{a.e.} t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$ and for all $\omega \in \Omega$. Hence, hypotheses (H_3) and (H_5) hold with $q(t,\omega) = q_r(t,\omega)$ for all $t \in J$ and $\omega \in \Omega$.

5 An Example

Example 5.1 Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$ and let $J = [0, 2\pi]$ be a closed and bounded interval in $\mathbb{R}$. Given a measurable function $x : \Omega \to C(J, \mathbb{R})$, consider the following random PBVP

$$\left.\begin{array}{l} -x''(t,\omega) = -x(t,\omega) + \frac{t\,\omega^2\,x^2(t,\omega)}{\pi^2(1+\omega^2)[1+x^2(t,\omega)]} \text{ a.e. } t \in J \\ x(0,\omega) = x(2\pi,\omega), \;\; x'(0,\omega) = x'(2\pi,\omega) \end{array}\right\} \tag{12}$$

for all $\omega \in \Omega$.

Here,

$$f(t, x, \omega) = -x + \frac{t\,\omega^2\,x^2}{\pi^2(1+\omega^2)[1+x^2]}$$

so that taking $m = 1$, we obtain

$$f_m(t, x, \omega) = f_1(t, x, \omega) = \frac{t\,\omega^2\,x^2}{\pi^2(1+\omega^2)[1+x^2]}.$$

Clearly, the map $(t, \omega) \mapsto f_1(t, x, \omega)$ is jointly continuous for all $x \in \mathbb{R}$ and hence jointly measurable for all $x \in \mathbb{R}$. Also the map $x \mapsto f_1(t, x, \omega)$ is continuous for all $t \in J$ and $\omega \in \Omega$. Moreover,

$$\left| \frac{t\,\omega^2\,x^2}{\pi^2(1+\omega^2)[1+x^2]} \right| \leq \frac{t}{\pi^2} = \gamma(t,\omega)\psi(|x|)$$

where, $\gamma(t, \omega) = \dfrac{t}{\pi^2}$ for all $t \in [0, 2\pi]$ and $\psi(r) = 1$ for all real number $r \geq 0$. Now

$$\|\gamma(\omega)\|_{L^1} = \int_0^{2\pi} \gamma(t,\omega)\,dt = \frac{1}{\pi^2}\int_0^{2\pi} t\,dt = 2.$$

Therefore, if we take $r = 2$, then

$$r = 2 > \frac{e^{2\pi}+1}{e^{2\pi}-1} = \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\|\gamma(\omega)\|_{L^1}\psi(r)$$

for all $\omega \in (0, \infty)$ and so, the condition (8) of Theorem 3.1 is satisfied. Hence the random PBVP (12) has a random solution in the closed ball $\overline{\mathcal{B}}_2(0)$ and defined on $[0, 2\pi]$.

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