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Conditional oscillation of half-linear Euler-type dynamic equations on time scales

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Abstract. We investigate second-order half-linear Euler-type dynamic equations on time scales with positive periodic coefficients. We show that these equations are conditionally oscillatory, i.e., there exists a sharp borderline (a constant given by the coefficients of the given equation) between oscillation and non-oscillation of these equations. In addition, we explicitly find this so-called critical constant. In the cases that the time scale is $\mathbb{R}$ or $\mathbb{Z}$, our result corresponds to the classical results as well as in the case that the coefficients are replaced by constants and we take into account the linear equations. An example and corollaries are provided as well.

Keywords: time scale, dynamic equation, oscillation theory, conditional oscillation, oscillation constant, Euler equation, Riccati technique, half-linear equation.

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1 Introduction

In this paper, we analyse oscillatory properties of second-order half-linear Euler-type dynamic equation

$$\left[r(t)\Phi(y^{\Delta})\right]^{\Delta} + c(t)\Phi(y^{\sigma}) = 0, \qquad \Phi(y) = |y|^{p-1}\operatorname{sgn} y, \qquad p > 1, \tag{1.1}$$

on time scale $\mathbb{T}$ with

$$c(t) = \frac{\gamma s(t)}{t^{(p-1)}\sigma(t)}, \tag{1.2}$$

where $t^{(p)}$ is generalized power function (for the definition see below), the functions r, s are rd-continuous, positive, α-periodic with $\inf\{r(t), t \in \mathbb{T}\} > 0$ and $\gamma \in \mathbb{R}$ is an arbitrary constant.

The designation *half-linear equations* was used for the first time in [3] (concerning the case $\mathbb{T} = \mathbb{R}$). Motivation of this term comes from the fact that the solution space of these equations

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is homogeneous (likewise in the linear case), but it is not additive. This difference is one of the reasons, why some methods and tools from the theory of linear equations are not available for half-linear equations. Nevertheless, it appears that the behavior of half-linear equations is in many ways similar to the behavior of the linear equations, and many results are extendable. Among others, the Sturmian theory extends verbatim for half-linear equations, therefore we can classify equations as oscillatory and non-oscillatory. For full theory background and comprehensive literature overview, we refer to [1, 2, 8].

Actually, we are interested in the conditional oscillation of equation (1.1) with (1.2). It means that our aim is to prove that there exists a so-called critical constant, dependent only on coefficients r and s, which establishes a sharp borderline between oscillation and non-oscillation of these equations. More precisely, let us consider the equation

$$\left[\hat{r}(t)\Phi(y^{\Delta})\right]^{\Delta} + \hat{\gamma}d(t)\Phi(y^{\sigma}) = 0, \qquad \hat{\gamma} \in \mathbb{R}. \tag{1.3}$$

We say that equation (1.3) is conditionally oscillatory, if there exists a constant Γ (> 0) such that equation (1.3) is oscillatory if $\hat{\gamma} > \Gamma$ and non-oscillatory if $\hat{\gamma} < \Gamma$. Since the Sturmian theory (especially the comparison theorem) is valid in the theory of half-linear dynamic equations, conditionally oscillatory equations are good testing equations. E.g., let $r, \hat{r} \equiv 1$, and let d be an arbitrary positive rd-continuous function. Then equation (1.1) is oscillatory if $\liminf_{t\to\infty} c(t)/d(t) > \Gamma$ and non-oscillatory if $\limsup_{t\to\infty} c(t)/d(t) < \Gamma$ (see Corollary 4.1).

We note that the case $\gamma = \Gamma$ is resolved for differential equations (i.e., for $\mathbb{T} = \mathbb{R}$) as non-oscillatory. However, the oscillation behavior of the discrete equation ($\mathbb{T} = \mathbb{Z}$) for $\gamma = \Gamma$ is generally not known. Moreover, it can be shown that even differential equations cannot be generally classified as (non-)oscillatory in the critical case for larger classes of coefficients. We give references and more detailed description below (in the concluding remarks at the end of the paper).

Now, we give a short history and literature overview on conditional oscillation, where Euler (resp. Euler-type) equations play an important role. It was proved in 1893 by A. Kneser (see [16]), that the Euler differential equation

$$y''(t) + \frac{\gamma}{t^2}y(t) = 0 \tag{1.4}$$

is conditionally oscillatory with critical constant $\Gamma = 1/4$. The corresponding discrete result with the same critical constant $\Gamma = 1/4$ comes from the paper [17], which was published in 1959, and deals with the discrete version of Euler differential equation

$$\Delta^2 y_k + \frac{\gamma}{k(k+1)}y_{k+1} = 0. \tag{1.5}$$

The first natural step was to replace the constant coefficients in (1.4) and (1.5) by periodic ones. The continuous case

$$[r(t)y'(t)]' + \frac{\gamma s(t)}{t^2}y(t) = 0, \tag{1.6}$$

where r, s are positive continuous α-periodic functions, was solved in [21]. Later, in the paper [10] from 2012, the discrete result appeared for an Euler-type equation

$$\Delta[r_k \Delta y_k] + \frac{\gamma s_k}{k(k+1)}y_{k+1} = 0 \tag{1.7}$$

with almost periodic coefficients which covers the case of α-periodic positive sequences r_k, s_k. Lately, the results mentioned for equations (1.6) and (1.7) have been unified in [27] for the Euler-type dynamic equation with α-periodic positive coefficients

$$[r(t)y^{\Delta}]^{\Delta} + \frac{\gamma s(t)}{t\sigma(t)} y^{\sigma} = 0 \tag{1.8}$$

and critical oscillation constant

$$\Gamma = \frac{\alpha^2}{4} \left(\int_a^{a+\alpha} \frac{\Delta t}{r(t)} \right)^{-1} \left(\int_a^{a+\alpha} s(t)\, \Delta t \right)^{-1}.$$

Note that the results for equations (1.6) and (1.7) have been, during the last few years, obtained also for differential and difference half-linear equations, see [9, 11, 25, 26]. Of course, once we know the oscillation properties of Euler-type equations, we can use them together with many comparison theorems to study other types of equations. The basic results of this kind for dynamic equations considered in this paper are mentioned in Section 4.

Our aim is to prove that equation (1.1) with (1.2) is conditionally oscillatory. We will also find its critical constant Γ. Evidently, this result covers the mentioned linear (i.e., $p = 2$) case and results for equations (1.6), (1.7), (1.8). Moreover, it covers also the mentioned half-linear cases from [11, 25] for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. We note that in the literature one can find Euler-type half-linear dynamic equation in forms different from the one treated in this paper. More precisely, the potential (1.2) is sometimes considered with the standard power function in the denominator (i.e., $c(t) = \gamma s(t)/t^p$ or $c(t) = \gamma s(t)/(\sigma(t))^p$) or in differential form (see, e.g., [18]). Nevertheless, we have chosen the potential in the form of (1.2), because there is a direct correspondence with the difference as well as with differential equations and for $p = 2$ it corresponds to Euler-type dynamic equation (1.8).

The paper is organized as follows. The notion of time scales is recalled in the next section together with the definition of the generalized power function. The (non-)oscillation theory for half-linear dynamic equation with lemmas that we need in the rest of the paper can the reader find in Section 2 as well. Then, in Section 3, we formulate and prove the main result concerning the conditional oscillation of the mentioned Euler-type half-linear dynamic equation (1.1) with (1.2) and illustrate it with an example. The paper is finished by corollaries and concluding remarks given in Section 4.

2 Preliminaries

At the beginning, let us remind a notation on time scales. The theory of time scales was introduced by Stefan Hilger in his Ph.D. thesis in 1988, see [14], in order to unify the continuous and discrete calculus. Nowadays, it is well-known calculus and it is often studied in applications. Remind that a time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of reals. Note that $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$ (resp. $[a,\infty)_{\mathbb{T}} := [a,\infty) \cap \mathbb{T}$) stands for an arbitrary finite (resp. infinite) time scale interval. Symbols σ, ρ, μ, f^{σ}, f^{Δ}, and $\int_a^b f(t)\,\Delta t$ stand for the forward jump operator, backward jump operator, graininess, $f \circ \sigma$, Δ-derivative of f, and Δ-integral of f from a to b, respectively. Further, we use the symbols $C_{\mathrm{rd}}(\mathbb{T})$ and $C^1_{\mathrm{rd}}(\mathbb{T})$ for the class of rd-continuous and rd-continuous Δ-differentiable functions defined on the time scale $\mathbb{T}$. Recall that the time scale $\mathbb{T}$ is α-periodic if there exists constant $\alpha > 0$ such that if $t \in \mathbb{T}$ then $t \pm \alpha \in \mathbb{T}$. We note, that any α-periodic time scale $\mathbb{T}$ is infinite and, naturally, unbounded from above. For further information and background on time scale calculus, see [13], which is the initiating paper of the time scale theory, and the books [4, 5], which contain a lot of information on time scale calculus.

For further reading, it is necessary to remind a definition of n-th composition of operator ρ, see also [4]. We define

$$\rho^{-1}(t) := \sigma(t),\ \rho^0(t) := t,\ \ \rho^1(t) := \rho(t),\ \rho^2(t) := \rho(\rho(t)), \dots,\ \rho^n(t) = \rho(\rho^{n-1}(t)).$$

If $-\infty < a = \min \mathbb{T}$, then we define $\rho^n(a) = a$ for each $n \in \mathbb{N}$.

Definition 2.1 (Generalized power function with natural exponent)**.** For arbitrary $t \in \mathbb{T}$ and $p \in \mathbb{N}$, we define the generalized power function on time scales as

$$t^{(p)} := t\rho(t)\cdots\rho^{p-1}(t).$$

For $p = 0$, we define $t^{(0)} := 1$.

The following definition naturally extends the previous one for arbitrary real $p \geq 0$.

Definition 2.2 (Generalized power function with real exponent)**.** Let $p \in \mathbb{R}$ and $\lfloor p \rfloor$ denote the greatest integer less then or equal to p (the floor function). For arbitrary $t \in \mathbb{T}$ and $p \geq 0$, we define the generalized power function on time scales as

$$t^{(p)} := t^{(\lfloor p \rfloor)} \left\{ \left(\rho^{\lfloor p-1 \rfloor}(t)\right)^{1-p+\lfloor p \rfloor} \cdot \left(\rho^{\lfloor p \rfloor}(t)\right)^{p-\lfloor p \rfloor} \right\}^{p-\lfloor p \rfloor}.$$

Example 2.3. Let us illustrate the generalized power function with two simple examples involving the backward and the forward jump operator, respectively.

(i) $t^{(7/3)} = t^{(2)} \left\{(\rho(t))^{2/3} \cdot (\rho^2(t))^{1/3}\right\}^{1/3} = t \cdot (\rho(t))^{11/9} \cdot (\rho^2(t))^{1/9}$,

(ii) $t^{(3/4)} = \left\{(\sigma(t))^{1/4} \cdot (t)^{3/4}\right\}^{3/4} = (\sigma(t))^{3/16} \cdot (t)^{9/16}$.

Note that for $\mathbb{T} = \mathbb{R}$ we get the "classic" power function and for $\mathbb{T} = \mathbb{Z}$, $p \in \mathbb{N}$, we get generalized discrete power function (also called the "falling factorial power"), see, e.g., [15, Chapter 2]. In the following, we show some properties of the generalized power function, which will be useful later.

Lemma 2.4. *Let $\mathbb{T}$ be an α-periodic time scale and $p \geq 0$. Then the function $f(p) = t^{(p)}$ is continuous and increasing in p for large $t \in \mathbb{T}$ and*

$$\lim_{t\to\infty} \frac{t^{(p)}}{t^p} = 1. \tag{2.1}$$

Proof. For the sake of clarity, we will use $p \in [1,2]$ in the first part of the proof and $p \in [1,2)$ in the second part. Nevertheless, for any other intervals $[k, k+1]$ and $[k, k+1)$, $k \in \mathbb{N} \cup \{0\}$, it can be verified analogously.

Let $p \in [1,2]$. We show a continuity from the right-side in a point $p = 1$ and a continuity from the left-side in a point $p = 2$ (for any other $p \in (1,2)$ the continuity is obvious):

$$\lim_{p\to1+} t^{(p)} = t \lim_{p\to1+} \left\{ t^{2-p} \cdot (\rho(t))^{p-1} \right\}^{p-1} = t = t^{(1)}$$

and
$$\lim_{p\to 2-} t^{(p)} = t \lim_{p\to 2-} \left\{ t^{2-p} \cdot (\rho(t))^{p-1} \right\}^{p-1} = t\rho(t) = t^{(2)}.$$

Next, we show that f is increasing for $p \in [1,2)$. Let $p_1, p_2 \in [1,2)$, $p_1 < p_2$. On the contrary, let $t^{(p_1)} > t^{(p_2)}$, i.e.,
$$\left\{ t^{2-p_1} \cdot (\rho(t))^{p_1-1} \right\}^{p_1-1} > \left\{ t^{2-p_2} \cdot (\rho(t))^{p_2-1} \right\}^{p_2-1}.$$

It is easy to see that the last inequality can be written in the form
$$t^{p_1-p_2} \cdot (t/\rho(t))^{(p_1-p_2)(2-p_1-p_2)} > 1. \tag{2.2}$$

Hence, for the arbitrary fixed p_1 and p_2, we can see that $t^{p_1-p_2} \to 0$ as $t \to \infty$ and
$$(t/\rho(t))^{(p_1-p_2)(2-p_1-p_2)} \to 1 \qquad \text{as } t \to \infty,$$
thus the inequality (2.2) is not valid for large $t \in \mathbb{T}$ and we get a contradiction.

Finally, for arbitrary fixed $p \in [1,2)$, we show that (2.1) holds. Let $p \in [1,2)$, then
$$\frac{t^{(p)}}{t^p} = \frac{t\left\{ t^{2-p} \cdot (\rho(t))^{p-1} \right\}^{p-1}}{t^p} = \frac{t\left\{ t^{2-p} \cdot t^{p-1}[1-(\mu(t)/t)]^{p-1} \right\}^{p-1}}{t^p} = [1-(\mu(t)/t)]^{(p-1)^2}.$$

Hence, in view of $\mu(t)/t \to 0$ as $t \to \infty$ (due to $\mu(t) \le \alpha$ for every t), we get (2.1). □

Now, we recall basic elements of the oscillation theory of dynamic equations on time scales. Throughout this paper, we assume that the time scale $\mathbb{T}$ is α-periodic, which implies $\sup \mathbb{T} = \infty$. Consider the second order half-linear dynamic equation
$$[r(t)\Phi(y^\Delta)]^\Delta + c(t)\Phi(y^\sigma) = 0, \qquad \Phi(y) = |y|^{p-1}\operatorname{sgn} y, \qquad p > 1, \tag{2.3}$$
on a time scale $\mathbb{T}$, where $c, r \in C_{\mathrm{rd}}(\mathbb{T})$ and $\inf\{r(t),\, t \in \mathbb{T}\} > 0$. We note that $\Phi^{-1}(y) = |y|^{q-1}\operatorname{sgn} y$, where $q > 1$ is the conjugate number of p, i.e., $p + q = pq$. It is easy to see that any solution y of (2.3) satisfies $r\,\Phi(y^\Delta) \in C^1_{\mathrm{rd}}(\mathbb{T})$.

Further, we note that it is not sufficient to assume only $r(t) > 0$ (instead of $\inf\{r(t),\, t \in \mathbb{T}\} > 0$), because it may happen that $\lim_{t\to t_0-} r(t) = 0$ and $r(t_0) > 0$, which would not be convenient in our case. Indeed, we need $1/r \in C_{\mathrm{rd}}(\mathbb{T})$ due to the integration of $1/r^{q-1}(t)$, which is now fulfilled, see also [19], where this and similar problems are discussed.

Definition 2.5. We say that a nontrivial solution y of (2.3) has a *generalized zero* at t if
$$r(t)y(t)y(\sigma(t)) \le 0.$$

If $y(t) = 0$, we say that solution y has a *common zero* at t (the common zero is a special case of the generalized zero).

Definition 2.6. We say that a solution y of equation (2.3) is *non-oscillatory* on $\mathbb{T}$ if there exists $\tau \in \mathbb{T}$ such that there does not exist any generalized zero at t for $t \in [\tau, \infty)_{\mathbb{T}}$. Otherwise, we say that it is *oscillatory*.

Remark 2.7. Oscillation may be equivalently defined as follows. A nontrivial solution y of (2.3) is called oscillatory on $\mathbb{T}$, if y has a generalized zero on $[\tau, \infty)_{\mathbb{T}}$ for every $\tau \in \mathbb{T}$.

From the Sturm-type separation theorem (see, e.g., [20]) it follows that if one solution of (2.3) is oscillatory (resp. non-oscillatory), then every solution of (2.3) is oscillatory (resp. non-oscillatory). Hence we can speak about *oscillation* or *non-oscillation* of equation (2.3).

Next, let us recall the well known Sturm-type comparison theorem, which will be useful later.

Theorem 2.8 (Sturm-type comparison theorem [20, p. 388])**.** *Consider the equation*

$$[R(t)\Phi(y^{\Delta})]^{\Delta} + C(t)\Phi(y^{\sigma}) = 0 \tag{2.4}$$

and equation (2.3)*, where* $R, C \in C_{\mathrm{rd}}(\mathbb{T})$ *with* $\inf\{|R(t)|,\, t \in \mathbb{T}\} > 0$*.*

(i) *Let* $R(t) \geq r(t)$ *and* $C(t) \leq c(t)$ *for every* $t \in \mathbb{T}$*. If* (2.3) *is non-oscillatory then* (2.4) *is also non-oscillatory.*

(ii) *Let* $R(t) \leq r(t)$ *and* $C(t) \geq c(t)$ *for every* $t \in \mathbb{T}$*. If* (2.3) *is oscillatory then* (2.4) *is also oscillatory.*

Our approach to the oscillatory and non-oscillatory problems of (2.3) is based mainly on the application of the generalized Riccati dynamic equation

$$w^{\Delta}(t) + c(t) + \mathcal{S}[w, r, \mu](t) = 0, \tag{2.5}$$

where

$$\mathcal{S}[w, r, \mu] = \lim_{\lambda \to \mu} \frac{w}{\lambda}\left(1 - \frac{r}{\Phi(\Phi^{-1}(r) + \lambda\Phi^{-1}(w))}\right).$$

It is not difficult to observe that

$$\mathcal{S}[w, r, \mu](t) = \begin{cases} \left\{\frac{p-1}{\Phi^{-1}(r)}|w|^{q}\right\}(t) & \text{at right-dense } t, \\ \left\{\frac{w}{\mu}\left(1 - \frac{r}{\Phi(\Phi^{-1}(r)+\mu\Phi^{-1}(w))}\right)\right\}(t) & \text{at right-scattered } t. \end{cases}$$

Note that using the Lagrange mean value theorem on time scales (see, e.g., [5]), one can show that the operator $\mathcal{S}$ can be written in the form

$$\mathcal{S}[w, r, \mu](t) = \frac{(p-1)|w(t)|^{q}|\eta(t)|^{p-2}}{\Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t))]}, \tag{2.6}$$

where $\eta(t)$ is between $\Phi^{-1}(r(t))$ and $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t))$. The form (2.6) will be convenient for our purpose.

The relation between (2.3) and (2.5) is the following. If $y(t)$ is a solution of (2.3) with $y(t)y^{\sigma}(t) \neq 0$ for $t \in [t_1, t_2]_{\mathbb{T}}$ and we denote

$$w(t) = \frac{r(t)\Phi(y^{\Delta}(t))}{\Phi(y(t))},$$

then, for $t \in [t_1, t_2]_{\mathbb{T}}$, $w = w(t)$ satisfies equation (2.5). Now, we are ready to formulate the so-called roundabout theorem, which can be understood as a central statement of the oscillation theory for equation (2.3).

Theorem 2.9 (Roundabout theorem [20, p. 383])**.** *Let $a \in \mathbb{T}$. The following statements are equivalent.*

(i) *Every nontrivial solution of* (2.3) *has at most one generalized zero on $[a, \infty)_{\mathbb{T}}$.*

(ii) *Equation* (2.5) ...

(iii) *Equation* (2.5) *has a solution w with*

$$\left\{\Phi^{-1}(r) + \mu\,\Phi^{-1}(w)\right\}(t) > 0 \quad \text{for } t \in [a, \infty)_{\mathbb{T}}. \tag{2.7}$$

The following theorem is a consequence of the roundabout theorem 2.9 and the Sturm-type comparison theorem 2.8. The method of oscillation theory for (2.3), which uses the ideas of this theorem, is usually referred to as the *Riccati technique*.

Theorem 2.10 (Riccati technique [20, p. 390])**.** *The following statements are equivalent.*

(i) *Equation* (2.3) *is non-oscillatory.*

(ii) *There is $a \in \mathbb{T}$ and a function $w\colon [a, \infty)_{\mathbb{T}} \to \mathbb{R}$ such that* (2.7) *holds and $w(t)$ satisfies* (2.5) *for $t \in [a, \infty)_{\mathbb{T}}$.*

(iii) *There is $a \in \mathbb{T}$ and a function $w\colon [a, \infty)_{\mathbb{T}} \to \mathbb{R}$ such that* (2.7) *holds and $w(t)$ satisfies*

$$w^{\Delta}(t) + c(t) + \mathcal{S}[w, r, \mu](t) \le 0 \quad \text{for } t \in [a, \infty)_{\mathbb{T}}.$$

For further considerations, the following lemma plays an important role (see also [20], where the similar result can be found).

Lemma 2.11. *Let the equation*

$$[r(t)\Phi(y^{\Delta})]^{\Delta} + c(t)\Phi(y^{\sigma}) = 0, \tag{2.8}$$

where coefficients $c, r \in C_{\mathrm{rd}}(\mathbb{T})$ are positive and

$$0 < \inf\{r(t),\, t \in \mathbb{T}\} \le \sup\{r(t),\, t \in \mathbb{T}\} < \infty, \tag{2.9}$$

be non-oscillatory. Then for every solution $w(t)$ of the associated generalized Riccati equation (2.5), *there exists $T \in \mathbb{T}$ such that $w(t) > 0$ for $t \in [T, \infty)_{\mathbb{T}}$. Moreover, $w(t)$ is decreasing for large t with*

$$\lim_{t \to \infty} w(t) = 0.$$

Proof. At first, let us suppose that y is a positive solution of non-oscillatory equation (2.8), i.e., $y(t) > 0$ for $t \in [S, \infty)_{\mathbb{T}}$, where $S \in \mathbb{T}$ is sufficiently large. By contradiction, we prove that there exists $T \in [S, \infty)_{\mathbb{T}}$ such that $y^{\Delta}(t) > 0$ for $t \in [T, \infty)_{\mathbb{T}}$.

(i) Let $y^{\Delta}(t) < 0$ for $t \in [S, \infty)_{\mathbb{T}}$. Because $c(t)\Phi(y^{\sigma}(t)) > 0$ for $t \in [S, \infty)_{\mathbb{T}}$, we have

$$\left[r(t)\Phi(y^{\Delta}(t))\right]^{\Delta} < 0 \quad \text{for } t \in [S, \infty)_{\mathbb{T}}.$$

Integrating the last inequality from S to t, we have

$$r(t)\Phi(y^{\Delta}(t)) - r(S)\Phi(y^{\Delta}(S)) = \int_S^t \left[r(s)\Phi(y^{\Delta}(s))\right]^{\Delta} \Delta s \le 0.$$

Hence

$$y^{\Delta}(t) \le \frac{r^{q-1}(S)y^{\Delta}(S)}{r^{q-1}(t)} \tag{2.10}$$

for $t \in [S,\infty)_{\mathbb{T}}$. Integrating (2.10) for $t \ge S$, we get

$$\left[\lim_{t\to\infty} y(t)\right] - y(S) = \int_S^{\infty} y^{\Delta}(s)\Delta s \le r^{q-1}(S)y^{\Delta}(S)\int_S^{\infty} \frac{\Delta s}{r^{q-1}(s)} = -\infty.$$

Note that the last integral is equal to infinity in view of (2.9). Hence $y(t) \to -\infty$ as $t \to \infty$, a contradiction. Therefore $y^{\Delta}(t) < 0$ cannot hold for large t.

(ii) Let $y^{\Delta}(t) \not> 0$ for large t, i.e., there exists $T_0 \in [S,\infty)_{\mathbb{T}}$ such that $y^{\Delta}(T_0) \le 0$. Thanks to $c(t) > 0$ for $t \in \mathbb{T}$, we have

$$\liminf_{t\to\infty} \int_S^t c(s)\,\Delta s > 0.$$

Since (2.8) is non-oscillatory, then due to Theorem 2.10, the function

$$w(t) = \frac{r(t)\Phi(y^{\Delta}(t))}{\Phi(y(t))} \tag{2.11}$$

satisfies (2.5) with $\left\{\Phi^{-1}(r) + \mu\,\Phi^{-1}(w)\right\}(t) > 0$ for $t \in [S,\infty)_{\mathbb{T}}$. Integrating (2.5) from T_0 to t, $t \ge T_0$, we get

$$w(t) = w(T_0) - \int_{T_0}^t c(s)\,\Delta s - \int_{T_0}^t \mathcal{S}[w,r,\mu](s)\,\Delta s. \tag{2.12}$$

Since $w(T_0) \le 0$, the first integral in (2.12) is positive for large t, and the second integral in (2.12) is nonnegative for large t, we obtain $\limsup_{t\to\infty} w(t) < 0$. For the nonnegativity of function $\mathcal{S}$ see [20, Lemma 13]. Hence, there exists $T_1 \in [S,\infty)_{\mathbb{T}}$ such that $w(t) < 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$, thus $y^{\Delta}(t) < 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$, which is a contradiction to the case (i). We proved that for positive y there exists $T \in \mathbb{T}$ such that $y^{\Delta}(t) > 0$ for $t \in [T,\infty)_{\mathbb{T}}$.

Let $y(t)$ be any negative solution of (2.8) for large t. Then $-y(t) > 0$ is a positive solution of (2.8) with just proven property (the solution space of half linear equations is homogeneous). Hence $y^{\Delta}(t) < 0$ for $t \in [T,\infty)_{\mathbb{T}}$.

In any case, we get (see (2.11)) that $w(t) > 0$ and satisfies (2.5) together with (2.7) for $t \in [T,\infty)_{\mathbb{T}}$. Moreover, since

$$w^{\Delta} = -c(t) - \mathcal{S}[w,r,\mu](t) < 0,$$

$w(t)$ is decreasing for $t \in [T,\infty)_{\mathbb{T}}$.

Finally, we show that $w(t) \to 0$ as $t \to \infty$. Suppose that a solution y is positive and increasing for large t (the case y is negative and decreasing can be proven analogically or with a help of trick as used above). Then it either converges to a positive constant L or diverges to ∞. First, we suppose that $y(t) \to \infty$ as $t \to \infty$. Then, since $r(t)\Phi(y^{\Delta}(t))$ is decreasing (see (2.8)), we have

$$w(t) = \frac{r(t)\Phi(y^{\Delta}(t))}{\Phi(y(t))} < \frac{r(T)\Phi(y^{\Delta}(T))}{\Phi(y(t))} \to 0 \quad \text{as } t \to \infty.$$

Hence $w(t) \to 0$ as $t \to \infty$. Second, if $y(t) \to L$ as $t \to \infty$, then $y^{\Delta}(t) \to 0$ as $t \to \infty$. Thus $r(t)\Phi(y^{\Delta}(t)) \to 0$ as $t \to \infty$ and consequently, $w(t)$ tends to zero as $t \to \infty$ (see (2.11)). □

In the proof of the main result, we use the so-called adapted generalized Riccati equation. Putting

$$z(t) = -t^{p-1} w(t)$$

and using the form of (2.5) with (2.6), a direct calculation leads to the adapted generalized Riccati equation

$$\begin{aligned} z^{\Delta}(t) = c(t)(\sigma(t))^{p-1} &+ \frac{(p-1)(\sigma(t))^{p-1}|\eta(t)|^{p-2}|z(t)|^{q}}{t^{p}\,\Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})]} \\ &+ \frac{(p-1)(\zeta(t))^{p-2} z(t)}{t^{p-1}}, \end{aligned} \tag{2.13}$$

where $\eta(t)$ is between $\Phi^{-1}(r(t))$ and $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})$ and $\zeta(t)$ is defined as

$$\zeta(t) := \left[\frac{\left(t^{p-1}\right)^{\Delta}}{p-1}\right]^{\frac{1}{p-2}}. \tag{2.14}$$

Note that using the Lagrange mean value theorem on time scales, we can (after rewriting (2.14) on $(t^{p-1})^{\Delta} = (p-1)(\zeta(t))^{p-2}$) see that $\zeta(t)$ exists and satisfies $t \le \zeta(t) \le \sigma(t)$.

Now we state two auxiliary lemmas concerning equation (2.13), which can be regarded as consequences of Lemma 2.11.

Lemma 2.12. *Let* (2.8) *be non-oscillatory. Then for every solution $z(t)$ of the associated adapted generalized Riccati equation* (2.13), *there exists sufficiently large $t_0 \in \mathbb{T}$ such that $z(t) < 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.*

Proof. The statement of the lemma follows from Lemma 2.11. □

Lemma 2.13. *If there exists a solution $z(t)$ of the equation* (2.13) *satisfying $z(t) < 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, then its original equation* (2.8) *is non-oscillatory. Moreover,*

$$z(t)/t^{p-1} \to 0 \quad \textit{as } t \to \infty.$$

Proof. From $z(t) < 0$ it follows that $\left\{\Phi^{-1}(r) + \mu\,\Phi^{-1}(w)\right\}(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Hence, thanks to Theorem 2.9, we get that every solution of (2.8) is non-oscillatory and (2.8) is non-oscillatory as well. Further, $z(t)/t^{p-1} = -w(t) \to 0$ as $t \to \infty$ follows from Lemma 2.11. □

3 Conditional oscillation

In this section, we formulate and prove the main result of the paper. At first, for reader's convenience, let us recall, that we deal with the Euler-type half-linear dynamic equation

$$\left[r(t)\Phi(y^{\Delta})\right]^{\Delta} + \frac{\gamma s(t)}{t^{(p-1)}\sigma(t)}\Phi(y^{\sigma}) = 0, \qquad \Phi(y) = |y|^{p-1}\operatorname{sgn} y, \qquad p > 1, \tag{3.1}$$

on an α-periodic ($\alpha > 0$) time scale interval $[a, \infty)_{\mathbb{T}}$, $a \in \mathbb{T}$ with $a > 0$, where $t^{(p)}$ is generalized power function, the functions r, s are rd-continuous, positive, α-periodic with $\inf\{r(t),\ t \in [a, \infty)_{\mathbb{T}}\} > 0$, and $\gamma \in \mathbb{R}$ is an arbitrary constant. Now, we can formulate the main theorem as follows.

Theorem 3.1. *Let $\gamma \in \mathbb{R}$ be a given constant and let $r, s \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}})$ be positive α-periodic functions satisfying* $\inf\{r(t),\ t \in [a,\infty)_{\mathbb{T}}\} > 0$. *Further let*

$$\Gamma := \left(\frac{\alpha}{q}\right)^p \left[\int_a^{a+\alpha} r^{1-q}(t)\,\Delta t\right]^{1-p} \left[\int_a^{a+\alpha} s(t)\,\Delta t\right]^{-1}. \tag{3.2}$$

Then the Euler-type half-linear dynamic equation (3.1) *is oscillatory if $\gamma > \Gamma$ and non-oscillatory if $\gamma < \Gamma$.*

Proof. Since the functions r and s are α-periodic, we have that $\mu(t) \le \alpha$ for every $t \in [a,\infty)_{\mathbb{T}}$ and that a written in limits of integrals in (3.2) can be replace by arbitrary $\tau \in [a,\infty)_{\mathbb{T}}$ with same resulting value Γ.

Throughout the proof, we will use the following estimates in which we assume that $\gamma > 0$ and $z(t) < 0$ for large t. Denote

$$r^+ := \sup\{r(t), t \in [a,\infty)_{\mathbb{T}}\}, \qquad r^- := \inf\{r(t), t \in [a,\infty)_{\mathbb{T}}\}$$

and

$$s^+ := \sup\{s(t), t \in [a,\infty)_{\mathbb{T}}\}, \qquad s^- := \inf\{s(t), t \in [a,\infty)_{\mathbb{T}}\}.$$

Note that due to rd-continuity and α-periodicity of the functions r and s,

$$0 < r^- \le r^+ < \infty \quad \text{and} \quad 0 \le s^- \le s^+ < \infty$$

hold. In view of (2.13), the adapted Riccati equation associated to (3.1) has the form

$$\begin{aligned} z^{\Delta}(t) = {} & \frac{\gamma s(t)(\sigma(t))^{p-2}}{t^{(p-1)}} + \frac{(p-1)(\sigma(t))^{p-1}|\eta(t)|^{p-2}|z(t)|^q}{t^p\,\Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})]} \\ & + \frac{(p-1)(\zeta(t))^{p-2}z(t)}{t^{p-1}}, \end{aligned} \tag{3.3}$$

where $\eta(t)$ is between $\Phi^{-1}(r(t))$ and $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})$, and $t \le \zeta(t) \le \sigma(t)$. Let us define the function

$$h(t) := \mu(t) r^{1-q}(t)\Phi^{-1}(-z(t)/t^{p-1}).$$

It is easy to see (in view of Lemma 2.13) that

$$0 \le h(t) \to 0 \quad \text{as } t \to \infty. \tag{3.4}$$

Therefore, equation (3.3) can be written in the form

$$\begin{aligned} z^{\Delta}(t) = {} & \frac{\gamma s(t)(\sigma(t))^{p-2}}{t^{(p-1)}} \\ & + (p-1)|z(t)|\frac{((\sigma(t))^{p-1}/t)|\eta(t)|^{p-2}|z(t)|^{q-1} - (\zeta(t))^{p-2}F(t)}{t^{p-1}F(t)}, \end{aligned} \tag{3.5}$$

where

$$F(t) := \Phi\left[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})\right] = r(t)[1+h(t)]^{p-1} > 0. \tag{3.6}$$

Hence, we get for large t and for $p \geq 2$

$$\begin{aligned} z^{\Delta}(t) &\geq \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)| \cdot \frac{(\sigma(t))^{p-2}|\eta(t)|^{p-2}|z(t)|^{q-1} - (\sigma(t))^{p-2}r(t)[1+h(t)]^{p-1}}{t^{p-1}r(t)[1+h(t)]^{p-1}} \\ &> \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)|(\sigma(t))^{p-2} \cdot \frac{r^{(q-1)(p-2)}(t)|z(t)|^{q-1} - 2^{p-1}r(t)}{t^{p-1}r(t)[1+h(t)]^{p-1}} \\ &= \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)|(\sigma(t))^{p-2}r^{2-q}(t) \cdot \frac{|z(t)|^{q-1} - 2^{p-1}r^{q-1}(t)}{t^{p-1}r(t)[1+h(t)]^{p-1}}. \end{aligned}$$

Analogously, for large t and for $p < 2$, we have

$$\begin{aligned} z^{\Delta}(t) &\geq \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)| \cdot \frac{(\sigma(t))^{p-2}|\eta(t)|^{p-2}|z(t)|^{q-1} - (\sigma(t))^{p-2}r(t)[1+h(t)]^{p-1}}{t^{p-1}r(t)[1+h(t)]^{p-1}} \\ &> \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)| \cdot \frac{(\sigma(t))^{p-2}(2r^{q-1}(t))^{p-2}|z(t)|^{q-1} - t^{p-2}2^{p-1}r(t)}{t^{p-1}r(t)[1+h(t)]^{p-1}} \\ &= \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)|(\sigma(t))^{p-2}2^{p-2}r^{2-q}(t) \cdot \frac{|z(t)|^{q-1} - (\sigma(t)/t)^{2-p}2r^{q-1}(t)}{t^{p-1}r(t)[1+h(t)]^{p-1}} \\ &\geq \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)|(\sigma(t))^{p-2}2^{p-2}r^{2-q}(t) \cdot \frac{|z(t)|^{q-1} - (1+\alpha)^{2-p}2r^{q-1}(t)}{t^{p-1}r(t)[1+h(t)]^{p-1}} \end{aligned}$$

and thus

$$z^{\Delta}(t) > \frac{\gamma s^-}{\sigma(t)} \quad \text{if} \quad z(t) < \min\left\{-2^{(p-1)^2}r^+, -2^{\frac{p}{q}}(1+\alpha)^{\frac{2-p}{q-1}}r^+\right\}. \tag{3.7}$$

Simultaneously, we estimate $|z^{\Delta}(t)|$ for $z(t) \in (-C, 0)$ and large t. We denote

$$D := \max\left\{\sup\left\{\frac{\sigma(t)}{t}, t \in [a,\infty)_{\mathbb{T}}\right\}, \sup\left\{\frac{(\sigma(t))^{p-2}}{t^{p-2}}, t \in [a,\infty)_{\mathbb{T}}\right\}, \right. \\ \left. \sup\left\{\frac{t(\sigma(t))^{p-2}}{t^{(p-1)}}, t \in [a,\infty)_{\mathbb{T}}\right\}\right\} > 0.$$

Then, we get thanks to (3.5) for $p \geq 2$ (i.e., $q \leq 2$)

$$\begin{aligned} |z^{\Delta}(t)| &< \frac{\gamma s^+ D}{t} + (p-1)C\frac{(\sigma(t))^{p-2}D \cdot [2r^{q-1}(t)]^{p-2} \cdot C^{q-1} + (\sigma(t))^{p-2}2^{p-1}r(t)}{t^{p-1}r(t)} \\ &\leq \frac{\gamma s^+ D}{t} + \frac{2^{p-2}C(p-1)(\sigma(t))^{p-2}[C^{q-1}Dr^{2-q}(t) + 2r(t)]}{t^{p-1}r^-} \\ &\leq \frac{\gamma s^+ D}{t} + \frac{2^{p-2}C(p-1)D[C^{q-1}D(r^+)^{2-q} + 2r^+]}{tr^-} \\ &= \frac{\gamma s^+ r^- D + 2^{p-2}C(p-1)D[C^{q-1}D(r^+)^{2-q} + 2r^+]}{tr^-}, \end{aligned} \tag{3.8}$$

and for $p < 2$ (i.e., $q > 2$)

$$\begin{aligned} |z^{\Delta}(t)| &< \frac{\gamma s^+ D}{t} + (p-1)C\frac{(\sigma(t))^{p-2}D \cdot [r^{q-1}(t)]^{p-2} \cdot C^{q-1} + t^{p-2}2^{p-1}r(t)}{t^{p-1}r(t)} \\ &\leq \frac{\gamma s^+ D}{t} + \frac{(p-1)C^q D^2(r^-)^{2-q}}{tr^-} + \frac{(p-1)2^{p-1}r^+ C}{tr^-} \\ &= \frac{\gamma s^+ r^- D + (p-1)C^q D^2(r^-)^{2-q} + (p-1)2^{p-1}r^+ C}{tr^-}. \end{aligned} \tag{3.9}$$

Therefore, we have

$$|z^{\Delta}(t)| < \frac{H(C)}{t}, \tag{3.10}$$

where

$$\begin{aligned} H(C) := \max\Bigg\{ &\frac{\gamma s^{+} r^{-} D + 2^{p-2} C(p-1) D [C^{q-1} D (r^{+})^{2-q} + 2r^{+}]}{r^{-}}, \\ &\qquad \frac{\gamma s^{+} r^{-} D + (p-1) C^{q} D^{2} (r^{-})^{2-q} + (p-1) 2^{p-1} r^{+} C}{r^{-}} \Bigg\} \end{aligned} \tag{3.11}$$

is a positive constant which exists due to (3.8) and (3.9).

Next, from (3.7) and (3.10) it follows that if $z(t) < 0$ for every $t \in [t_0, \infty)_{\mathbb{T}}$, $t_0 \geq a$, then there exists a constant $K > 0$ such that

$$z(t) \in (-K, 0) \quad \text{for every } t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.12}$$

Indeed, according to (3.7), $z(t)$ is increasing if $z(t)$ is sufficiently small. Otherwise, thanks to (3.10), $z(t)$ cannot drop arbitrarily low.

Next, using the fact that the graininess $\mu(t) \leq \alpha$ for all $t \in [a, \infty)_{\mathbb{T}}$ together with the definition of ζ given in (2.14) and taking into the account that $\eta(t)$ is between $\Phi^{-1}(r(t))$ and $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})$, we obtain (see also Lemma 2.4), that there exists a constant $\varepsilon \in [0, 1/2)$ such that

$$\begin{gathered} 1 - \varepsilon \leq \frac{(\sigma(t))^{p-2}}{t^{(p-1)}/t} \leq 1 + \varepsilon, \qquad 1 - \varepsilon \leq \frac{(\sigma(t))^{p-1} |\eta(t)|^{p-2}}{t^{p-1} r^{2-q}(t)} \leq 1 + \varepsilon, \\ 1 - \varepsilon \leq \frac{(\zeta(t))^{p-2}}{t^{p-2}} \leq 1 + \varepsilon \end{gathered} \tag{3.13}$$

are fulfilled for arbitrary $p > 1$ and large t. More precisely, ε can be chosen arbitrarily near to zero in (3.13), if t is sufficiently large.

Using the above estimates, we can turn our attention to the proof of the theorem. We start with the oscillatory part. In this part of the proof, let $\gamma > \Gamma$. By contradiction, we suppose that (3.1) is non-oscillatory. According to Lemma 2.12, for every solution $z(t)$ of the associated adapted Riccati equation (3.3) there exists sufficiently large $t_0 \in \mathbb{T}$ such that $z(t) < 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Moreover, from previous estimates, there exists $K > 0$, such that (3.12) holds. Using (3.10) and (3.11), we get

$$|z^{\Delta}(t)| < \frac{H(K)}{t}, \qquad t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.14}$$

Now, we introduce the average value $\xi(t)$ of the function $z(t)$ on an arbitrary interval $[t, t+\alpha]_{\mathbb{T}}$, where t is sufficiently large. Using $\xi(t)$, we will obtain a contradiction with $z(t) \in (-K, 0)$. Obviously,

$$\xi(t) \in (-K, 0) \quad \text{and} \quad \xi(t) := \frac{1}{\alpha} \int_{t}^{t+\alpha} z(\tau)\, \Delta\tau, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.15}$$

Using (3.5), (3.6), (3.13), and (3.15) we get

$$\begin{aligned}
\xi^{\Delta}(t) &= \frac{1}{\alpha}\int_t^{t+\alpha} z^{\Delta}(\tau)\,\Delta\tau \\
&= \frac{1}{\alpha}\int_t^{t+\alpha} \frac{1}{\tau}\left[\frac{\gamma s(\tau)(\sigma(\tau))^{p-2}}{\tau^{(p-1)}/\tau} + \frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^{q}}{\tau^{p-1}r(\tau)[1+h(\tau)]^{p-1}}\right]\Delta\tau \\
&\quad + \frac{1}{\alpha}\int_t^{t+\alpha} \frac{1}{\tau}\cdot\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\,\Delta\tau \\
&\geq \frac{1}{\alpha}\cdot\frac{1-\varepsilon}{t+\alpha}\int_t^{t+\alpha}\left[\gamma s(\tau) + \frac{(p-1)r^{1-q}(\tau)|z(\tau)|^{q}}{[1+h(\tau)]^{p-1}}\right]\Delta\tau + \frac{1}{\alpha}\cdot\frac{1+\varepsilon}{t}\int_t^{t+\alpha}(p-1)z(\tau)\,\Delta\tau \\
&= \frac{1-\varepsilon}{t+\alpha}\left[\frac{\gamma}{\alpha}\int_t^{t+\alpha} s(\tau)\,\Delta\tau + \frac{1}{\alpha}\int_t^{t+\alpha}\frac{(p-1)r^{1-q}(\tau)|z(\tau)|^{q}}{[1+h(\tau)]^{p-1}}\,\Delta\tau\right] + \frac{(1+\varepsilon)(p-1)\xi(t)}{t} \\
&= \frac{1-\varepsilon}{t+\alpha}\left\{\frac{\gamma}{\alpha}\int_t^{t+\alpha} s(\tau)\,\Delta\tau - \frac{A^{p}(t)}{p} + \frac{1}{\alpha}\int_t^{t+\alpha}\frac{(p-1)r^{1-q}(\tau)|z(\tau)|^{q}}{[1+h(\tau)]^{p-1}}\,\Delta\tau - \frac{B^{q}(t)}{q}\right. \\
&\qquad\qquad \left. + \frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{t+\alpha}{t}(p-1)\xi(t) + \frac{A^{p}(t)}{p} + \frac{B^{q}(t)}{q}\right\},
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
A(t) &= (p-1)\left(\frac{p}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{-1/q}, \qquad t \geq t_0, \\
B(t) &= |\xi(t)|\left(\frac{p}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{1/q}, \qquad t \geq t_0.
\end{aligned} \tag{3.17}$$

We will estimate $\xi^{\Delta}(t)$ using (3.16) in three steps.
Step I. We show that there exists $M > 0$ such that

$$\frac{\gamma}{\alpha}\int_t^{t+\alpha} s(\tau)\,\Delta\tau - \frac{A^{p}(t)}{p} = M \tag{3.18}$$

holds for every $t \in [t_0, \infty)_{\mathbb{T}}$. Using $p/q = p-1$, we have for every $t \in [t_0, \infty)_{\mathbb{T}}$

$$\begin{aligned}
&\frac{\gamma}{\alpha}\int_t^{t+\alpha} s(\tau)\,\Delta\tau - \frac{A^{p}(t)}{p} \\
&\quad = \frac{\gamma}{\alpha}\int_t^{t+\alpha} s(\tau)\,\Delta\tau - \frac{(p-1)^{p}}{p}\left(\frac{p}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{-p/q} \\
&\quad = \left(\frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\,\Delta\tau\right)\left[\gamma - \frac{(p-1)^{p}\alpha^{1+p/q}}{p^{1+p/q}}\left(\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{-p/q}\left(\int_t^{t+\alpha} s(\tau)\,\Delta\tau\right)^{-1}\right]
\end{aligned}$$

$$= \left(\frac{1}{\alpha}\int\limits_t^{t+\alpha} s(\tau)\,\Delta\tau\right)\left[\gamma - \left(\frac{p-1}{p}\right)^p \alpha^p \left(\int\limits_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{1-p}\left(\int\limits_t^{t+\alpha} s(\tau)\,\Delta\tau\right)^{-1}\right]$$
$$= \left(\frac{1}{\alpha}\int\limits_t^{t+\alpha} s(\tau)\,\Delta\tau\right)\left[\gamma - \left(\frac{\alpha}{q}\right)^p \left(\int\limits_a^{a+\alpha} r^{1-q}(t)\,\Delta t\right)^{1-p}\left(\int\limits_a^{a+\alpha} s(t)\Delta t\right)^{-1}\right] = S(\gamma-\Gamma),$$

where

$$S := \frac{1}{\alpha}\int\limits_t^{t+\alpha} s(\tau)\Delta\tau > 0. \tag{3.19}$$

Hence there exists $M = S(\gamma - \Gamma) > 0$ such that (3.18) holds for $t \in [t_0, \infty)_{\mathbb{T}}$.
Step II. We prove the existence of $t_1 \in \mathbb{T}$, $t_1 \geq t_0$, satisfying

$$\frac{1}{\alpha}\int\limits_t^{t+\alpha} \frac{(p-1)r^{1-q}(\tau)|z(\tau)|^q}{[1+h(\tau)]^{p-1}}\Delta\tau - \frac{B^q(t)}{q} \geq -\frac{M}{4}, \qquad t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.20}$$

where M is taken from Step I. To do it, we need three further auxiliary estimates. First, in view of (3.4), we can write

$$\frac{1}{[1+h(t)]^{p-1}} = \frac{1}{1+\tilde{h}(t)} = 1 - \frac{\tilde{h}(t)}{1+\tilde{h}(t)} = 1 - \hat{h}(t), \tag{3.21}$$

where $\tilde{h}(t)$ and $\hat{h}(t)$ are convenient functions. It is obvious that $0 \leq \tilde{h}(t) \to 0$ as $t \to \infty$ and

$$0 \leq \hat{h}(t) \to 0 \quad \text{as } t \to \infty. \tag{3.22}$$

Second, since the function $y = |x|^q$ is continuously differentiable on $(-K, 0)$, there exists $\lambda > 0$ for which

$$|z|^q - |\xi|^q \geq -\lambda|z - \xi|, \quad \text{where } z, \xi \in (-K, 0). \tag{3.23}$$

Third, from (3.14) we have

$$|z(t_m) - z(t_n)| = \left|\int\limits_{t_n}^{t_m} z^\Delta(\tau)\,\Delta\tau\right| \leq \int\limits_{t_n}^{t_m} |z^\Delta(\tau)|\,\Delta\tau \leq \int\limits_t^{t+\alpha} |z^\Delta(\tau)|\,\Delta\tau$$
$$< \int\limits_t^{t+\alpha} \frac{H(K)}{\tau}\,\Delta\tau \leq \frac{1}{t}\int\limits_t^{t+\alpha} H(K)\,\Delta\tau = \frac{H(K)\alpha}{t} \tag{3.24}$$

for every $t_m, t_n \in [t, t+\alpha]_{\mathbb{T}}$, where $t \geq t_0$ and (see (3.11)) $H(K) > 0$. Because (see (3.15))

$$\xi(t) \in [z_{\min}(t), z_{\max}(t)],$$

where

$$z_{\min}(t) := \min\{z(\tau), \tau \in [t, t+\alpha]_{\mathbb{T}}\}, \qquad z_{\max}(t) := \max\{z(\tau), \tau \in [t, t+\alpha]_{\mathbb{T}}\},$$

there exist $t_m, t_n \in [t, t+\alpha]_{\mathbb{T}}$ (see (3.24)) such that for every $\tau \in [t, t+\alpha]_{\mathbb{T}}$

$$|z(\tau) - \xi(t)| \leq |z(t_m) - z(t_n)| < \frac{H(K)\alpha}{t}. \tag{3.25}$$

Now we are ready to finish Step II. Using (3.17), (3.21), (3.23), (3.25), and again $p/q = p - 1$, we can estimate

$$
\begin{aligned}
&\frac{1}{\alpha}\int_t^{t+\alpha} \frac{(p-1)r^{1-q}(\tau)|z(\tau)|^q}{[1+h(\tau)]^{p-1}}\,\Delta\tau - \frac{B^q(t)}{q}\\
&= \frac{1}{\alpha}\int_t^{t+\alpha}(p-1)r^{1-q}(\tau)|z(\tau)|^q(1-\hat{h}(\tau))\,\Delta\tau - \frac{|\xi(t)|^q p}{\alpha q}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\\
&= \frac{p-1}{\alpha}\int_t^{t+\alpha}\left[r^{1-q}(\tau)|z(\tau)|^q(1-\hat{h}(\tau)) - r^{1-q}(\tau)|\xi(t)|^q\right]\Delta\tau\\
&= \frac{p-1}{\alpha}\int_t^{t+\alpha}\frac{|z(\tau)|^q - |\xi(t)|^q - \hat{h}(\tau)|z(\tau)|^q}{r^{q-1}(\tau)}\,\Delta\tau\\
&\geq -\frac{\lambda(p-1)}{\alpha}\int_t^{t+\alpha}\frac{|z(\tau)-\xi(t)|}{r^{q-1}(\tau)}\,\Delta\tau - \frac{(p-1)}{\alpha}\int_t^{t+\alpha}\frac{\hat{h}(\tau)|z(\tau)|^q}{r^{q-1}(\tau)}\,\Delta\tau\\
&> -\frac{\lambda(p-1)H(K)}{t}\int_t^{t+\alpha}\frac{1}{r^{q-1}(\tau)}\,\Delta\tau - \frac{(p-1)}{\alpha}\int_t^{t+\alpha}\frac{\hat{h}(\tau)|z(\tau)|^q}{r^{q-1}(\tau)}\,\Delta\tau\\
&\geq -\frac{\lambda(p-1)H(K)\alpha}{t(r^-)^{q-1}} - \frac{(p-1)}{\alpha(r^-)^{q-1}}\int_t^{t+\alpha}\hat{h}(\tau)|z(\tau)|^q\,\Delta\tau, \qquad t\in[t_0,\infty)_{\mathbb{T}}.
\end{aligned}
\tag{3.26}
$$

Finally, (3.26) (see also (3.22), which ensures that the value of the last integral in (3.26) tends to zero for large t) implies that there exists $t_1 \geq t_0$ such that (3.20) is fulfilled.

Step III. From Young's inequality ($A^p/p + B^q/q \geq AB$), from the fact that $(p-1)|\xi(t)| = A(t)B(t)$ (see (3.17)), and from (3.15), we obtain that

$$
\begin{aligned}
\frac{t+\alpha}{t}(p-1)\xi(t) + \frac{A^p(t)}{p} + \frac{B^q(t)}{q} &= \frac{A^p(t)}{p} + \frac{B^q(t)}{q} - (p-1)|\xi(t)| - \frac{\alpha(p-1)}{t}|\xi(t)|\\
&= \frac{A^p(t)}{p} + \frac{B^q(t)}{q} - A(t)B(t) + \frac{\alpha(p-1)\xi(t)}{t} > -\frac{\alpha(p-1)K}{t}
\end{aligned}
$$

is fulfilled for $t \in [t_0,\infty)_{\mathbb{T}}$. Hence there exists $t_2 \in \mathbb{T}$, $t_2 \geq t_1$, such that

$$
\frac{t+\alpha}{t}(p-1)\xi(t) + \frac{A^p(t)}{p} + \frac{B^q(t)}{q} \geq -\frac{M}{8}, \qquad t\in[t_2,\infty)_{\mathbb{T}}, \tag{3.27}
$$

where M is taken from Step I. Finally, we know that the constant ε in (3.16) can be taken arbitrarily near to zero for sufficiently large t. Hence, and in view of (3.27), there exists $t_3 \in \mathbb{T}$, $t_3 \geq t_2$, such that

$$
\frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{t+\alpha}{t}(p-1)\xi(t) + \frac{A^p(t)}{p} + \frac{B^q(t)}{q} \geq -\frac{M}{4}, \qquad t\in[t_3,\infty)_{\mathbb{T}}. \tag{3.28}
$$

Altogether, from the previous three steps, we show that $\xi(t) \to \infty$ if $t \to \infty$. Indeed, in view of (3.16) and estimates (3.18), (3.20), and (3.28), we can easily see that

$$
\xi^{\Delta}(t) \geq \frac{1-\varepsilon}{t+\alpha}\left(M - \frac{M}{4} - \frac{M}{4}\right) = \frac{M(1-\varepsilon)}{2(t+\alpha)} > \frac{M}{4(t+\alpha)}, \qquad t\in[t_3,\infty)_{\mathbb{T}}. \tag{3.29}
$$

Integrating (3.29) from t_3 to ∞, we get (thanks to $\mu(t) \le \alpha$)

$$\left[\lim_{t\to\infty} \xi(t)\right] - \xi(t_3) \ge \frac{M}{4} \int_{t_3}^{\infty} \frac{\Delta t}{t+\alpha} \ge \frac{M}{4} \sum_{n=1}^{\infty} \frac{\alpha}{n\alpha + t_3 + \alpha} = \infty,$$

thus $\xi(t) \to \infty$ if $t \to \infty$. Therefore, $\xi(t) > 0$ for every sufficiently large $t \in \mathbb{T}$, which means that $z(t) > 0$ for every sufficiently large $t \in \mathbb{T}$. This contradiction gives that equation (3.1) is oscillatory for $\gamma > \Gamma$.

To prove the non-oscillatory part of the theorem, we start with $\gamma \le 0$. In this case, (3.1) is non-oscillatory in view of Theorem 2.8, part (i). It suffices to consider the non-oscillatory equation $\left[r(t)\Phi(y^\Delta)\right]^\Delta = 0$. Then

$$c(t) = 0 \ge \frac{\gamma s(t)}{t^{(p-1)}\sigma(t)} = C(t), \qquad t \in [a,\infty)_{\mathbb{T}}.$$

Therefore, using this comparison, (3.1) is non-oscillatory as well.

To prove the last part of the theorem, we show that (3.1) is non-oscillatory for $0 < \gamma < \Gamma$. To do it, we show that there exists $t^* \in \mathbb{T}$ such that a solution $z(t)$ of (3.3) with

$$z(t^*) = -\left(\frac{q}{\alpha} \int_{t^*}^{t^*+\alpha} \frac{\Delta\tau}{r^{q-1}(\tau)}\right)^{1-p} =: -Z \tag{3.30}$$

is negative for every $t \in [t^*,\infty)_{\mathbb{T}}$. Since

$$-r^+ < -\frac{r^+}{q^{p-1}} \le -\left(\frac{q}{\alpha} \int_{t^*}^{t^*+\alpha} \frac{\Delta\tau}{r^{q-1}(\tau)}\right)^{1-p},$$

and using (3.7) and (3.10), there exists $T_1 \in \mathbb{T}$ sufficiently large such that

$$z(t) \in (-2r^+, 0) \quad \text{for } t \in [t^*, t^*+\alpha]_{\mathbb{T}},\ t^* \ge T_1. \tag{3.31}$$

More precisely, according to (3.7), $z(t)$ is increasing if $z(t) \in (-2r^+, -r^+)$. Otherwise, from (3.10), we have that $z(t) \in (-2r^+, 0)$ is varying arbitrarily small for large t. Hence, for $t \in [t^*, t^*+\alpha]_{\mathbb{T}}$, (3.31) holds.

Next, using (3.31) (see also (3.10) and (3.24)), there exists constant $c > 0$ such that

$$|z(t_m) - z(t_n)| < \frac{c}{t^*} \quad \text{for } t_m, t_n \in [t^*, t^*+\alpha]_{\mathbb{T}},\ t^* \ge T_1. \tag{3.32}$$

Analogically as in the first part of the proof, we use the average value $\xi(t^*)$, i.e.,

$$\xi(t^*) \in (-2r^+, 0) \quad \text{and} \quad \xi(t^*) := \frac{1}{\alpha} \int_{t^*}^{t^*+\alpha} z(\tau)\,\Delta\tau, \qquad t^* \ge T_1. \tag{3.33}$$

From (3.32) it follows (compare with (3.25))

$$|\xi(t^*) - z(\tau)| < \frac{c}{t^*}, \qquad \tau \in [t^*, t^*+\alpha]_{\mathbb{T}},\ t^* \ge T_1. \tag{3.34}$$

Now (similarly as before, see (3.16)), we estimate $\xi^\Delta(t^*)$. Using (3.3), (3.13), and (3.33), we get

$$
\begin{aligned}
\xi^{\Delta}(t^*) &= \frac{1}{\alpha}\int_{t^*}^{t^*+\alpha} z^{\Delta}(\tau)\,\Delta\tau \\
&\le \frac{1}{\alpha}\cdot\frac{1}{t^*}\int_{t^*}^{t^*+\alpha}\left[\frac{\gamma s(\tau)(\sigma(\tau))^{p-2}}{\tau^{(p-1)}/\tau}+\frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^{q}}{\tau^{p-1}r(\tau)[1+h(\tau)]^{p-1}}\right]\Delta\tau \\
&\quad+\frac{1}{\alpha}\cdot\frac{1}{t^*+\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\,\Delta\tau \\
&= \frac{1}{t^*}\Bigg\{\frac{\gamma}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{s(\tau)(\sigma(\tau))^{p-2}}{\tau^{(p-1)}/\tau}\,\Delta\tau-\frac{A^{p}(t^*)}{p} \\
&\qquad-\frac{1}{t^*+\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\,\Delta\tau \\
&\qquad+\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^{q}}{\tau^{p-1}r(\tau)[1+h(\tau)]^{p-1}}\,\Delta\tau-\frac{B^{q}(t^*)}{q} \\
&\qquad+\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\,\Delta\tau+\frac{A^{p}(t^*)}{p}+\frac{B^{q}(t^*)}{q}\Bigg\},
\end{aligned}
\tag{3.35}
$$

where $A(t)$ and $B(t)$ are given in (3.13). Again, we will estimate $\xi^{\Delta}(t^*)$ using (3.35) in three steps.

Step I. Let $S>0$ be defined by (3.19) for t^*. Then, using (3.13), (3.17), and (3.33), we get

$$
\begin{aligned}
&\frac{\gamma}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{s(\tau)(\sigma(\tau))^{p-2}}{\tau^{(p-1)}/\tau}\,\Delta\tau-\frac{A^{p}(t^*)}{p}-\frac{1}{t^*+\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\,\Delta\tau \\
&\qquad\le \gamma S(1+\varepsilon)-\Gamma S+\frac{2\alpha r^{+}(p-1)(1+\varepsilon)}{t^*+\alpha}=S\big[(1+\varepsilon)\gamma-\Gamma\big]+\frac{2\alpha r^{+}(p-1)(1+\varepsilon)}{t^*+\alpha}.
\end{aligned}
$$

Therefore, there exist $T_2\in\mathbb{T}$, $T_2\ge T_1$, and $N>0$ such that for $t^*\ge T_2$ ($t^*\in\mathbb{T}$) we have

$$
\frac{\gamma}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{s(\tau)(\sigma(\tau))^{p-2}}{\tau^{(p-1)}/\tau}\,\Delta\tau-\frac{A^{p}(t^*)}{p}-\frac{1}{t^*+\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\,\Delta\tau\le -N. \tag{3.36}
$$

Note that we use the fact that ε tends to zero for large t.

Step II. Using (3.13), (3.21), (3.31), and (3.34), we have

$$
\begin{aligned}
&\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^{q}}{\tau^{p-1}r(\tau)[1+h(\tau)]^{p-1}}\Delta\tau-\frac{B^{q}(t^*)}{q} \\
&\quad=\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^{q}}{\tau^{p-1}r(\tau)[1+h(\tau)]^{p-1}}\,\Delta\tau-|\xi(t^*)|^{q}\frac{p}{q\alpha}\int_{t^*}^{t^*+\alpha}r^{1-q}(\tau)\,\Delta\tau \\
&\quad\le\frac{(1+\varepsilon)(p-1)}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{r^{1-q}(\tau)|z(\tau)|^{q}}{[1+h(\tau)]^{p-1}}\,\Delta\tau-(p-1)\frac{|\xi(t^*)|^{q}}{\alpha}\int_{t^*}^{t^*+\alpha}r^{1-q}(\tau)\,\Delta\tau
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1+\varepsilon)(p-1)}{\alpha} \int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)|z(\tau)|^q (1-\hat{h}(\tau))\,\Delta\tau \\
&\quad -(p-1)\frac{|\xi(t^*)|^q}{\alpha} \int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\,\Delta\tau \\
&= \frac{p-1}{\alpha} \int_{t^*}^{t^*+\alpha} r^{1-q}(\tau) \left[|z(\tau)|^q(1-\hat{h}(\tau)) - |\xi(t^*)|^q\right] \Delta\tau \\
&\quad + \frac{\varepsilon(p-1)}{\alpha} \int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)|z(\tau)|^q(1-\hat{h}(\tau))\,\Delta\tau \le \frac{N}{4}
\end{aligned}
\tag{3.37}
$$

for $t^* \in [T_3, \infty)_T$, where $T_3 \ge T_2$ is sufficiently large. Indeed, T_3 exists due to the facts, that r, z, ξ are bounded, $\hat{h}, \varepsilon$ tend to zero, and due to the continuity of the function $|x|^q$ (compare (3.23)). Of course, the constant N is taken from Step I.

Step III. Using (3.13), (3.17), and (3.33) in this part of the proof, we have

$$
\begin{aligned}
&\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha} \frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\,\Delta\tau + \frac{A^p(t^*)}{p} + \frac{B^q(t^*)}{q} \\
&\qquad \le (1+\varepsilon)(p-1)\xi(t^*) + \frac{(p-1)^p}{p}\left(\frac{p}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{-p/q} \\
&\qquad\quad + (p-1)\,|\xi(t^*)|^q\,\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\,\Delta\tau,
\end{aligned}
$$

which is, according to (3.34), asymptotically the same as

$$
\begin{aligned}
&(1+\varepsilon)(p-1)z(t^*) + \frac{(p-1)^p}{p}\left(\frac{p}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{-p/q} + (p-1)\,|z(t^*)|^q\,\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\,\Delta\tau \\
&= -(1+\varepsilon)(p-1)q^{1-p}\left(\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{1-p} \\
&\quad + \left(\frac{p-1}{p}\right)^p \left(\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{1-p} + (p-1)q^{-p}\left(\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{1-p} \\
&= \left(\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\,\Delta\tau\right)^{1-p}\left[-(1+\varepsilon)(p-1)q^{1-p} + \left(\frac{p-1}{p}\right)^p + (p-1)q^{-p}\right].
\end{aligned}
$$

By a direct calculation one can verify, that $p + q = pq$ implies

$$
(p-1)q^{-p} - (p-1)q^{1-p} + \left(\frac{p-1}{p}\right)^p = 0.
$$

Therefore, there exists $T_4 \in \mathbb{T}$, $T_4 \ge T_3$, such that

$$
\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha} \frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\,\Delta\tau + \frac{A^p(t^*)}{p} + \frac{B^q(t^*)}{q} \le \frac{N}{4}, \qquad t^* \in [T_4,\infty)_{\mathbb{T}} \tag{3.38}
$$

where N is, again, taken from Step I.

Finally, using (3.36), (3.37), and (3.38) in (3.35), we have

$$\zeta^{\Delta}(t^*) \leq \frac{1}{t^*}\left(-N + \frac{N}{4} + \frac{N}{4}\right) = -\frac{N}{2t^*}, \qquad t^* \in [T_4, \infty)_{\mathbb{T}}, \tag{3.39}$$

and taking into account (3.39), we obtain

$$\zeta^{\Delta}(t^*) = \frac{1}{\alpha}\int\limits_{t^*}^{t^*+\alpha} z^{\Delta}(\tau)\Delta\tau = \frac{z(t^*+\alpha) - z(t^*)}{\alpha} < 0, \qquad t^* \in [T_4, \infty)_{\mathbb{T}},$$

i.e.,

$$z(t^*+\alpha) < z(t^*), \qquad t^* \in [T_4, \infty)_{\mathbb{T}}. \tag{3.40}$$

In particular, if (3.30) holds for some $t^* \in [T_4, \infty)_{\mathbb{T}}$, then (3.31) and (3.40) assure the negativity of $z(t)$ for the whole period, more precisely,

$$z(t) < 0 \quad \text{for} \quad t \in [t^*, t^*+\alpha]_{\mathbb{T}} \quad \text{with} \quad z(t^*+\alpha) < z(t^*).$$

To finish the proof, it suffices to show the existence of $\vartheta > 0$ (depending only on r and α) such that if $z(t) \in (-\vartheta - Z, -Z)$ for some $t \in (t^*, \infty)_{\mathbb{T}}, t^* > T_4$, then $z(t+\alpha) < z(t)$. Immediately, we have that if $z(t) \in (-\vartheta - Z, -Z)$ then $z(t+\alpha) < -Z$. Next, using (3.32), if $z(t) \leq -\vartheta - Z$ then $z(t+\alpha) \leq -\vartheta - Z$ as well. Further, the initial value $-Z$ was not used in (3.35), (3.36), and (3.37). Moreover, (3.38) is valid for (3.30) with a sufficiently small negative perturbation depending only on the coefficient r and the period α. Therefore, the number ϑ exists, which guarantees the existence of negative solution $z(t)$ of (3.3) for large t.

Altogether, we have shown, that the initial value problem (3.3), (3.30) has a solution $z(t)$ satisfying $z(t) < 0$ for every $t \in [t^*, \infty)_{\mathbb{T}}$ (where t^* is sufficiently large), which, combined with Lemma 2.13, means that equation (3.1) is non-oscillatory. □

The following example demonstrates the previous theorem.

Example 3.2. Consider an arbitrary finite time scale interval $[3, 3+\alpha]_{\mathbb{T}}$ with $\alpha > 0$, where $3 \in \mathbb{T}$ and $3 + \alpha \in \mathbb{T}$. Let us define infinite time scale interval $[3, \infty)_{\mathbb{T}}$ such that

$$\text{if } t \in [3, 3+\alpha]_{\mathbb{T}}, \text{ then } \{t + \alpha n\}_{n=1}^{\infty} \subseteq [3, \infty)_{\mathbb{T}}$$

and moreover, $[3, \infty)_{\mathbb{T}}$ does not contain any other points. Consider the dynamic equation

$$\left[\left(3 - 2\cos\left(\frac{2\pi t}{\alpha}\right)\right)\Phi(y^{\Delta})\right]^{\Delta} + \frac{\gamma\left(1 + \frac{2}{3}\sin\left(\frac{2\pi t}{\alpha}\right)\right)}{t^{(p-1)}\sigma(t)}\Phi(y^{\sigma}) = 0 \tag{3.41}$$

on $[3, \infty)_{\mathbb{T}}$. Then (3.41) is oscillatory if $\gamma > \tilde{\Gamma}$ and non-oscillatory if $\gamma < \tilde{\Gamma}$, where

$$\tilde{\Gamma} = \left(\frac{\alpha}{q}\right)^p \left[\int\limits_3^{3+\alpha}\left(3 - 2\cos\left(\frac{2\pi t}{\alpha}\right)\right)^{1-q}\Delta t\right]^{1-p}\left[\int\limits_3^{3+\alpha}\left(1 + \frac{2}{3}\sin\left(\frac{2\pi t}{\alpha}\right)\right)\Delta t\right]^{-1}.$$

For the concrete time scale interval $[3, \infty)_{\mathbb{T}}$ and numbers α and p, we can compute the exact value of constant $\tilde{\Gamma}$. We illustrate this fact, e.g., for

$$\mathbb{T} = \left\{\bigcup_{k=0}^{\infty}[3+3k, 4+3k]\right\} \cup \{5+3k\}_{k=0}^{\infty},$$

$\alpha = 3$, and $p = 3/2$ (which implies $q = 3$). For this choice we get

$$\begin{aligned}\tilde{\Gamma} &= \left[\int_3^6 \left(3 - 2\cos\left(\frac{2\pi t}{3}\right)\right)^{-2} \Delta t\right]^{-1/2} \left[\int_3^6 \left(1 + \frac{2}{3}\sin\left(\frac{2\pi t}{3}\right)\right) \Delta t\right]^{-1} \\ &= \left[\int_3^4 \left(3 - 2\cos\left(\frac{2\pi t}{3}\right)\right)^{-2} \mathrm{d}t\right]^{-1/2} \left[\int_3^4 \left(1 + \frac{2}{3}\sin\left(\frac{2\pi t}{3}\right)\right) \mathrm{d}t\right]^{-1} \\ &\quad + \left[\sum_{k=4}^5 \left(3 - 2\cos\left(\frac{2k\pi}{3}\right)\right)^{-2}\right]^{-1/2} \left[\sum_{k=4}^5 \left(1 + \frac{2}{3}\sin\left(\frac{2k\pi}{3}\right)\right)\right]^{-1} \\ &= \sqrt{2} + \frac{20\sqrt{6\pi^3}}{3(2\pi + 3)\sqrt{5\sqrt{3} + 24\sqrt{5}\arctan\sqrt{15}}} \doteq 2.513492637.\end{aligned}$$

Note that we used a software to obtain this value (namely, we used Maple 16).

4 Applications and concluding remarks

As we mentioned in the introduction, the result of Theorem 3.1 can be used as an oscillation test also to equations that are not Euler-type. For example, we can combine Theorem 3.1 and Sturm-type comparison theorem 2.8 to obtain the following Kneser-type oscillation criteria.

Corollary 4.1. *Let us consider the equation*

$$\left[\Phi(y^\Delta)\right]^\Delta + d(t)\Phi(y^\sigma) = 0, \tag{4.1}$$

where $d \in C_{\mathrm{rd}}([a,\infty)_\mathbb{T})$, $a \in \mathbb{T}$, $a > 0$.

(i) If there exists a positive α-periodic function $s \in C_{\mathrm{rd}}([a,\infty)_\mathbb{T})$ such that

$$\limsup_{t\to\infty} \frac{t^{(p-1)}\sigma(t)d(t)}{s(t)} < \frac{\alpha}{q^p}\left[\int_a^{a+\alpha} s(t)\,\Delta t\right]^{-1},$$

then Eq. (4.1) is non-oscillatory.

(ii) If there exists a positive α-periodic function $s \in C_{\mathrm{rd}}([a,\infty)_\mathbb{T})$ such that

$$\liminf_{t\to\infty} \frac{t^{(p-1)}\sigma(t)d(t)}{s(t)} > \frac{\alpha}{q^p}\left[\int_a^{a+\alpha} s(t)\,\Delta t\right]^{-1},$$

then Eq. (4.1) is oscillatory.

Proof. Let the assumptions of the first part hold. We consider the Euler-type equation

$$\left[\Phi(y^\Delta)\right]^\Delta + \frac{\gamma s(t)}{t^{(p-1)}\sigma(t)}\Phi(y^\sigma) = 0 \tag{4.2}$$

together with its oscillation constant

$$\Gamma = \frac{\alpha}{q^p}\left[\int_a^{a+\alpha} s(t)\,\Delta t\right]^{-1}.$$

Then, for some positive number $\varepsilon \in \mathbb{R}$, we have

$$d(t) < \frac{(\Gamma - \varepsilon)s(t)}{t^{(p-1)}\sigma(t)}.$$

From Theorem 3.1 we have that (4.2) is non-oscillatory for $\gamma = \Gamma - \varepsilon$. Using the Sturm-type comparison theorem 2.8, part (i), we obtain that (4.1) is non-oscillatory.

The second part follows from an analogical idea and the Sturm-type comparison theorem 2.8, part (ii). □

Next, let us mention a corollary that (partially) covers the cases of negative coefficients.

Corollary 4.2. *Let us consider* (3.1) *with rd-continuous, α-periodic functions r, s satisfying*

$$\inf\{|r(t)|,\, t \in [a,\infty)_{\mathbb{T}}\} > 0, \qquad s(t) \not\equiv 0,\; t \in [a,\infty)_{\mathbb{T}}.$$

Further denote

$$\bar{\Gamma} := \left(\frac{\alpha}{q}\right)^p \left[\int_a^{a+\alpha} |r(t)|^{1-q}\,\Delta t\right]^{1-p} \left[\int_a^{a+\alpha} |s(t)|\,\Delta t\right]^{-1}.$$

Then the following statements hold.

(i) If $r(t)$ is positive for $t \in [a,\infty)_{\mathbb{T}}$ and $\gamma < \bar{\Gamma}$, then (3.1) *is non-oscillatory.*

(ii) If $s(t)$ is positive for $t \in [a,\infty)_{\mathbb{T}}$ and $\gamma > \bar{\Gamma}$, then (3.1) *is oscillatory.*

Proof. The corollary comes directly from Theorem 3.1, the Sturm-type comparison theorem 2.8, and the fact that the absolute value preserves periodicity. □

Finally, as a possible direction of future research, we conjecture that (3.1) with more general coefficients remains conditionally oscillatory. This conjecture is based on continuous and discrete cases. More precisely, in [25], there is found the oscillation constant for Euler-type half-linear difference equations with asymptotically almost periodic coefficients. Concerning the continuous case, in [26] is shown that Euler-type half-linear differential equations with coefficients having mean values (which covers periodic and almost periodic cases) are conditionally oscillatory. However, extension of these types for dynamic equations on time scales appear to be much more technical difficult.

For another natural possible direction, we should mention papers [6, 7, 12], where perturbed half-linear differential equations are studied. Typically, the perturbations are placed in the potential of the given equation, which leads to the equations of the form

$$\left[r(t)\Phi(y')\right]' + \left[\frac{c(t)}{t^2} + \frac{d(t)}{t^2\log^2 t}\right]\Phi(y) = 0, \qquad \mathbb{T} = \mathbb{R},$$

which is referred to as the Riemann–Weber half-linear equation. Eventually, the perturbation in the potential can be replaced by a more complex one involving the iterated logarithms (i.e., $\log(\log(\ldots(\log t)))$). In the above mentioned papers is proved that such equations are conditionally oscillatory and from the behavior of the "more perturbed" equation, there is shown, that the "less perturbed" equation with the critical constant is non-oscillatory, e.g., the results concerning the Riemann–Weber equation give that the border case of the Euler equation is non-oscillatory.

We should also emphasize, that the above mentioned results about the critical case, i.e., the non-oscillation of the equation (3.1) with $\gamma = \Gamma$ and $\mathbb{T} = \mathbb{R}$, was obtained only for (differential) equations with periodic coefficients. The step to more general class, e.g., the almost periodic coefficients, is probably not possible. The reason is that, using the methods described in [22, 23, 24], it is possible to construct almost periodic functions (and sequences), such that the equation is oscillatory in the critical case. Together with its non-oscillation for periodic coefficients and due to the fact, that any periodic function (sequence) is almost periodic as well, we have that it is not possible to decide in general. For more details, we refer to [11, Section 5] and [25, Remark 19].

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