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Sharp Oscillation Criteria for Fourth Order Sub-half-linear and Super-half-linear Differential Equations

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Abstract. This paper is concerned with the oscillatory behavior of the fourth-order nonlinear differential equation

$$(p(t)|x''|^{\alpha-1}x'')'' + q(t)|x|^{\beta-1}x = 0\,, \tag{E}$$

where $\alpha > 0$, $\beta > 0$ are constants and $p, q : [a,\infty) \to (0,\infty)$ are continuous functions satisfying conditions

$$\int_a^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt < \infty, \quad \int_a^\infty \frac{t}{(p(t))^{\frac{1}{\alpha}}}\, dt < \infty\,.$$

We will establish necessary and sufficient condition for oscillation of all solutions of the sub-half-linear equation (E) (for $\beta < \alpha$) as well as of the super-half-linear equation (E) (for $\beta > \alpha$).

Keywords and phrases. fourth order nonlinear differential equation, positive solution, oscillation, sub-halflinear, super-halflinear

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1. Introduction

We consider the fourth-order nonlinear differential equation

$$(p(t)|x''|^{\alpha-1}x'')'' + q(t)|x|^{\beta-1}x = 0\,, \tag{E}$$

where $\alpha > 0$, $\beta > 0$ are constants and $p, q : [a,\infty) \to (0,\infty)$, $a > 0$ are continuous functions. If we use the notation

$$\varphi_\gamma(\xi) = |\xi|^{\gamma-1}\xi, \quad \xi \in \mathbb{R},\ \gamma > 0\,,$$

the equation (E) can be expressed in the form

$$(p(t)\varphi_\alpha(x''))'' + q(t)\varphi_\beta(x) = 0\,.$$

The equation (E) is called *super-half-linear* if $\beta > \alpha$ and *sub-half-linear* if $\beta < \alpha$.

Solution of (E) is a function $x : [T_x,\infty) \to \mathbb{R}$, such that $p(t)\varphi_\alpha(x''(t))$ is twice continuously differentiable and x satisfies the equation (E) for every $t \in [T_x.\infty)$. We exclude solution of (E) that vanish identically in some neighborhood of infinity. A solution of (E) is called *oscillatory* if it has an infinite sequence of zeros tending to infinity; otherwise it is called *nonoscillatory*.

The oscillatory and asymptotic behavior of solutions of the nonlinear differential equation of the form (E) were first considered by Wu [5] and Kamo and Usami [2]. In [5], the equation (E) is discussed under the condition

$$(P_1) \qquad \int_a^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt = \infty, \quad \int_a^\infty \frac{t}{(p(t))^{\frac{1}{\alpha}}}\, dt = \infty\,;$$

and in [2] the equation (E) is considered under the condition

$$(P_3) \qquad \int_a^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt = \infty, \quad \int_a^\infty \frac{t}{(p(t))^{\frac{1}{\alpha}}}\, dt < \infty\,.$$

The thing that is naturally imposed is that we proceed further with investigation of the oscillatory and asymptotic behavior of solutions of (E) under the following two conditions:

$$(P_2) \qquad \int_a^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt < \infty, \quad \int_a^\infty \frac{t}{(p(t))^{\frac{1}{\alpha}}}\, dt = \infty\,,$$

$$(P_4) \qquad \int_a^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt < \infty, \quad \int_a^\infty \frac{t}{(p(t))^{\frac{1}{\alpha}}}\, dt < \infty\,,$$

which has not yet been discussed in the literature by our knowledge .

Note, that in the paper of Kusano, Tanigawa [3] and in the paper of Kusano, Tanigawa and Manojlovic [4], the equation (E) was discussed under the following condition

$$(P_5) \qquad \int_a^\infty t\left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt < \infty\,.$$

These two papers together provide the complete characterization of the oscillatory solutions of (E) under the assumption (P_5). Essentially, Kusano, Tanigawa and Manojlovic in [4] established sharp criteria for oscillation of all solutions of (E). Also, Kusano and Tanigawa in [3] have made a detailed analysis of the structure of nonoscillatory solutions of (E) and established that there are six possible cases of the asymptotic behavior of positive solutions of the equation (E):

(A) $x(t) \sim c_1\psi_1(t), \quad t \to \infty\,,$ (B) $x(t) \sim c_2\psi_2(t), \quad t \to \infty\,,$

(C) $x(t) \sim c_3\psi_3(t), \quad t \to \infty\,,$ (D) $x(t) \sim c_4\psi_4(t), \quad t \to \infty\,,$

(E) $\psi_1(t) \prec x(t) \prec \psi_2(t), \quad t \to \infty\,,$ (F) $\psi_3(t) \prec x(t) \prec \psi_4(t), \quad t \to \infty\,,$

where the functions $\psi_i(t),\ i=1,2,3,4$ defined as

$$\psi_1(t)=\int_t^\infty \frac{s-t}{p^{\frac{1}{\alpha}}(s)}\,ds\,,\quad \psi_2(t)=\int_t^\infty \frac{(s-t)s^{\frac{1}{\alpha}}}{p^{\frac{1}{\alpha}}(s)}\,ds\,,\quad \psi_3(t)=1\,,\quad \psi_4(t)=t\,,$$

$c_i>0$, $i=1,2,3,4$, are constants. Here the symbol $f(t)\sim g(t),\quad t\to\infty$ is used to mean the asymptotic equivalence:

$$f(t)\sim g(t),\quad t\to\infty \quad\Longleftrightarrow\quad \lim_{t\to\infty}\frac{f(t)}{g(t)}=1\,,$$

while the symbol $f(t)\prec g(t),\quad t\to\infty$ is used to express

$$\lim_{t\to\infty}\frac{g(t)}{f(t)}=\infty\,.$$

The necessary and sufficient condition for the existence of nonoscillatory solutions of (E) belonging to each of the six classes has been establish in [3].

The main objective of this paper is to establish necessary and sufficient conditions for oscillation of all solutions of (E) under the condition (P_4). Throughout the paper we always assume that (P_4) holds. To prove our main results, we are going to use the following two theorems stated and proved in [3]:

Theorem 1.1 *Let* (P_5) *hold. The equation* (E) *has a positive solution* $x(t)$ *of type* (A) *if and only if*

$$(C_1)\qquad \int_a^\infty t\,q(t)\,(\psi_1(t))^\beta\,dt<\infty\,.$$

Theorem 1.2 *Let* (P_5) *hold. The equation* (E) *has a positive solution* $x(t)$ *of type* (D) *if and only if*

$$(C_2)\qquad \int_a^\infty\left(\frac{1}{p(t)}\int_a^t (t-s)\,s^\beta\,q(s)\,ds\right)^{\frac{1}{\alpha}}dt<\infty\,.$$

Evidently, the condition (P_5) implies the condition (P_4). However, the technique of proving the oscillation criteria for the equation (E) under the condition (P_5) can not be applied when (P_4) holds. Namely, in order to prove that the desired condition is sufficient for the oscillation of all solutions of (E) under the condition (P_5) authors in [4], in fact, prove that neither of positive solution of type (A) - (E) can exists if the desired condition is assumed to hold. Naturally, we can not apply such a procedure when (P_4) holds, because we actually do not have complete characterization of asymptotic properties of nonoscillatory solutions of (E) under this assumption. Therefore, our original work is based on different technique of proving the oscillation criteria for (E).

The paper is organized as follows. In Section 2. we begin with the classification of all nonoscillatory solutions of (E) according to their signs of derivatives $x'(t)$, $x''(t)$ and $(p(t)|x''(t)|^{\alpha-1}x''(t))'$. In Section 3. we collect auxiliary lemmas and propositions which are used later in proofs of our main results. In Section 4. the main results will be stated and proved. The example illustrating the main results will be presented in Section 5.

2. Classification of positive solutions

In this section we state some of the basic results regarding the classification of nonoscillatory solutions of (E). There is no loss of generality in restricting our attention to the set of positive solutions, because if $x(t)$ satisfied (E), then so does $-x(t)$.

Let $x(t)$ be a positive solution of the equation (E). Since, from (E), $(p(t)|x''(t)|^{\alpha-1}x''(t))'$ is eventually monotone, it follows, that all of the functions $(p(t)|x''(t)|^{\alpha-1}x''(t))'$, $x''(t)$ and $x'(t)$ are eventually monotone and one-signed. Hence, the next eight cases can be considered:

	$(p(t)\varphi_\alpha(x''))'$	x''	x'		$(p(t)\varphi_\alpha(x''))'$	x''	x'
(a)	$+$	$+$	$+$	(e)	$-$	$+$	$+$
(b)	$+$	$+$	$-$	(f)	$-$	$+$	$-$
(c)	$+$	$-$	$+$	(g)	$-$	$-$	$+$
(d)	$+$	$-$	$-$	(h)	$-$	$-$	$-$

If $x' < 0$ and $x'' < 0$ eventually, then $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the positivity of the solution $x(t)$. Therefore, cases (d) and (h) never hold. Similarly, since $(p(t)\varphi_\alpha(x''(t))'' < 0$, if $(p(t)\varphi_\alpha(x''(t))' < 0$, then $\lim_{t\to\infty} p(t)\varphi_\alpha(x''(t)) = -\infty$, that is, $x''(t) < 0$ for large t. This observation rules out the cases (e) and (f).

Accordingly, the following four types of combination of the signs of $x'(t)$, $x''(t)$ and $(p(t)|x''|^{\alpha-1}\,x'')'$ are possible for an eventually positive solution $x(t)$ of the equation (E):

$$
\begin{array}{llll}
\text{(I)} & (p(t)|x''(t)|^{\alpha-1}x''(t))' > 0, & x''(t) > 0, & x'(t) > 0 \\
\text{(II)} & (p(t)|x''(t)|^{\alpha-1}x''(t))' > 0, & x''(t) > 0, & x'(t) < 0 \\
\text{(III)} & (p(t)|x''(t)|^{\alpha-1}x''(t))' > 0, & x''(t) < 0, & x'(t) > 0 \\
\text{(IV)} & (p(t)|x''(t)|^{\alpha-1}x''(t))' < 0, & x''(t) < 0, & x'(t) > 0
\end{array}
$$

3. Auxiliary lemmas and propositions

In this section, we first introduce the useful inequalities for positive solutions of (E) belonging to type (II) and (IV), respectively.

Lemma 3.1 *Let $x(t)$ be a positive solution of equation* (E) *of type* (II)*. Then there exists a positive number c such that the following inequalities hold for all large t:*

$$x(t) \geq (p(t))^{\frac{1}{\alpha}}\, \psi_1(t)\, x''(t)\,, \tag{3.1}$$

$$(x(t))^\alpha \geq c\, t\, (\psi_1(t))^\alpha\, (p(t)(x''(t))^\alpha)'\,. \tag{3.2}$$

Lemma 3.1 has been stated and proved in [2]. The statement of this lemma holds under the condition (P_4), because its proof is independent of the conditions (P_3) or (P_4).

Lemma 3.2 *Let $x(t)$ be a positive solution of equation* (E) *of type* (IV)*. Then there exists a positive number c such that the following inequality holds for all large t:*

$$x(t) \geq c\, t\, x'(t)\,. \tag{3.3}$$

Proof: Suppose that for a positive solution $x(t)$ of (E) (IV) holds for all $t \geq t_0$. Since $x'(t)$ is decreasing, we have

$$x(t) > x(t) - x(t_0) = \int_{t_0}^{t} x'(s)\, ds \geq x'(t) \int_{t_0}^{t} ds = x'(t)\ (t - t_0),\ \ t \geq t_0.$$

Then, there is a constant $c > 0$, and a sufficiently large t, such that (3.3) holds. This completes the proof. ⊠

Lemma 3.3 *Let $\alpha \leq 1$. If $x(t)$ is a positive solution of equation* (E) *of type* (IV)*, then there exists a positive number c such that the following inequality holds for all large t:*

$$x(t) \geq c\, t\, \left|(p(t)|x''(t)|^{\alpha-1}x''(t))'\right|^{\frac{1}{\alpha}}\ \psi_1(t)\,. \tag{3.4}$$

Proof: Since $(p(t)|x''(t)|^{\alpha-1}x''(t))'$ is decreasing, we find that

$$(p(\xi)|x''(\xi)|^{\alpha-1}x''(\xi))' \leq (p(t)|x''(t)|^{\alpha-1}x''(t))', \quad \xi \geq t\,.$$

Integrating the last inequality from t to s we have

$$p(s)|x''(s)|^{\alpha-1}x''(s) \leq p(t)|x''(t)|^{\alpha-1}x''(t) + (p(t)|x''(t)|^{\alpha-1}x''(t))'\,(s-t),\ \ s \geq t\,,$$

and so

$$x''(s) \leq -\left|(p(t)|x''(t)|^{\alpha-1}x''(t))'\right|^{\frac{1}{\alpha}} \left(\frac{s-t}{p(s)}\right)^{\frac{1}{\alpha}},\ \ s \geq t\,.$$

Since the limit $\lim_{t\to\infty} x'(t) = \omega_1 \geq 0$ is finite and noting that $1/\alpha \geq 1$, integrating the last inequality from t to ∞ we have

$$x'(t) \geq \left|(p(t)|x''(t)|^{\alpha-1}x''(t))'\right|^{\frac{1}{\alpha}} \int_t^\infty \frac{s-t}{(p(s))^{1/\alpha}}\, ds = \left|(p(t)|x''(t)|^{\alpha-1}x''(t))'\right|^{\frac{1}{\alpha}}\ \psi_1(t)\,.$$

Combining the last inequality and the inequality (3.3) we obtain (3.4). This completes the proof. ⊠

Lemma 3.4 *Let $\beta < 1 \leq \alpha$. Then the condition*

$$\int_a^\infty \left(\frac{1}{p(t)} \int_a^t (t-s)s^\beta q(s)\, ds\right)^{\frac{1}{\alpha}}\ dt = \infty \tag{3.5}$$

implies that

$$\int_a^\infty t^{\frac{\beta}{\alpha}} \psi_1^\beta(t)\, q(t)\, dt = \infty. \tag{3.6}$$

Proof: If (3.5) holds, it implies that for any $t_0 > a$

$$\int_{t_0}^{\infty} (p(t))^{-\frac{1}{\alpha}} \left(\int_{t_0}^{t} (t-s) s^{\beta} q(s)\, ds \right)^{\frac{1}{\alpha}} dt = \infty\,. \tag{3.7}$$

Choose $t_0 > 1$, such that $\psi_1(t) \le 1$ for $t \ge t_0$. In the view of the basic integral condition (P_4), the equation (3.7) implies that

$$\lim_{t\to\infty} \int_{t_0}^{t} (t-s) s^{\beta} q(s)\, ds = \infty\,,$$

so that, by L'Hospital's rule, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (t-s) s^{\beta} q(s)\, ds = \lim_{t\to\infty} \int_{t_0}^{t} s^{\beta} q(s)\, ds \in (0, \infty]\,,$$

which shows that there exists some constant $k > 0$ and some $t_1 > t_0$ such that

$$\int_{t_0}^{t} (t-s) s^{\beta} q(s)\, ds \ge k\, t \quad \text{for every } t \ge t_1\,. \tag{3.8}$$

For all $t > t_1$, using integration by parts, we obtain

$$\begin{aligned}
\int_{t_1}^{t} (p(s))^{-\frac{1}{\alpha}} \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{\frac{1}{\alpha}} ds &= \int_{t_1}^{t} \psi_1''(s) \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{\frac{1}{\alpha}} ds \\
= \psi_1'(s) \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{\frac{1}{\alpha}} \Bigg|_{s=t_1}^{s=t} & \\
- \frac{1}{\alpha} \int_{t_1}^{t} \psi_1'(s) \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{\frac{1}{\alpha}-1} & \left(\int_{t_0}^{s} r^{\beta} q(r)\, dr \right) ds\,.
\end{aligned}$$

From the last equality, using (3.8), and the fact that $1/\alpha \le 1$ as well as that $\psi_1'(t)$ is a negative function, we get

$$\begin{aligned}
&\int_{t_1}^{t} (p(s))^{-\frac{1}{\alpha}} \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{\frac{1}{\alpha}} ds \\
&\quad \le k_1 - k_2 \int_{t_1}^{t} \psi_1'(s)\, s^{\frac{1}{\alpha}-1} \int_{t_0}^{s} r^{\beta}\, q(r)\, dr\, ds \\
&\quad \le k_1 - k_2 \int_{t_1}^{t} \psi_1'(s) \int_{t_0}^{s} r^{\frac{1}{\alpha}-1+\beta}\, q(r)\, dr\, ds\,, \quad t \ge t_1\,,
\end{aligned} \tag{3.9}$$

where

$$k_1 = -\psi_1'(t_1) \left(\int_{t_0}^{t_1} (t_1 - r)\, r^{\beta} q(r)\, dr \right)^{\frac{1}{\alpha}} > 0, \quad k_2 = \frac{1}{\alpha}\, k^{\frac{1}{\alpha}-1} > 0\,.$$

Now, from (3.9), using the fact that

$$0 \le \frac{(1-\alpha)\,(\beta-1)}{\alpha} = \frac{\beta}{\alpha} - \left(\frac{1}{\alpha} - 1 + \beta \right)\,,$$

we have

$$\int_{t_1}^{t} (p(s))^{-\frac{1}{\alpha}} \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{\frac{1}{\alpha}} ds \le k_1 - k_2 \int_{t_1}^{t} \psi_1'(s) \int_{t_0}^{s} r^{\frac{\beta}{\alpha}}\, q(r)\, dr\, ds$$

$$= k_1 + k_2 \left[\left(-\psi_1(s) \int_{t_0}^{s} r^{\frac{\beta}{\alpha}}\, q(r)\, dr \right) \Bigg|_{s=t_1}^{s=t} + \int_{t_1}^{t} \psi_1(s)\, s^{\frac{\beta}{\alpha}}\, q(s)\, ds \right]$$

$$\le k_3 + k_2 \int_{t_1}^{t} \psi_1(s)\, s^{\frac{\beta}{\alpha}}\, q(s)\, ds \le k_3 + k_2 \int_{t_1}^{t} s^{\frac{\beta}{\alpha}}\, \psi_1^{\beta}(s)\, q(s)\, ds\,, \quad t \ge t_1\,,$$

where $k_3 = k_1 + k_2\, \psi_1(t_1) \int_{t_0}^{t_1} r^{\beta/\alpha}\, q(r)\, dr$. Letting $t \to \infty$, we conclude that (3.5) implies (3.6). This completes the proof. ⊠

Next we introduce some useful propositions, that play an important role in some parts of the proof of our main results given in Section 4. The following two propositions have been proved in [2]. Their statements hold under assumption (P_4), because their proofs are independent of conditions (P_3) or (P_4).

Proposition 3.1 *Let $\beta > \alpha$. If there exists a positive solution $x(t)$ of the equation* (E) *of type* (II)*, then*

$$\int_{a}^{\infty} t\, q(t)\, (\psi_1(t))^{\beta}\, dt < \infty\,. \tag{3.10}$$

Proposition 3.2 *Let $\beta < \alpha$. If there exists a positive solution $x(t)$ of the equation* (E) *of type* (II)*, then*

$$\int_{a}^{\infty} t^{\frac{\beta}{\alpha}}\, q(t)\, (\psi_1(t))^{\beta}\, dt < \infty\,. \tag{3.11}$$

Proposition 3.3 *Let $\alpha < 1 \le \beta$. If there exists a positive solution $x(t)$ of the equation* (E) *of type* (IV)*, then*

$$\int_{a}^{\infty} t\, q(t)\, (\psi_1(t))^{\beta}\, dt < \infty\,. \tag{3.12}$$

Proof: For a positive solution $x(t)$ of (E) lets choose $t_0 \ge a$ such that (IV) and (3.4) hold for all $t \ge t_0$. Denote by $P(t) = (p(t)|x''(t)|^{\alpha-1}x''(t))'$. Then, from the equation (E) we have

$$-\left(|P(t)|^{1-\frac{\beta}{\alpha}}\right)' = -\frac{\alpha-\beta}{\alpha}\, |P(t)|^{-\frac{\beta}{\alpha}} \left(-P'(t)\right) = -\frac{\alpha-\beta}{\alpha}\, |P(t)|^{-\frac{\beta}{\alpha}}\, q(t)\, (x(t))^{\beta}\,.$$

Applying that to the last equality, we see that $c_2 > 0$ exists such that

$$-\left(|P(t)|^{1-\frac{\beta}{\alpha}}\right)' \ge \frac{\beta-\alpha}{\alpha}\, |P(t)|^{-\frac{\beta}{\alpha}}\, q(t)\, c_2^{\beta}\, t^{\beta}\, |P(t)|^{\frac{\beta}{\alpha}}\, (\psi_1(t))^{\beta},\ \ t \ge t_0.$$

Using the fact that $\beta \ge 1$ i.e. $t^{\beta} \ge t$ for all $t \ge \max\{1, t_0\} = t_1$, we have

$$-\left(|P(t)|^{1-\frac{\beta}{\alpha}}\right)' \ge \frac{\beta-\alpha}{\alpha}\, c_2^{\beta}\, t\, q(t)\, (\psi_1(t))^{\beta},\ \ t \ge t_1.$$

Integrating this inequality from t_1 to t we find that

$$|P(t_1)|^{\frac{\alpha-\beta}{\alpha}} \ge -\,|P(t)|^{\frac{\alpha-\beta}{\alpha}} + |P(t_1)|^{\frac{\alpha-\beta}{\alpha}} \ge \frac{\beta-\alpha}{\alpha}\ c_2^{\beta} \int_{t_1}^{t} s\, q(s)\, (\psi_1(s))^{\beta}\, ds,$$

witch gives (3.12). This completes the proof. ⊠

Proposition 3.4 *Let $\beta < 1 \le \alpha$. If there exists a positive solution $x(t)$ of equation* (E) *of type* (IV)*, then*

$$\int_a^\infty \frac{1}{(p(t))^{\frac{1}{\alpha}}} \left(\int_a^t (t-s)\, s^\beta \, q(s)\, ds \right)^{\frac{1}{\alpha}} dt < \infty. \tag{3.13}$$

Proof: Suppose that for a positive solution $x(t)$ of (E) (IV) holds for all $t \ge t_0$. By Lemma 3.2, there exist $c_1 > 0$ and $t_1 \ge t_0$ such that (3.3) holds for all $t \ge t_1$. Integrating the equation (E) from t_1 to t we see that

$$-(p(t)|x''(t)|^{\alpha-1}x''(t))' \ge \int_{t_1}^t q(s)(x(s))^\beta \, ds, \;\; t \ge t_1 \,.$$

Integrating the last inequality from t_1 to t we have

$$-x''(t) \ge \frac{1}{(p(t))^{\frac{1}{\alpha}}} \left(\int_{t_1}^t (t-s)\, q(s)\, (x(s))^\beta \, ds \right)^{\frac{1}{\alpha}}, \quad t \ge t_1 \,.$$

From the last inequality we see that

$$\begin{aligned} -\left((x'(t))^{1-\beta} \right)' &= (1-\beta)(x'(t))^{-\beta}(-x''(t)) \\ &\ge (1-\beta)(x'(t))^{-\beta} \frac{1}{(p(t))^{\frac{1}{\alpha}}} \left(\int_{t_1}^t (t-s)\, q(s)\, (x(s))^\beta \, ds \right)^{\frac{1}{\alpha}}, \quad t \ge t_1 \,. \end{aligned}$$

Since $x'(t)$ is decreasing i.e. $(x'(t))^{-\beta} \ge (x'(s))^{-\beta}$ for $t \ge s$, we have

$$\begin{aligned} -\left((x'(t))^{1-\beta} \right)' &\ge (1-\beta) \frac{1}{(p(t))^{\frac{1}{\alpha}}} \left(\int_{t_1}^t (t-s)\, q(s)\, (x'(s))^{-\beta\alpha}\, (x(s))^\beta \, ds \right)^{\frac{1}{\alpha}} \\ &= (1-\beta) \frac{1}{(p(t))^{\frac{1}{\alpha}}} \left(\int_{t_1}^t (t-s)\, q(s)\, (s\, x'(s))^{-\beta\alpha}\, s^{\beta\alpha}\, (x(s))^\beta \, ds \right)^{\frac{1}{\alpha}}, \quad t \ge t_1 \,. \end{aligned}$$

Now, using (3.3) we get

$$-\left((x'(t))^{1-\beta} \right)' \ge (1-\beta) \frac{c_1^\beta}{(p(t))^{\frac{1}{\alpha}}} \left(\int_{t_1}^t (t-s)\, q(s)\, (x(s))^{-\beta\alpha}\, s^{\beta\alpha}\, (x(s))^\beta \, ds \right)^{\frac{1}{\alpha}}, \quad t \ge t_1 \,.$$

Since $x'(t) \le x'(t_1)$, $t \ge t_1$, there exists some constant $c > 0$ and $t_2 \ge t_1$ such that $x(t) \le c\,t$ for $t \ge t_2$. Therefore, the fact that $\beta(1-\alpha) \le 0$ implies $(x(t))^{\beta(1-\alpha)} \ge c^{\beta(1-\alpha)}\, t^{\beta(1-\alpha)}$ for $t \ge t_2$. Consequently, we have

$$-\left((x'(t))^{1-\beta} \right)' \ge (1-\beta)\, c^{\frac{\beta(1-\alpha)}{\alpha}} \frac{c_1^\beta}{(p(t))^{\frac{1}{\alpha}}} \left(\int_{t_2}^t (t-s)\, q(s)\, s^{\beta(1-\alpha)}\, s^{\beta\alpha} \, ds \right)^{\frac{1}{\alpha}}, \quad t \ge t_2.$$

Integrating the last inequality from t_2 to ∞ we have

$$(x'(t_2))^{1-\beta} \ge K \int_{t_2}^\infty \frac{1}{(p(s))^{\frac{1}{\alpha}}} \left(\int_{t_2}^s (s-r)\, q(r)\, r^\beta \, dr \right)^{\frac{1}{\alpha}} ds \,,$$

where $K = (1-\beta)\, c^{\frac{\beta(1-\alpha)}{\alpha}}\, c_1^\beta > 0$, which gives (3.13). This completes the proof. ⊠

4. Oscillation theorems

In this section we state and prove the sharp criteria for the oscillation of all solutions of super-half-linear and sub-half-linear equation (E), respectively. Necessity part of proof is easier to handle, since it is an immediate consequence of Theorem 1.1 and Theorem 1.2. To prove the sufficiency part, we use the following technique: we suppose that the condition (Ω_1) or (Ω_2) holds, and then we eliminate the positive solutions of (E) of the four possible types (I)-(IV) mentioned in Section 2. Therefore, we conclude the nonexistence of any positive solution of (E), or equivalently, all solution of (E) are oscillatory.

Theorem 4.1 *Let $\beta \geq 1 > \alpha$. The equation (E) is oscillatory if and only if*

$$\int_a^\infty t\, q(t)\, (\psi_1(t))^\beta\, dt = \infty\,. \tag{Ω_1}$$

Proof: ($\Rightarrow$:) Suppose on the contrary that (Ω_1) is not satisfied. Then, by the Theorem 1.1 the equation (E) has a positive solution $x(t)$ of type (A), which contradicts the assumption that all solutions of (E) are oscillatory.

($\Leftarrow$:) We will prove that (Ω_1) ensures the oscillation of all solutions of (E), or equivalently, the nonexistence of some positive solution of (E). Since any positive solution of (E) falls into one of the four types (I)-(IV) mentioned in the Section 2., it is sufficient to verify that (Ω_1) eliminates the positive solutions of (E) of all these four types.

Elimination of solution $x(t)$ of type (I): Suppose that for a positive solution $x(t)$ of (E) (I) holds for all $t \geq t_0$. Since $x'(t) \geq x'(t_0)$, $t \geq t_0$, there exists some constant $c > 0$ and $t_1 \geq t_0$ such that $x(t) \geq c\,t$ for $t \geq t_1$. Integrating the equation (E) from t_1 to ∞ and using the previous we have

$$c^\beta \int_{t_1}^\infty s^\beta\, q(s)\, ds \leq \int_{t_1}^\infty (x(s))^\beta\, q(s)\, ds \leq (p(t_1)|x''(t_1)|^{\alpha-1}x''(t_1))'.$$

Therefore we conclude that

$$\int_a^\infty t^\beta\, q(t)\, dt < \infty\,. \tag{4.1}$$

On the other hand, from the condition (P_4) there exists $T_0 \geq a$ such that $\psi_1(t) \leq 1$ for $t \geq T_0$. The fact that $\beta \geq 1$ implies $t^\beta \geq t$ for $t \geq T_1 = \max\{T_0, 1\}$, and consequently we have

$$\int_{T_1}^\infty t\, q(t)\, (\psi_1(t))^\beta\, dt \leq \int_{T_1}^\infty t\, q(t)\, dt \leq \int_{T_1}^\infty t^\beta\, q(t)\, dt\,, \qquad t \geq T_1\,. \tag{4.2}$$

Combining (4.1) i (4.2), we conclude that (Ω_1) is not satisfied. The obtained contradiction eliminates the positive solution of type (I).

Elimination of solution $x(t)$ of type (II): If there exists a positive solution $x(t)$ of type (II), from Proposition 3.1 we see that (Ω_1) fails to hold. The positive solution of type (II) is eliminated by this contradiction.

Elimination of solution $x(t)$ of type (III): Multiplying equation (E) by t and integrating the resulting equation from t_0 to t, using integration by parts, we have

$$\int_{t_0}^t s\, q(s)\, (x(s))^\beta\, ds = c_3 - t(p(t)|x''(t)|^{\alpha-1}x''(t))' + p(t)|x''(t)|^{\alpha-1}x''(t) < c_3\,,$$

where $c_3 = t_0(p(t_0)|x''(t_0)|^{\alpha-1}x''(t_0))' - p(t_0)|x''(t_0)|^{\alpha-1}x''(t_0) > 0$ is constant. Consequently, we conclude that

$$\int_a^\infty t\, q(t)\ (x(t))^\beta\, dt < \infty.$$

Since $x(t)$ is increasing, this implies that

$$\int_a^\infty t\, q(t)\, dt < \infty. \tag{4.3}$$

Combining (4.2) and (4.3) we are led to contradiction with the assumption (Ω_1) and to the conclusion that the equation (E) cannot have a positive solution of type (III).

Elimination of solution $x(t)$ of type (IV): If there exists a positive solution $x(t)$ of type (IV), from Proposition 3.3 we conclude that (Ω_1) is not satisfied. The obtained contradiction eliminates the positive solution of type (IV). This completes the proof. ⊠

Theorem 4.2 *Let $\beta < 1 \le \alpha$. The equation* (E) *is oscillatory if and only if*

$$\int_a^\infty \left(\frac{1}{p(t)}\int_a^t (t-s)s^\beta q(s)\, ds\right)^{\frac{1}{\alpha}} dt = \infty. \tag{Ω_2}$$

Proof: ($\Rightarrow$:) Suppose on the contrary that (Ω_2) fails to hold. Then, by the Theorem 1.2 the equation (E) has a positive solution $x(t)$ of type (D), which contradicts the assumption that all solutions of (E) are oscillatory.

($\Leftarrow$:) Assume that (Ω_2) is satisfied. We will show that (Ω_2) is sufficient to eliminate all four types (I)-(IV) of positive solutions of the sub-half-linear equation (E).

Elimination of solution $x(t)$ of type (I): Suppose that for a positive solution $x(t)$ of (E) (I) holds for all $t \ge t_0$. As in the proof of Theorem 4.1 we can obtain (4.1). Then

$$\int_a^t (t-s)\, s^\beta\, q(s)\, ds \le (t-a)\int_a^t s^\beta\, q(s)\, ds \le Q\cdot t \quad \text{for} \quad t \ge t_0\,,$$

where $Q = \displaystyle\int_a^\infty t^\beta q(t)\, dt$. Using the previous inequality we see that

$$\int_{t_0}^\infty \frac{1}{(p(t))^{\frac{1}{\alpha}}}\left[\int_a^t (t-s)\, q(s)\, s^\beta\, ds\right]^{\frac{1}{\alpha}} dt \le Q^{\frac{1}{\alpha}} \int_{t_0}^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt \tag{4.4}$$

In view of the basic integral condition (P_4), (4.4) implies that (Ω_2) fails to hold. The positive solution of type (I) is eliminated by this contradiction.

Elimination of solution $x(t)$ of type (II): If there exists a positive solution $x(t)$ of type (II), from Proposition 3.2 we conclude that (Ω_2) is not satisfied. The obtained contradiction eliminates the positive solution of type (II).

Elimination of solution $x(t)$ of type (III): As in the proof of Theorem 4.1 we can obtain (4.3). Using (4.3) and the fact that $\beta < 1$ implies $t^\beta < t$ for all $t \ge \max\{1, a\} = t_0$, we get (4.1). Now, following exactly the same steps of elimination of type (I), using (4.4), we are led to a contradiction with the assumption (Ω_2) and therefore, to the conclusion that the equation (E) cannot have a positive solution of type (III).

Elimination of solution $x(t)$ of type (IV): If there exists a positive solution $x(t)$ of type (IV), from Proposition 3.4 we find that (Ω_2) fails to hold. The positive solution of type (IV) is eliminated by this contradiction. This completes the proof. ⊠

5. Example

In this section we present an example that illustrates the results given in the previous section and the already known results from the papers [3] and [5]. Consider the equation

$$(E_1) \qquad \left(t^{\mu}\,|x''(t)|^{\alpha-1}x''(t)\right)'' + t^{-\lambda}\,|x(t)|^{\beta-1}x(t) = 0.$$

(1) Suppose that $\beta > 1 \geq \alpha$, $\lambda > 2$ and $\mu \leq 2\alpha$. The assumption $\mu \leq 2\alpha$ ensures that the condition (P_1) is satisfied for the function $p(t) = t^{\mu}$. By Theorem 3.2 in [5], the super-half-linear equation (E_1) is oscillatory if and only if

$$\int_a^{\infty} t\left(\frac{1}{p(t)}\int_t^{\infty}(s-t)\,q(s)\,ds\right)^{\frac{1}{\alpha}} dt = \infty. \tag{5.1}$$

It is easy to verify that for $q(t) = t^{-\lambda}$ and $\lambda > 2$

$$\int_t^{\infty}\int_s^{\infty} q(r)\,dr\,ds \sim t^{2-\lambda},\ \ t \to \infty,$$

which leads to

$$\int_a^{\infty} t\left(\frac{1}{p(t)}\int_t^{\infty}(s-t)\,q(s)\,ds\right)^{\frac{1}{\alpha}} dt \sim \int_a^{\infty} t^{1+\frac{2-\mu-\lambda}{\alpha}}\,dt,\ \ t \to \infty.$$

Consequently, for the equation (E_1) the condition (5.1) holds if $\lambda < 2\alpha + 2 - \mu$. Since the assumption $\alpha \in \left(\frac{\mu}{2}, 1\right]$ implies $2 < 2\alpha + 2 - \mu$, the super-half-linear equation (E_1) is oscillatory if $2 < \lambda < 2\alpha + 2 - \mu$.

(2a) Let $\beta < 1 \leq \alpha$. If $\mu \leq 1 + \alpha$ then the condition (P_1) holds. From Theorem 3.1 in [5], we get that the sub-half-linear equation (E_1) is oscillatory if and only if

$$\int_a^{\infty} q(t)\left[\int_a^t (t-s)\left(\frac{s}{p(s)}\right)^{\frac{1}{\alpha}} ds\right]^{\beta} dt = \infty. \tag{5.2}$$

Since $\displaystyle\int_a^t (t-s)\left(\frac{s}{p(s)}\right)^{\frac{1}{\alpha}} ds \sim t^{2+\frac{1-\mu}{\alpha}}$, $t \to \infty$, it is easy to verify that

$$\int_a^{\infty} q(t)\left[\int_a^t (t-s)\left(\frac{s}{p(s)}\right)^{\frac{1}{\alpha}} ds\right]^{\beta} dt \sim \int_a^{\infty} t^{-\lambda+(2+\frac{1-\mu}{\alpha})\beta}\,dt,\ \ t \to \infty,$$

and for the equation (E_1) the condition (5.2) holds if $\lambda < 1 + \beta\left(2 + \dfrac{1-\mu}{\alpha}\right)$. Finally, the sub-half-linear equation (E_1) is oscillatory if $\lambda < 1 + \beta\left(2 + \dfrac{1-\mu}{\alpha}\right)$.

(2b) Suppose that $\beta < \alpha < 1$. If $2\alpha < \mu \leq 1 + \alpha$ then the condition (P_3) holds. From Theorem 5.3 in [2], we have that the sub-half-linear equation (E_1) is oscillatory if and only if

(5.2) holds. Using the same reasoning as in (2a) we conclude that the sub-half-linear equation (E_1) is oscillatory if $\lambda < 1+\beta\left(2+\dfrac{1-\mu}{\alpha}\right)$.

(3a) Suppose that $\beta \geq 1 > \alpha$ and $\mu > 1+\alpha$. The assumption $\mu > 1+\alpha$ ensures that the condition (P_4) is satisfied for the function $p(t)=t^{\mu}$. Then, by using Theorem 4.1, we get that the super-half-linear equation (E_1) is oscillatory if and only if

$$(\Omega_1) \qquad \int_a^{\infty} t\, q(t)\, (\psi_1(t))^{\beta}\, dt = \infty\,.$$

Since $\psi_1(t) = \displaystyle\int_t^{\infty} \frac{s-t}{(p(s))^{\frac{1}{\alpha}}}\, ds \sim t^{2-\frac{\mu}{\alpha}}$, $t \sim \infty$, it follows by an easy calculation that

$$\int_a^{\infty} t\, q(t)\, (\psi_1(t))^{\beta}\, dt \sim \int_a^{\infty} t^{1-\lambda+\beta(2-\frac{\mu}{\alpha})}\, dt,\ \ t\to\infty.$$

Consequently, the condition (Ω_1) holds if $2-\lambda+\left(2-\dfrac{\mu}{\alpha}\right)\beta > 0$. Therefore, the super-half-linear equation (E_1) is oscillatory if $\lambda < 2+\left(2-\dfrac{\mu}{\alpha}\right)\beta$. Using the assumptions $\beta \geq 1 > \alpha$ and $\mu > 1+\alpha$ we have that $\lambda < 2+\left(1-\dfrac{1}{\alpha}\right)\beta < 2$.

(3b) Let $\beta \geq 1 > \alpha$. The condition (P_3) holds if $\mu \in (2\alpha, 1+\alpha]$. From Theorem 5.1 in [2], we get that the super-half-linear equation (E_1) is oscillatory if and only if (Ω_1) holds. Using the same reasoning as in (3a) we conclude that the super-half-linear equation (E_1) is oscillatory if $\lambda < 2+\left(2-\dfrac{\mu}{\alpha}\right)\beta < 2$.

(4) Suppose that $\beta < 1 \leq \alpha$. The assumption $\mu > 2\,\alpha$ ensures that the condition (P_4) is satisfied. Then, by using Theorem 4.2, we get that the sub-half-linear equation (E_1) is oscillatory if and only if

$$(\Omega_2) \qquad \int_a^{\infty} \left(\frac{1}{p(t)}\int_a^t (t-s)s^{\beta} q(s)\, ds\right)^{\frac{1}{\alpha}} dt = \infty.$$

.

For $q(t)=t^{-\lambda}$ it is easy to verify that $\displaystyle\int_a^t\int_a^s r^{\beta}\, q(r)\, dr\, ds \sim t^{\beta-\lambda+2}$, $t\to\infty$, and so

$$\int_a^{\infty} \left(\frac{1}{p(t)}\int_a^t (t-s)s^{\beta} q(s)\, ds\right)^{\frac{1}{\alpha}} dt \sim \int_a^{\infty} t^{\frac{\beta-\lambda+2-\mu}{\alpha}}\, dt,\ \ t\to\infty.$$

Then, the condition (Ω_2) holds if $\lambda < \alpha+\beta-\mu+2$. Finally, the sub-half-linear equation (E_1) is oscillatory if $\lambda < \alpha+\beta-\mu+2$. Using the assumptions $\beta < 1 \leq \alpha$ and $\mu > 2\,\alpha$ we get that $\lambda < \beta-\alpha+2 < 2$.

Note, that if we assume $\beta > 1 > \alpha$, we can use (1), (3a), (3b) to conclude that the super-half-linear equation (E_1) is oscillatory if:

(i) $\mu > 2\alpha$ and $\lambda \in \left(-\infty\,,\ 2+\left(2-\dfrac{\mu}{\alpha}\right)\beta\right)$

or

(ii) $\mu \leq 2\alpha$ and $2 < \lambda < 2\alpha + 2 - \mu$.

Therefore, (E_1) is oscillatory for all $\mu \in \mathbb{R}$ and $\lambda \in \left(-\infty\ ,\ 2 + \left(2 - \frac{\mu}{\alpha}\right)\beta\right) \cup (2, 2\alpha + 2 - \mu)$.
Also, if we assume $\beta < 1 < \alpha$, we can use (2a) and (4) to conclude that the sub-half-linear equation (E_1) is oscillatory if:

(i) $\mu \leq 1 + \alpha$ and $\lambda < 1 + \beta \left(2 + \frac{1-\mu}{\alpha}\right)$

or

(ii) $\mu > 2\alpha$ i $\lambda < \alpha + \beta - \mu + 2$.

Using (2b) we conclude that the sub-half-linear equation (E_1) is oscillatory for $\beta < \alpha < 1$ if

$\mu \in (2\alpha, 1 + \alpha]$ and $\lambda < 1 + \beta \left(2 + \frac{1-\mu}{\alpha}\right)$. ▲

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