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First Order Impulsive Differential Inclusions with Periodic Conditions

John R Graef[1] **and Abdelghani Ouahab**[2]

[1]Department of Mathematics, University of Tennessee at Chattanooga
Chattanooga, TN 37403-2504 USA
e-mail: John-Graef@utc.edu
[2] Laboratoire de Mathématiques, Université de Sidi Bel Abbès
BP 89, 22000 Sidi Bel Abbès, Algérie
e-mail: agh_ouahab@yahoo.fr

Abstract

In this paper, we present an impulsive version of Filippov's Theorem for the first-order nonresonance impulsive differential inclusion

$$\begin{aligned} y'(t)-\lambda y(t) &\in F(t,y(t)), \quad \text{a.e. } t\in J\backslash\{t_1,\dots,t_m\},\\ y(t_k^+)-y(t_k^-) &= I_k(y(t_k^-)), \quad k=1,\dots,m,\\ y(0) &= y(b), \end{aligned}$$

where $J=[0,b]$ and $F: J\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ is a set-valued map. The functions I_k characterize the jump of the solutions at impulse points t_k $(k=1,\dots,m.)$. Then the relaxed problem is considered and a Filippov-Wasewski result is obtained. We also consider periodic solutions of the first order impulsive differential inclusion

$$\begin{aligned} y'(t) &\in \varphi(t,y(t)), \quad \text{a.e. } t\in J\backslash\{t_1,\dots,t_m\},\\ y(t_k^+)-y(t_k^-) &= I_k(y(t_k^-)), \quad k=1,\dots,m,\\ y(0) &= y(b), \end{aligned}$$

where $\varphi: J\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ is a multi-valued map. The study of the above problems use an approach based on the topological degree combined with a Poincaré operator.

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1 Introduction

The dynamics of many processes in physics, population dynamics, biology, medicine, and other areas may be subject to abrupt changes such as shocks or perturbations (see for instance [1, 30] and the references therein). These perturbations may be viewed as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations. Important contributions to the study of the mathematical aspects of such equations can be found in the works by Bainov and Simeonov [7], Lakshmikantham, Bainov, and Simeonov [31], Pandit and Deo [36], and Samoilenko and Perestyuk [37] among others.

During the last couple of years, impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensely studied; see, for example, the monographs by Aubin [4] and Benchohra *et al.* [10], as well as the thesis of Ouahab [35], and the references therein.

In this paper, we will consider the problem

$$y'(t) - \lambda y(t) \in F(t, y(t)), \quad \text{a.e. } t \in J := [0, b], \tag{1}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k)), \quad k = 1, \dots, m, \tag{2}$$

$$y(0) = y(b), \tag{3}$$

where $\lambda \neq 0$ is a parameter, $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multi-valued map, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, \dots, m$, $t_0 = 0 < t_1 < \dots < t_m < t_{m+1} = b$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$, and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$.

First, we shall be concerned with Filippov's theorem for first order nonresonance impulsive differential inclusions. This is the aim of Section 3. Section 4 is devoted to the relaxed problem associated with problem (1)–(3), that is, the problem where we consider the convex hull of the right-hand side. The compactness of the solution sets is examined in Section 5. In Section 6, we study the existence of solutions of first order impulsive differential inclusions with periodic conditions.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multi-valued analysis that are used throughout this paper. Here, $C(J,\mathbb{R})$ will denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the Tchebyshev norm

$$\|x\|_\infty = \sup\{|x(t)| \,:\, t \in J\}.$$

In addition, we let $L^1(J,\mathbb{R})$ be the Banach space of measurable functions $x : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable with the norm

$$|x|_1 = \int_0^b |x(s)|ds.$$

If (X,d) is a metric space, the following notations will be used throughout this paper:

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$.
- $\mathcal{P}_p(X) = \{Y \in \mathcal{P}(X) : Y$ has the property "p"$\}$ where p could be: *cl*=closed, *b*=bounded, *cp*=compact, *cv*=convex, etc. Thus,
- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$.
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$.
- $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$.
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$.
- $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$.

Let $(X, \|.\|)$ be a Banach space and $F : J \to \mathcal{P}_{cl}(X)$ be a multi-valued map. We say that F is *measurable* provided for every open $U \subset X$, the set $F^{+1}(U) = \{t \in J : F(t) \subset U\}$ is Lebesgue measurable in J. We will need the following lemma.

Lemma 2.1 ([13, 17]) *The mapping F is measurable if and only if for each $x \in X$, the function $\zeta : J \to [0, +\infty)$ defined by*

$$\zeta(t) = \operatorname{dist}(x, F(t)) = \inf\{\|x - y\| \,:\, y \in F(t)\}, \quad t \in J,$$

is Lebesgue measurable.

Let $(X, \|\cdot\|)$ be a Banach space and $F : X \to \mathcal{P}(X)$ be a multi-valued map. We say that F has a *fixed point* if there exists $x \in X$ such that $x \in F(x)$. The set of fixed points of F will be denoted by $Fix\, F$. We will say that F has *convex (closed) values* if $F(x)$ is convex (closed) for all $x \in X$, and that F is *totally bounded* if $F(A) = \bigcup_{x \in A}\{F(x)\}$ is bounded in X for each bounded set A of X, i.e.,

$$\sup_{x \in A}\{\sup\{\|y\| \,:\, y \in F(x)\}\} < \infty.$$

Let (X, d) and (Y, ρ) be two metric spaces and $F : X \to \mathcal{P}_{cl}(Y)$ be a multi-valued mapping. Then F is said to be *lower semi-continuous* (*l.s.c.*) if the inverse image of V by F

$$F^{-1}(V) = \{x \in X \,:\, F(x) \cap V \neq \emptyset\}$$

is open for any open set V in Y. Equivalently, F is *l.s.c.* if the core of V by F

$$F^{+1}(V) = \{x \in X \,:\, F(x) \subset V\}$$

is closed for any closed set V in Y.

Likewise, the map F is called *upper semi-continuous* (*u.s.c.*) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subseteq Y$. That is, if the set $F^{-1}(V)$ is closed for any closed set V in Y. Equivalently, F is *u.s.c.* if the set $F^{+1}(V)$ is open for any open set V in Y.

The mapping F is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $F(A)$ is relatively compact, i.e., there exists a relatively compact set $K = K(A) \subset X$ such that

$$F(A) = \bigcup \{F(x) : x \in A\} \subset K.$$

Also, F is *compact* if $F(X)$ is relatively compact, and it is called *locally compact* if for each $x \in X$, there exists an open set U containing x such that $F(U)$ is relatively compact.

We denote the graph of F to be the set $\mathcal{G}r(F) = \{(x,y) \in X \times Y,\ y \in F(x)\}$ and recall the following facts.

Lemma 2.2 ([12], [15, Proposition 1.2]) *If $F : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $\mathcal{G}r(F)$ is a closed subset of $X \times Y$, i.e., for any sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$, if $x_n \to x_*$ and $y_n \to y_*$ as $n \to \infty$, and $y_n \in F(x_n)$, then $y_* \in F(x_*)$. Conversely, if F has nonempty compact values, is locally compact, and has a closed graph, then it is u.s.c.*

The following two lemmas are concerned with the measurability of multi-functions; they will be needed in this paper. The first one is the well known Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 2.3 ([17, Theorem 19.7]) *Let E be a separable metric space and G a multi-valued map with nonempty closed values. Then G has a measurable selection.*

Lemma 2.4 ([40]) *Let $G : J \to \mathcal{P}(E)$ be a measurable multifunction and let $g : J \to E$ be a measurable function. Then for any measurable $v : J \to \mathbb{R}_+$ there exists a measurable selection u of G such that*

$$|u(t) - g(t)| \leq d(g(t), G(t)) + v(t).$$

For any multi-valued function $G : J \times \mathbb{R}^n \to \mathcal{P}(E)$, we define

$$\|G(t,z)\|_{\mathcal{P}} := \sup\{|v| : v \in G(t,z)\}.$$

Definition 2.5 *The mapping G is called a multi-valued Carathéodory function if:*

(a) *The function $t \mapsto G(t,z)$ is measurable for each $z \in \mathcal{D}$;*

(b) *For a.e. $t \in J$, the map $z \mapsto G(t,z)$ is upper semi-continuous.*

Furthermore, it is an L^1-Carathéodory if it is locally integrably bounded, i.e., for each positive r, there exists some $h_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|G(t,z)\|_{\mathcal{P}} \le h_r(t) \ \textit{ for a.e. } \ t \in J \ \textit{ and all } \ \|z\| \le r.$$

Consider the Hausdorf pseudo-metric $H_d\colon\ \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ defined by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(E), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [28]). In particular, H_d satisfies the triangle inequality. Also, notice that if $x_0 \in E$, then

$$d(x_0, A) = \inf_{x\in A} d(x_0, x) \ \text{ whereas } \ H_d(\{x_0\}, A) = \sup_{x\in A} d(x_0, x).$$

Definition 2.6 *A multi-valued operator $N\colon\ E \to \mathcal{P}_{cl}(E)$ is called:*

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad \textit{for each } \ x, y \in E;$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Notice that if N is $\gamma-$Lipschitz, then

$$H_d(F(x), F(y)) \leq kd(x, y) \quad \text{for all} \quad x, y \in E.$$

For more details on multi-valued maps, we refer the reader to the works of Aubin and Cellina [5], Aubin and Frankowska [6], Deimling [15], Gorniewicz [17], Hu and Papageorgiou [25], Kamenskii [27], Kisielewicz [28], and Tolstonogov [38].

3 Filippov's Theorem

Let $J_k = (t_k, t_{k+1}]$, $k = 0, \ldots, m$, and let y_k be the restriction of a function y to J_k. In order to define mild solutions for problem (1)–(3), consider the space

$$PC = \{y\colon\ J \to \mathbb{R}^n \mid y_k \in C(J_k, \mathbb{R}^n),\ k = 0, \ldots, m,\ \text{and}$$
$$y(t_k^-)\ \text{and}\ y(t_k^+)\ \text{exist and satisfy}\ y(t_k^-) = y(t_k)\ \text{for}\ k = 1, \ldots, m\}.$$

Endowed with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_\infty :\ k = 0, \ldots, m\},$$

this is a Banach space.

Definition 3.1 *A function $y \in PC \cap \cup_{k=0}^m AC(J_k, \mathbb{R}^n)$ is said to be a solution of problem (1)–(3) if there exists $v \in L^1(J, \mathbb{R}^n)$ such that $v(t) \in F(t, y(t))$ a.e. $t \in J$, $y'(t) - \lambda y(t) = v(t)$ for $t \in J\backslash\{t_1, \ldots, t_m\}$, $y(t_k^+) - y(t_k) = I_k(y(t_k))$, $k = 1, \ldots, m$, and $y(0) = y(b)$.*

We will need the following auxiliary result in order to prove our main existence theorems.

Lemma 3.2 ([21]) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Then y is the unique solution of the problem*

$$y'(t) - \lambda y(t) = f(y(t)), \quad t \in J, \ t \neq t_k, \quad k = 1, \dots, m, \tag{4}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k)), \quad k = 1, \dots, m, \tag{5}$$

$$y(0) = y(b), \tag{6}$$

if and only if

$$y(t) = \int_0^b H(t,s) f(y(s))ds + \sum_{k=1}^{m} H(t,t_k) I_k(y(t_k)), \tag{7}$$

where

$$H(t,s) = (e^{-\lambda b} - 1)^{-1} \begin{cases} e^{-\lambda(b+s-t)}, & \text{if } 0 \le s \le t \le b, \\ \\ e^{-\lambda(s-t)}, & \text{if } 0 \le t < s \le b. \end{cases}$$

In the case of both differential equations and inclusions, existence results for problem (1)–(3) can be found in [20, 21, 35]. The main result of this section is a Filippov type result for problem (1)–(3).

Let $g \in L^1(J, \mathbb{R}^n)$ and let $x \in PC$ be a solution to the linear impulsive problem

$$\begin{cases} x'(t) - \lambda x(t) &= g(t), \qquad \text{a.e. } t \in J \backslash \{t_1, \dots, t_m\}, \\ x(t_k^+) - x(t_k) &= I_k(x(t_k^-)), \quad k = 1, \dots, m, \\ x(0) &= x(b). \end{cases} \tag{8}$$

Our main result in this section is contained in the following theorem.

Theorem 3.3 *Assume the following assumptions hold.*

($\mathcal{H}_1$) *The function $F\colon J \times \mathbb{R}^n \to \mathcal{P}_{cl}(\mathbb{R}^n)$ satisfies*

(a) for all $y \in \mathbb{R}^n$, the map $t \mapsto F(t,y)$ is measurable, and

(b) the map $t \mapsto \gamma(t) = d(g(t), F(t, x(t))$ is integrable.

($\mathcal{H}_2$) *There exist constants $c_k \ge 0$ such that*

$$|I_k(u) - I_k(z)| \le c_k |u - z| \quad \textit{for each} \quad u, z \in \mathbb{R}^n.$$

$(\mathcal{H}_3)$ *There exist a function $p \in L^1(J, \mathbb{R}^+)$ such that*

$$H_d(F(t,z_1), F(t,z_2)) \leq p(t)|z_1 - z_2| \ \ \textit{for all} \ \ z_1,\, z_2 \in \mathbb{R}^n.$$

If

$$\frac{H_*\|p\|_{L^1}}{1 - H_* \sum_{k=1}^{m} c_k} < 1,$$

then the problem (1)–(3) has at least one solution y satisfying the estimates

$$\|y - x\|_{PC} \quad \leq \quad \frac{\|\gamma\|_{L1}}{\left(1 - H_* \sum_{k=1}^{m} c_k - H_*\|p\|_{L^1}\right)}.$$

and

$$|y'(t) - \lambda y(t) - g(t)| \leq \widetilde{H} p(t) + |\gamma(t)|,$$

where

$$\widetilde{H} = \frac{H_*\|\gamma\|_{L^1}}{\left(1 - H_* \sum_{k=1}^{m} c_k - H_*\|p\|_{L^1}\right)}$$

and

$$H_* = \sup\{H(t,s) \mid (t,s) \in J \times J\}.$$

Proof Let $f_0 = g$ and

$$y_0(t) = \int_0^b H(t,s) f_0(s) ds + \sum_{k=0}^{m} H(t,t_k) I_k(x(t_k)), \qquad y_0(t_k) = x(t_k).$$

Let $U_1\colon\ J \to \mathcal{P}(\mathbb{R}^n)$ be given by $U_1(t) = F(t, y_0(t)) \cap \mathcal{B}(g(t), \gamma(t))$. Since g and γ are measurable, Theorem III.4.1 in [13] implies that the ball $\mathcal{B}(g(t), \gamma(t))$ is measurable. Moreover, $F(t, y_0)$ is measurable and U_1 is nonempty. Indeed, since $v = 0$ is a measurable function, from Lemma 2.4, there exists a function u which is a measurable selection of $F(t, y_0)$ and such that

$$|u(t) - g(t)| \leq d(g(t), F(t, y_0)) = \gamma(t).$$

Then $u \in U_1(t)$, proving our claim. We conclude that the intersection multi-valued operator $U_1(t)$ is measurable (see [6, 13, 17]). By the Kuratowski-Ryll-Nardzewski selection theorem (Lemma 2.3), there exists a function $t \to f_1(t)$ which is a measurable selection for U_1. Hence, $U_1(t) = F(t, y_0(t)) \cap \mathcal{B}(g(t), \gamma(t)) \neq \emptyset$. Consider

$$y_1(t) = \int_0^b H(t,s) f_1(s) ds + \sum_{k=0}^{m} H(t, t_k) I_k(y_1(t_k)), \qquad t \in J,$$

where y_1 is a solution of the problem

$$\left\{ \begin{array}{rcll} y'(t) - \lambda y(t) & = & f_1(t), & \text{a.e. } t \in J \backslash \{t_1, \ldots, t_m\}, \\ y(t_k^+) - y(t_k) & = & I_k(y(t_k^-)), & k = 1, \ldots, m, \\ y(0) & = & y(b). & \end{array} \right. \tag{9}$$

For every $t \in J$, we have

$$\begin{aligned} |y_1(t) - y_0(t)| & \leq \int_0^b |H(t,s)| |f_1(s) - f_0(s)| \, ds \\ & \quad + \sum_{k=1}^{m} |I_k(y_0(t_k)) - I_k(y_1(t_k))| \\ & \leq H_* \|\gamma\|_{L^1} + H_* \sum_{k=1}^{m} c_k |y_1(t_k) - y_0(t_k)|. \end{aligned}$$

Then,

$$\|y_1 - y_0\|_{PC} \leq \frac{H_*}{1 - H_* \sum_{k=1}^{m} c_k} \|\gamma\|_{L^1}.$$

Define the set-valued map $U_2(t) = F(t, y_1(t)) \cap \mathcal{B}(f_1(t), p(t)|y_1(t) - y_0(t)|)$. The multifunction $t \to F(t, y_1)$ is measurable and the ball $\mathcal{B}(f_1(t), p(t)\|y_1 - y_0\|_{\mathcal{D}})$ is measurable by Theorem III.4.1 in [13]. To see that the set $U_2(t) = F(t, y_1) \cap \mathcal{B}(f_1(t), p(t)\|y_1 - y_0\|_{\mathcal{D}})$ is nonempty, observe that since f_1 is a measurable function, Lemma 2.4 yields a measurable selection u of $F(t, y_1)$ such that

$$|u(t) - f_1(t)| \leq d(f_1(t), F(t, y_1))$$

Moreover, $\|y_1 - y_0\|_{\mathcal{D}} \leq \eta_0(t_1) \leq \beta$. Then, using $(\mathcal{H}_2)$, we have

$$\begin{aligned} |u(t) - f_1(t)| &\leq d(f_1(t), F(t, y_1)) \\ &\leq H_d(F(t, y_0), F(t, y_1)) \\ &\leq p(t)\|y_0 - y_1\|_{\mathcal{D}}, \end{aligned}$$

i.e., $u \in U_2(t) \neq \emptyset$. Since the multi-valued operator U_2 is measurable (see [6, 13, 17]), there exists a measurable function $f_2(t) \in U_2(t)$. Then define

$$y_2(t) = \int_0^b H(t,s)f_2(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y_2(t_k)), \qquad t \in J,$$

where y_2 is a solution of the problem

$$\left\{ \begin{array}{rcll} y'(t) - \lambda y(t) &=& f_2(t), & \text{a.e. } t \in J\backslash\{t_1, \ldots, t_m\}, \\ y(t_k^+) - y(t_k) &=& I_k(y(t_k^-)), & k = 1, \ldots, m, \\ y(0) &=& y(b). & \end{array} \right. \tag{10}$$

We then have

$$\begin{aligned} |y_2(t) - y_1(t)| &\leq H_* \int_0^b |f_2(s) - f_1(s)|\,ds + H_* \sum_{k=1}^m c_k|y_2(t_k) - y_1(t_k)| \\ &\leq H_* \int_0^b p(s)|y_1(s) - y_0(s)|\,ds + H_* \sum_{k=1}^m c_k|y_2(t_k) - y_1(t_k)| \\ &\leq \frac{H_*^2}{1 - H_* \sum_{k=1}^m c_k} \|\gamma\|_{L^1}\|p\|_{L^1} + H_* \sum_{k=1}^m c_k|y_2(t_k) - y_1(t_k)|. \end{aligned}$$

Thus,

$$\|y_2 - y_1\|_{PC} \leq \frac{H_*^2}{\left(1 - H_* \sum_{k=1}^m c_k\right)^2} \|p\|_{L^1}\|\gamma\|_{L^1}.$$

Let $U_3(t) = F(t, y_2(t)) \cap \mathcal{B}(f_2(t), p(t)|y_2(t) - y_1(t)|)$. Arguing as we did for U_2 shows that U_3 is a measurable multi-valued map with nonempty values, so there exists a measurable selection $f_3(t) \in U_3(t)$. Consider

$$y_3(t) = \int_0^b H(t,s)f_3(s)ds + \sum_{k=1}^m I_k(y_3(t_k)), \qquad t \in J,$$

where y_3 is a solution of the problem

$$\left\{\begin{array}{rcll} y'(t)-\lambda y(t) &=& f_3(t), & \text{a.e. } t\in J\backslash\{t_1,\dots,t_m\},\\ y(t_k^+)-y(t_k) &=& I_k(y(t_k^-)), & k=1,\dots,m,\\ y(0) &=& y(b). & \end{array}\right. \tag{11}$$

We have

$$|y_3(t)-y_2(t)| \;\leq\; H_*\int_0^t |f_3(s)-f_2(s)|\,ds + H_*\sum_{k=1}^m c_k|y_3(t_k)-y_2(t_k)|.$$

Hence, from the estimates above, we have

$$\|y_3-y_2\|_{PC}\leq \frac{H_*^3}{\left(1-H_*\sum\limits_{k=1}^m c_k\right)^3}\|p\|_{L^1}^2\|\gamma\|_{L^1}.$$

Repeating the process for $n=1,2,\dots$, we arrive at the bound

$$\|y_n-y_{n-1}\|_{PC}\leq \frac{H_*^n}{\left(1-H_*\sum\limits_{k=1}^m c_k\right)^n}\|p\|_{L^1}^{n-1}\|\gamma\|_{L^1}. \tag{12}$$

By induction, suppose that (12) holds for some n. Let

$$U_{n+1}(t)=F(t,y_n(t))\cap\mathcal{B}(f_n(t),p(t)|y_n(t)-y_{n-1}(t)|).$$

Since again U_{n+1} is measurable (see [6, 13, 17]), there exists a measurable function $f_{n+1}(t)\in U_{n+1}(t)$ which allows us to define

$$y_{n+1}(t)=\int_0^b H(t,s)f_{n+1}(s)ds+\sum_{k=1}^m H(t,t_k)I_k(y_{n+1}(t_k)), \qquad t\in J. \tag{13}$$

Therefore,

$$|y_{n+1}(t)-y_n(t)| \;\leq\; H_*\int_0^b |f_{n+1}(s)-f_n(s)|\,ds + H_*\sum_{k=1}^m c_k|y_n(t_k)-y_{n-1}(t_k)|.$$

Thus, we arrive at

$$\|y_{n+1}-y_n\|_{PC}\leq \frac{H_*^{n+1}}{\left(1-H_*\sum\limits_{k=1}^m c_k\right)^{n+1}}\|p\|_{L^1}^{n}\|\gamma\|_{L^1}. \tag{14}$$

Hence, (12) holds for all $n \in \mathbb{N}$, and so $\{y_n\}$ is a Cauchy sequence in PC, converging uniformly to a function $y \in PC$. Moreover, from the definition of $U_n,\ n \in \mathbb{N}$,

$$|f_{n+1}(t) - f_n(t)| \leq p(t)|y_n(t) - y_{n-1}(t)| \quad \text{for a.e. } \ t \in J.$$

Therefore, for almost every $t \in J$, $\{f_n(t) : n \in \mathbb{N}\}$ is also a Cauchy sequence in $\mathbb{R}^n$ and so converges almost everywhere to some measurable function $f(\cdot)$ in $\mathbb{R}^n$. Moreover, since $f_0 = g$, we have

$$\begin{aligned} |f_n(t)| &\leq |f_n(t) - f_{n-1}(t)| + |f_{n-1}(t) - f_{n-2}(t)| + \ldots + |f_2(t) - f_1(t)| \\ &\quad + |f_1(t) - f_0(t)| + |f_0(t)| \\ &\leq \sum_{k=2}^{n} p(t)|y_{k-1}(t) - y_{k-2}(t)| + \gamma(t) + |f_0(t)| \\ &\leq p(t)\sum_{k=2}^{\infty} |y_{k-1}(t) - y_{k-2}(t)| + \gamma(t) + |g(t)| \\ &\leq \widetilde{H}p(t) + \gamma(t) + |g(t)|. \end{aligned}$$

Then, for all $n \in \mathbb{N}$,

$$|f_n(t)| \leq \widetilde{H}p(t) + \gamma(t) + g(t) \text{ a.e. } t \in J. \tag{15}$$

From (15) and the Lebesgue Dominated Convergence Theorem, we conclude that f_n converges to f in $L^1(J, \mathbb{R}^n)$. Passing to the limit in (13), the function

$$y(t) = \int_0^b H(t,s)f(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)), \quad t \in J,$$

is a solution to the problem (1)–(3).

Next, we give estimates for $|y'(t) - \lambda y(t) - g(t)|$ and $|x(t) - y(t)|$. We have

$$\begin{aligned} |y'(t) - \lambda y(t) - g(t)| &= |f(t) - f_0(t)| \\ &\leq |f(t) - f_n(t)| + |f_n(t) - f_0(t)| \end{aligned}$$

$$\leq |f(t) - f_n(t)| + \sum_{k=2}^{n} |f_k(t) - f_{k-1}(t)| + \gamma(t)$$
$$\leq |f(t) - f_n(t)| + \sum_{k=2}^{n} p(t)|y_{k-1}(t) - y_{k-2}(t)| + \gamma(t).$$

Using (14) and passing to the limit as $n \to +\infty$, we obtain

$$\begin{aligned}|y'(t) - \lambda y(t) - g(t)| &\leq p(t)\sum_{k=2}^{\infty} |y_{k-1}(t) - y_{k-2}(t)| + \gamma(t) + |f(t) - f_n(t)| \\ &\leq p(t)\sum_{k=2}^{\infty} \frac{H_*^{k-1}\|p\|_{L^1}^{k-2}\|\gamma\|_{L^1}}{\left(1 - H_*\sum_{i=1}^{m} c_i\right)^{k-1}} + |\gamma|,\end{aligned}$$

so

$$|y' - \lambda y - g| \leq \widetilde{H}p(t) + \gamma(t), \quad t \in J.$$

Similarly,

$$\begin{aligned}|x(t) - y(t)| &= \Big|\int_0^b H(t,s)g(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(x(t_k)) \\ &\quad - \int_0^b H(t,s)f(s)ds - \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k))\Big| \\ &\leq H_*\int_0^b |f(s) - f_0(s)|ds + H_*\sum_{k=1}^{m} c_k|x(t_k) - y(t_k)| \\ &\leq H_*\int_0^b |f(s) - f_n(s)|ds + H_*\int_0^b |f_n(s) - f_0(s)|ds \\ &\quad + H_*\sum_{k=1}^{m} c_k|x(t_k) - y(t_k)|.\end{aligned}$$

As $n \to \infty$, we arrive at

$$\|x - y\|_{PC} \quad \leq \quad \frac{H_*\|p\|_{L^1}\|\gamma\|_{L^1}}{(1 - H_*\sum_{k=1}^{m} c_k)(1 - H_*\sum_{k=1}^{m} c_k - H_*\|p\|_{L^1})} + \frac{\|\gamma\|_{L^1}}{1 - H_*\sum_{k=1}^{m} c_k}$$

$$= \frac{\|\gamma\|_{L^1}}{1 - H_* \sum_{k=1}^{m} c_k - H_*\|p\|_{L^1}},$$

completing the proof of the theorem.

4 Relaxation Theorem

In this section, we examine to what extent the convexification of the right-hand side of the inclusion introduces new solutions. More precisely, we want to find out if the solutions of the nonconvex problem are dense in those of the convex one. Such a result is known in the literature as a Relaxation theorem and has important implications in optimal control theory. It is well-known that in order to have optimal state-control pairs, the system has to satisfy certain convexity requirements. If these conditions are not present, then in order to guarantee existence of optimal solutions we need to pass to an augmented system with convex structure by introducing the so-called relaxed (generalized, chattering) controls. The resulting relaxed problem has a solution. The Relaxation theorems tell us that the relaxed optimal state can be approximated by original states, which are generated by a more economical set of controls that are much simpler to build. In particular, "strong relaxation" theorems imply that this approximation can be achieved using states generated by bang-bang controls. More precisely, we compare trajectories of (1)–(3) to those of the relaxation impulsive differential inclusion

$$x'(t) - \lambda x(t) \in \overline{co}F(t, x(t)), \quad a.e.\ t \in J\backslash\{t_1, \ldots, t_m\}, \tag{16}$$

$$x(t_k^+) - x(t_k) = I_k(x(t_k^-)), \quad k = 1, \ldots, m, \tag{17}$$

$$x(0) = x(b). \tag{18}$$

Theorem 4.1 ([24]) *Let $U : J \to \mathcal{P}_{cl}(E)$ be a measurable, integrably bounded set-valued map and $t \to d(0, U(t))$ be an integrable map. Then, the integral $\int_0^b U(t)dt$ is* *convex and* $t \longrightarrow \overline{co}U(t)$ *is measurable. Moreover, for every* $\epsilon > 0$ *and every measurable selection of* u *of* $\overline{co}U(t)$*, there exists a measurable selection* $\overline{u}$ *of* U *such that*

$$\sup_{t\in J}\Big|\int_0^t u(s)ds - \int_0^t \overline{u}(s)ds\Big| \leq \epsilon$$

and

$$\overline{\int_0^b \overline{co}U(t)dt} = \overline{\int_0^b U(t)dt} = \int_0^b \overline{co}U(t)dt.$$

We will need the following lemma to prove our main result in this section.

Lemma 4.2 (Covitz and Nadler [14]) *Let* (X,d) *be a complete metric space. If* $G : X \to \mathcal{P}_{cl}(X)$ *is a contraction, then* $FixG \neq \emptyset$.

We now present a relaxation theorem for the problem (1)–(3).

Theorem 4.3 *Assume that* $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$ *hold and*

$(\overline{\mathcal{H}}_1)$ *The function* $F\colon J\times\mathbb{R}^n \to \mathcal{P}_{cl}(\mathbb{R}^n)$ *satisfies that for all* $y \in \mathbb{R}^n$ *the map* $t \mapsto F(t,y)$ *is measurable, and the map* $t \mapsto \gamma(t) = d(g(t), F(t,0))$ *is integrable.*

Assume that

$$\frac{H_*\|p\|_{L^1}}{1 - H_*\sum_{k=1}^m c_k} < 1,$$

and let x *be a solution of (16)–(18). Then, for every* $\epsilon > 0$ *there exists a solution* y *to (1)–(3) on* J *satisfying*

$$\|x-y\|_\infty \leq \epsilon.$$

This implies that $S^{co} = \overline{S^F}$*, where*

$$S^{co} = \{x \,|\, x \textit{ is a solution to (16)–(18)}\}$$

and

$$S^F = \{y \,|\, y \textit{ is a solution to (1)–(3)}\}.$$

Proof. First, we prove that $S^{co} \neq \emptyset$. We transform the problem (16)–(18) into a fixed point problem. Consider the operator $N : PC(J, \mathbb{R}^n) \to \mathcal{P}(PC(J, \mathbb{R}^n))$ defined by

$$N(y) = \{h \in PC : h(t) = \int_0^b H(t,s)g(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)),\ g \in S_{\overline{co}F,y}\}.$$

We shall show that N satisfies the assumptions of Lemma 4.2. The proof will be given in two steps.

Step 1: *$N(y) \in P_{cl}(PC)$ for each $y \in PC$.*

Let $(y_n)_{n\geq 0} \in N(y)$ be such that $y_n \longrightarrow \tilde{y}$ in PC. Then $\tilde{y} \in PC$ and there exists $g_n \in S_{\overline{co}F,y}$ such that

$$y_n(t) = \int_0^b H(t,s)g_n(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)).$$

From $\overline{\mathcal{H}}_1$ and $\mathcal{H}_3$, we have

$$|g_n(t)| \leq p(t)\|y\|_{PC} + d(0, F(t,0)) := M(t),$$

and observe that

$$n \in \mathbb{N} \quad \text{implies} \quad g_n(t) \in M(t)B(0,1) \quad \text{for} \quad t \in J.$$

Now $B(0,1)$ is compact in $\mathbb{R}^n$, so passing to a subsequence if necessary, we have that $\{g_n\}$ converges to some function g. An application of the Lebesgue Dominated Convergence Theorem shows that

$$\|g_n - g\|_{L^1} \to 0 \text{ as } n \to \infty.$$

Using the continuity of I_k, we have

$$y_n(t) \longrightarrow \tilde{y}(t) = \int_0^b H(t,s)g(s)ds + \sum_{k=1}^{m} H_k(t,t_k)I_k(y(t_k)),$$

and so $\tilde{y} \in N(y)$.

Step 2: *There exists* $\gamma < 1$ *such that* $H(N(y), N(\overline{y})) \leq \gamma \|y - \overline{y}\|_{PC}$ *for each* y, $\overline{y} \in PC$.

Let y, $\overline{y} \in PC$ and $h_1 \in N(y)$. Then there exists $g_1(t) \in F(t, y(t))$ such that for each $t \in J$,

$$h_1(t) = \int_0^b H(t,s) g_1(s) ds + \sum_{k=1}^m H(t, t_k) I_k(y(t_k)).$$

From $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, it follows that

$$H(F(t, y(t)), F(t, \overline{y}(t)) \leq p(t)|y(t) - \overline{y}(t)|.$$

Hence, there is $w \in F(t, \overline{y}(t))$ such that

$$|g_1(t) - w| \leq p(t)|y(t) - \overline{y}(t)|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R}^n)$ given by

$$U(t) = \{w \in \mathbb{R}^n : |g_1(t) - w| \leq p(t)|y(t) - \overline{y}(t)|\}.$$

Since the multi-valued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see [6, 13]), there exists a function $g_2(t)$, which is a measurable selection for V. So, $g_2(t) \in F(t, \overline{y}(t))$ and

$$|g_1(t) - g_2(t)| \leq p(t)|y(t) - \overline{y}(t)|, \quad \text{for each } \ t \in J.$$

Let us define for each $t \in J$,

$$h_2(t) = \int_0^b H(t,s) g_2(s) ds + \sum_{k=1}^m H(t, t_k) I_k(\overline{y}(t_k)).$$

Then, we have

$$|h_1(t) - h_2(t)| \ \leq \ \int_0^b H(t,s)|g_1(s) - g_2(s)| ds + \sum_{k=1}^m |H(t, t_k)| c_k |y(t_k) - \overline{y}(t_k)|$$

$$\begin{aligned} &\leq H_* \int_0^b p(s)|y(s)-\overline{y}(s)|ds + H_* \sum_{k=1}^m c_k \|y-\overline{y}\|_{PC} \\ &\leq H_* \int_0^b p(s)ds\|y-\overline{y}\|_{PC} + H_* \sum_{k=1}^m c_k \|y-\overline{y}\|_{PC}. \end{aligned}$$

Thus,

$$\|h_1 - h_2\|_{PC} \leq \frac{H_*\|p\|_{L^1}}{1 - H_* \sum_{k=1}^m c_k} \|y-\overline{y}\|_{PC}.$$

By an analogous relation obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \leq \frac{H_*\|p\|_{L^1}}{1 - H_* \sum_{k=1}^m c_k} \|y-\overline{y}\|_{PC}.$$

Therefore, N is a contraction, and so by Lemma 4.2, N has a fixed point y that is solution to (16)–(18).

We next prove that $S^{co} = \overline{S^F}$. Let x be a solution of Problem (16)–(18); then there exists $g \in S_{\overline{co}F,x}$ such that

$$x(t) = \int_0^b H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(x(t_k)),\ t \in J.$$

Hence, x is a solution of the problem

$$\left\{ \begin{array}{rcll} x'(t) - \lambda x(t) &=& g(t), & \text{a.e. } t \in J\backslash\{t_1,\dots,t_m\}, \\ x(t_k^+) - x(t_k) &=& I_k(x(t_k^-)), & k = 1,\dots,m, \\ x(0) &=& x(b). & \end{array} \right. \tag{19}$$

From Theorem 4.1, we have that for every $\epsilon > 0$ and $g \in \overline{co}F(t,x(t))$ there exists a measurable selection f_* of $t \to F(t,x(t))$ such that

$$\begin{aligned} \Big|\int_0^b H(t,s)f_*(s)ds - \int_0^b H(t,s)g(s)ds\Big| &\leq H_*\Big|\int_0^t f_*(s)ds - \int_0^t g(s)ds\Big| \\ &\leq H_* \sup_{t\in J}\Big|\int_0^t f_*(s)ds - \int_0^t g(s)ds\Big| \\ &\leq \frac{H_*\epsilon\left(1 - H_* \sum_{k=1}^m c_k - H_*\|p\|_{L^1}\right)}{2\|p\|_{L^1}} := \delta. \end{aligned}$$

Let

$$z(t) = \int_0^b H(t,s)f_*(s)ds + \sum_{0<t_k<t} H_k(t,t_k)I_k(x(t_k)),\ t \in J.$$

Observing that $z(t_k) = x(t_k)$, we see that for $t \in J$,

$$|x(t) - z(t)| \le \delta.$$

It follows that for all $u \in B(x(t), \delta)$,

$$\begin{aligned}
\gamma(t) := d(g(t), F(t, x(t)) &\le d(g(t), u) + H_d(F(t, z(t)), F(t, x(t))), \\
&\le H_d(\overline{co}F(t, x(t)), \overline{co}F(t, z(t))) + H_d(F(t, z(t)), F(t, x(t))) \\
&\le 2p(t)|x(t) - z(t)| \le 2p(t)\delta.
\end{aligned}$$

Since γ is measurable (see [6]), the above inequality also shows that $\gamma \in L^1(J, \mathbb{R}^n)$. From Theorem 3.3, problem (1)–(3) has a solution y such that

$$\|x - y\|_{PC} \le \frac{\|\gamma\|_{L^1}}{1 - H_* \sum_{k=1}^{m} c_k - H_*\|p\|_{L^1}}.$$

Since $\gamma(t) \le 2\delta p(t)$, this becomes

$$\|x - y\|_{PC} \le \frac{2\delta\|p\|_{L^1}}{1 - H_* \sum_{k=1}^{m} c_k - H_*\|p\|_{L^1}},$$

so

$$\|x - y\|_{PC} \le H_*\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $S^{co} = \overline{S^F}$, which completes the proof of the theorem.

5 Compactness of the Solution Set

Let us introduce the following hypotheses. Notice that the first part of condition $(\mathcal{A}_2)$ below is actually condition $(\mathcal{H}_3)$ above, and condition $(\mathcal{A}_3)$ is the same as $(\mathcal{H}_2)$ above. We list them here in this form for the convenience of the reader.

$(\mathcal{A}_1)$ $F : J \times \mathbb{R}^n \longrightarrow \mathcal{P}_{cl,cv}(\mathbb{R}^n)$; $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}^n$.

$(\mathcal{A}_2)$ There exists a function $p \in L^1(J, \mathbb{R}^+)$ such that, for *a.e.* $t \in J$ and all $x, y \in \mathbb{R}^n$,

$$H_d(F(t,x), F(t,y)) \leq p(t)|x-y|$$

and

$$H_d(0, F(t,0)) \leq p(t) \text{ for a.e. } t \in J.$$

$(\mathcal{A}_3)$ There exist constants $c_k \geq 0$ such that

$$|I_k(u) - I_k(z)| \leq c_k|u-z|, \text{ for each } \quad u, z \in \mathbb{R}^n.$$

Our first compactness result is the following.

Theorem 5.1 *Suppose that hypotheses $(\mathcal{A}_1) - (\mathcal{A}_3)$ are satisfied. If*

$$H_*\|p\|_{L^1} + H_* \sum_{k=1}^{m} c_k < 1,$$

then the solution set of the problem (1)–(3) is nonempty and compact.

Proof. Let $N : PC(J, \mathbb{R}^n) \to \mathcal{P}(PC(J, \mathbb{R}^n))$ be defined by

$$N(y) = \{h \in PC : h(t)) = \int_0^b H(t,s)v(s)ds + \sum_{k=0}^{m} H(t,t_k)I_k(x(t_k)), \ v \in S_{F,y}\},$$

where

$$S_{F,y} = \{v \in L^1(J, \mathbb{R}^n) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}.$$

First we show that $N(y) \in \mathcal{P}_{cl}(PC)$ for each $y \in PC$. To do this, let $(y_n)_{n\geq 1} \in N(y)$ be such that $y_n \longrightarrow \tilde{y}$ in PC. Then, there exists $v_n \in S_{F,y}$, $n = 0, 1, \ldots$, such that for each $t \in J$,

$$y_n(t) = \int_0^b H(t,s)v_n(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y_n(t_k)).$$

From $(\mathcal{A}_2)$, we have $v_n(t) \in \overline{B}(0, p(t)|y(t)| + p(t))$, where

$$\overline{B}(0, p(t)|y(t)| + p(t)) = \{w \in \mathbb{R}^n : |w| \leq p(t)|y(t)| + p(t)\} := \varphi(t).$$

It is clear that $\varphi : J \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is a multi-valued map that is integrably bounded. Since $\{v_n(\cdot) : n \geq 1\} \in \varphi(\cdot)$, we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w(J, \mathbb{R}^n)$. From Mazur's lemma, there exists

$$v \in \overline{conv}\{v_n(t) : n \geq 1\},$$

so there exists a subsequence $\{\bar{v}_n(t) : n \geq 1\}$ in $\overline{conv}\{v_n(t) : n \geq 1\}$, such that $\bar{v}_n$ converges strongly to $v \in L^1(J, \mathbb{R}^n)$. From $(\mathcal{A}_2)$, we have for every $\epsilon > 0$ there exists $n_0(\epsilon)$ such that for every $n \geq n_0(\epsilon)$, we have

$$v_n(t) \in F(t, y_n(t)) \subseteq F(t, \widetilde{y}(t)) + \epsilon p(t) B(0, 1).$$

This implies that $v(t) \in F(t, y(t))$, a.e. $t \in J$. Thus, we have

$$\widetilde{y}(t) = \int_0^b H(t, s) v(s) ds + \sum_{k=1}^{m} H(t, t_k) I_k(\widetilde{y}(t_k)).$$

Hence, $\tilde{y} \in N(y)$. By the same method used in [8, 20, 35], we can prove that N has at least one fixed point.

Now we prove that $S^F \in \mathcal{P}_{cp}(PC)$, where

$$S^F = \{y \in PC \,|\, y \text{ is a solution of the problem (1)–(3)}\}.$$

Let $(y_n)_{n \in \mathbb{N}} \in S^F$; then there exist $v_n \in S_{F, y_n}$, $n \in \mathbb{N}$, such that

$$y_n(t) = \int_0^b H(t, s) v_n(s) ds + \sum_{k=1}^{m} H(t, t_k) I_k(y(t_k)), \quad t \in J.$$

From $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$, we have

$$|y_n(t)| \;\; \leq \;\; H_* \int_0^b p(s) |y_n(s)| ds + H_* \|p\|_{L^1}$$

$$+H_* \sum_{k=1}^{m} c_k |y_n(t_k)| + H_* \sum_{k=1}^{m} c_k |I_k(0)|.$$

Hence,

$$\|y_n\|_{PC} \leq \frac{1}{1 - H_* \sum_{k=1}^{m} c_k - H_* \|p\|_{L^1}} \left(H_* \|p\|_{L^1} + H_* \sum_{k=1}^{m} |I_k(0)| \right) := M, \text{ for all } n \in \mathbb{N}.$$

Next, we prove that $\{y_n : n \in \mathbb{N}\}$ is equicontinuous in PC. Let $0 < \tau_1 < \tau_2 \leq b$; then we have

$$\begin{aligned} |y_n(\tau_2) - y_n(\tau_1)| &\leq \int_0^b |H(\tau_2, s) - H(\tau_1, s)||v_n(s)|ds \\ &\quad + \sum_{k=1}^{m} |H(\tau_2, t_k) - H(\tau_1, t_k)|[Mc_k + |I_k(0)|] \\ &\leq (M+1) \int_0^b |H(\tau_2, s) - H(\tau_1, s)|p(s)ds \\ &\quad + \sum_{k=1}^{m} |H(\tau_2, t_k) - H(\tau_1, t_k)|[Mc_k + |I_k(0)|]. \end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$. This proves the equicontinuity for the case where $t \neq t_i \;\; i = 1, \ldots, m$. It remains to examine the equicontinuity at $t = t_i$.

Set

$$h_1(t) = \sum_{k=1}^{m} H(t, t_k) I_k(y_n(t_k))$$

and

$$h_2(t) = \int_0^b H(t, s) y_n(s) ds.$$

First, we prove equicontinuity at $t = t_i^-$. Fix $\delta_1 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$ and

$$h_1(t_i) = \sum_{k=1}^{m} H(t_i, t_k) I_k(y(t_k))$$

For $0 < h < \delta_1$, we have

$$\begin{aligned}|h_1(t_i - h) - h_1(t_i)| &\leq \sum_{k=1,k\neq i}^{m} |[H(t_i - h, t_k) - H(t_i, t_k)]I(y_n(t_k^-))| \\ &\leq \sum_{k=1,k\neq i}^{m} |H(t_i - h, t_k) - H(t_i, t_k)|[Mc_k + |I_k(0)|].\end{aligned}$$

The right-hand side tends to zero as $h \to 0$. Moreover,

$$|h_2(t_i - h) - h_2(t_i)| \leq (M + 1) \int_0^b |H(t_i - h, s) - H(t_i, s)|p(s)ds$$

which tends to zero as $h \to 0$.

Next, we prove equicontinuity at $t = t_i^+$. Fix $\delta_2 > 0$ such that $\{t_k : \ k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. Then, for $0 < h < \delta_2$, we have

$$\begin{aligned}|h_1(t_i + h) - h_1(t_i)| &\leq \sum_{k=1,k\neq i}^{m} |[H(t_i + h, t_k) - H(t_i, t_k)]I(y_n(t_k^-))| \\ &\leq \sum_{k=1,k\neq i}^{m} |H(t_i + h, t_k) - H(t_i, t_k)|[Mc_k + |I_k(0)|]\end{aligned}$$

Again, the right-hand side tends to zero as $h \to 0$. Similarly,

$$|h_2(t_i + h) - h_2(t_i)| \leq (M + 1) \int_0^b |H(t_i + h, s) - H(t_i, s)|p(s)ds$$

tends to zero as $h \to 0$.

Thus, the set $\{y_n : n \in \mathbb{N}\}$ is equicontinuous in PC. As a consequence of the Arzelá-Ascoli Theorem, we conclude that there exists a subsequence of $\{y_n\}$ converging to y in PC. As we did above, we can easily prove that there exists $v(\cdot) \in F(\cdot, y_\cdot)$ such that

$$y(t) = \int_0^b H(t, s)v(s)ds + \sum_{k=1}^{m} H(t, t_k)I_k(y(t_k)), \quad t \in J.$$

Hence, $S^F \in \mathcal{P}_{cp}(PC)$. This completes the proof of the theorem.

Our next theorem yields the same conclusion under the somewhat different hypotheses.

Theorem 5.2 *Assume that the following conditions hold.*

($\mathcal{H}_4$) The multifunction $F : J \times \mathbb{R}^n \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is L^1-Carathéodory.

($\mathcal{H}_5$) There exist functions $\bar{p}$, $\bar{q} \in L^1(J, \mathbb{R}_+)$ and $\alpha \in [0,1)$ such that

$$\|F(t,y)\|_{\mathcal{P}} \leq \bar{p}(t)|y|^{\alpha} + \bar{q}(t) \ \text{ for each } (t,y) \in J \times \mathbb{R}^n.$$

In addition, suppose that there exist constants $c_k^*, b_k^* \in \mathbb{R}_+$ and $\alpha_k \in [0,1)$ such that

$$|I_k(y)| \leq c_k^* + b_k^*|y|^{\alpha_k}, \quad y \in \mathbb{R}^n.$$

Then the solution set of the problem (1)–(3) is nonempty and compact.

Proof. Let $S^F = \{y \in PC \,|\, y \ \text{ is a solution of the problem (1)–(3)}\}$. From results in [9, 20, 35], it follows that $S^F \neq \emptyset$. Now, we prove that S^F is compact. Let $(y_n)_{n\in\mathbb{N}} \in S^F$; then there exist $v_n \in S_{F,y_n}$, $n \in \mathbb{N}$, such that

$$y_n(t) = \int_0^b H(t,s)v_n(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)), \quad t \in J.$$

From ($\mathcal{H}_4$), we can prove that there exists an $M_1 > 0$ such that

$$\|y_n\|_{PC} \leq M_1, \quad \text{for every } n \geq 1.$$

Similar to what we did in the proof of Theorem 5.1, we can use ($\mathcal{H}_5$) to show that the set $\{y_n : n \geq 1\}$ is equicontinuous in PC. Hence, by the Arzelá-Ascoli Theorem, we can conclude that there exists a subsequence of $\{y_n\}$ converging to y in PC. We shall show that there exist $v(\cdot) \in F(\cdot, y(\cdot))$ such that

$$y(t) = \int_0^b H(t,s)v(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)), \quad t \in J.$$

Since $F(t,\cdot)$ is upper semicontinuous, for every $\varepsilon > 0$, there exist $n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y(t)) + \varepsilon B(0,1), \quad \text{a.e.} \quad t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v(\cdot) \quad \text{as} \quad m \to \infty$$

and

$$v(t) \in F(t, y(t)), \ \text{a.e. } t \in J, \ \text{and for all } m \in \mathbb{N}.$$

It is clear that

$$|v_{n_m}(t)| \leq \bar{p}(t), \ \text{ a.e. } \ t \in J.$$

By the Lebesgue Dominated Convergence Theorem and the continuity of I_k, we conclude that $v \in L^1(J, \mathbb{R}^n)$ so $v \in S_{F,y}$. Thus,

$$y(t) = \int_0^b H(t,s)v(s)\,ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)), \ \ t \in J.$$

Therefore, $S^F \in \mathcal{P}_{cp}(PC)$, and this completes the proof of the theorem.

6 Periodic Solutions

In this section, we consider the impulsive periodic problem

$$y'(t) \in \varphi(t, y(t)), \quad \text{a.e. } t \in J\backslash\{t_1, \ldots, t_m\}, \tag{20}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{21}$$

$$y(0) = y(b), \tag{22}$$

where $\varphi : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multifunction.

A number of papers have been devoted to the study of initial and boundary value problems for impulsive differential inclusions. Some basic results in the theory of periodic boundary value problems for first order impulsive differential equations and inclusions may be found in [21, 32, 33, 34, 35] and the references therein. Our goal in this section is to give an existence result for the above problem by using topological degree combined with a Pointcaré operator.

6.1 Background in Geometric Topology

First, we begin with some elementary concepts from geometric topology. For additional details, we recommend [11, 19, 22, 26]. In what follows, (X, d) denotes a metric space. A set $A \in \mathcal{P}(X)$ is called a *contractible* set provided there exists a continuous homotopy $h : A \times [0, 1] \to A$ such that

(i) $h(x, 0) = x$, for every $x \in A$, and

(ii) $h(x, 1) = x_0$, for every $x \in A$.

Note that if $A \in \mathcal{P}_{cv,cp}(X)$, then A is contractible. Clearly, the class of contractible sets is much larger than the class of all compact convex sets.

Definition 6.1 *A space X is called an absolute retract (written as $X \in AR$) provided that for every space Y, a closed subset $B \subseteq Y$, and a continuous map $f : B \to X$, there exists a continuous extension $\widetilde{f} : Y \to X$ of f over Y, i.e., $\widetilde{f}(x) = f(x)$ for every $x \in B$.*

Definition 6.2 *A space X is called an absolute neighborhood retract (written as $X \in ANR$) if for every space Y, any closed subset $B \subseteq Y$, and any continuous map $f : B \to X$, there exists a open neighborhood U of B and a continuous map $\widetilde{f} : U \to X$ such that $\widetilde{f}(x) = f(x)$ for every $x \in B$.*

Definition 6.3 *A space X is called an $R_\delta-$set provided there exists a sequence of nonempty compact contractible spaces $\{X_n\}$ such that:*

$$X_{n+1} \subset X_n \text{ for every } n;$$

$$X = \bigcap_{n=1}^{\infty} X_n.$$

It is well known that any contractible set is acyclic and that the class of acyclic sets is larger then that of contractible sets. From the continuity of the Čech cohomology functor, we have the following lemma.

Lemma 6.4 ([19]) *Let X be a compact metric space. If X is an R_δ–set, then it is an acyclic space.*

Set

$$K^n(r) = K^n(x,r), \quad S^{n-1}(r) = \partial K^n(r), \quad \text{and} \quad P^n = \mathbb{R}^n \backslash \{0\},$$

where $K^n(r)$ is a closed ball in $\mathbb{R}^n$ with center x and radius r, and $\partial K^n(r)$ stands for the boundary of $K^n(r)$ in $\mathbb{R}^n$. For any $X \in ANR-$space X, we set

$$J(K^n(r), X) = \{F : X \to \mathcal{P}(X) \mid F \text{ u.s.c with } R_\delta\text{–values}\}.$$

Moreover, for any continuous $f : X \to \mathbb{R}^n$, where $X \in ANR$, we set

$$J_f(K^n(r), X) = \{\varphi : K^n(r) \to \mathcal{P}(X) \mid \varphi = f \circ F \text{ for some } F \in J(K^n(r), X) \text{ and } \varphi(S^{n-1}(r)) \subset P^n\}.$$

Finally, we define

$$CJ(K^n(r), \mathbb{R}^n) = \cup\{J_f(K^n(r), \mathbb{R}^n) \mid f : X \to \mathbb{R}^n \text{ is continuous and } X \in ANR\}.$$

It is well known that (see [17]) that for the multi-valued maps in this class, the notion of topological degree is available. To define it, we need an appropriate concept of homotopy in $CJ(K^n(r), \mathbb{R}^n))$.

Definition 6.5 *Let $\phi_1,\ \phi_2 \in CJ(K^n(r), \mathbb{R}^n)$ be two maps of the form*

$$\phi_1 = f_1 \circ F_1 : K^n(r) \xrightarrow{F_1} \mathcal{P}(X) \xrightarrow{f_1} \mathbb{R}^n$$

$$\phi_2 = f_2 \circ F_2 : K^n(r) \xrightarrow{F_2} \mathcal{P}(X) \xrightarrow{f_2} \mathbb{R}^n.$$

We say that ϕ_1 *and* ϕ_2 *are homotopic in* $CJ(K^n(r), \mathbb{R}^n)$ *if there exist an u.s.c.* R_δ*–valued homotopy* $\chi : [0,1]\times K^n(r) \to \mathcal{P}(X)$ *and a continuous homotopy* $h : [0,1]\times X \to \mathbb{R}^n$ *satisfying*

(i) $\chi(0, u) = F_1(u),\ \chi(1, u) = F_2(u)$ *for every* $u \in K^n(r)$*,*

(ii) $h(0, x) = f_1(x),\ h(1, x) = f_2(x)$ *for every* $x \in X$*,*

(iii) for every $(u, \lambda) \in [0, 1] \times S^{n-1}(r)$ *and* $x \in \chi(\lambda, u)$*, we have* $h(x, \lambda) \neq 0$*.*

The map $H : [0, 1] \times K^n(r) \to \mathcal{P}(\mathbb{R}^n)$ *given by*

$$H(\lambda, u) = h(\lambda, \chi(\lambda, u))$$

is called a homotopy in $CJ(K^n(r), \mathbb{R}^n)$ *between* ϕ_1 *and* ϕ_2*.*

Theorem 6.6 ([17]) *There exist a map* $Deg : CJ(K^n(r), \mathbb{R}^n) \to \mathbb{Z}$*, called the topological degree function, satisfying the following properties:*

($\mathcal{C}_1$) *If* $\varphi \in CJ(K^n(r), \mathbb{R}^n)$ *is of the form* $\varphi = f \circ F$ *with* F *single valued and continuous, then* $Deg(\varphi) = deg(\varphi)$*, where* $deg(\varphi)$ *stands for the ordinary Brouwer degree of the single valued continuous map* $\varphi : K^n(r) \to \mathbb{R}^n$*.*

($\mathcal{C}_2$) *If* $Deg(\varphi) = 0$*, where* $\varphi \in CJ(K^n(r), \mathbb{R}^n)$*, then there exists* $u \in K^n(r)$ *such that* $0 \in \varphi(u)$*.*

($\mathcal{C}_3$) *If* $\varphi \in CJ(K^n(r), \mathbb{R}^n)$ *and* $\{u \in K^n(r) | 0 \in \varphi(u)\} \subset IntK^n(r_0)$ *for some* $0 < r_0 < r$*, then the restriction* φ_0 *of* φ *to* $K^n(r_0)$ *is in* $CJ(K^n(r), \mathbb{R}^n)$ *and* $Deg(\varphi_0) = Deg(\varphi)$*.*

Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$; $CJ_0(A, B)$ will denote the class of mappings

$$CJ_0(K^n(r), \mathbb{R}^n) = \{\varphi : A \to \mathcal{P}(B) \mid \varphi = f \circ F, \quad F : A \to \mathcal{P}(X), \quad F \text{ is u.s.c.}$$

with R_δ–values and $f : X \to B$ is continuous},

where $X \in ANR$. The next two definitions were introduced in [18]

Definition 6.7 *A metric space X is called acyclically contractible if there exists an acyclic homotopy $\Pi : X \times [0,1] \to \mathcal{P}(X)$ such that*

(a) $x_0 \in \Pi(x,1)$ for every $x \in X$ and for some $x_0 \in X$;

(b) $x \in \Pi(x,0)$ for every $x \in X$.

Notice that any contractible space and any acyclic, compact metric space are acyclically contractible (see [3], Theorem 19). Also, from [17], any acyclically contractible space is acyclic.

Definition 6.8 *A metric space X is called $R_\delta-$contractible if there exists a multivalued homotopy $\Pi : X \times [0,1] \to \mathcal{P}(X)$ which is u.s.c. and satisfies:*

(a) $x \in \Pi(x,1)$ for every $x \in X$;

(b) $\Pi(x,0) = B$ for every $x \in X$ and for some $B \subset X$;

(c) $\Pi(x,\alpha)$ is an $R_\delta-$set for every $\alpha \in [0,1]$ and $x \in X$.

6.2 Poincaré translation operator

By Poincaré operators we mean the translation operator along the trajectories of the associated differential system, and the first return (or section) map defined on the cross section of the torus by means of the flow generated by the vector field. The translation operator is sometimes also called the Poincaré-Andronov, or Levinson, or simply the T-operator. In the classical theory (see [29, 39] and the references therein), both these operators are defined to be single-valued, when assuming, among other things, the uniqueness of solutions of initial value problems. In the absence of uniqueness, it is often possible to approximate the right-hand sides of the given systems by locally lipschitzian ones (implying uniqueness already), and then apply a standard limiting argument. This might be, however, rather complicated and is impossible for discontinuous right-hand sides. On the other hand, set-valued analysis allows us to handle effectively such classically troublesome situations. For additional background details, see [2, 17].

Let $\varphi : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a Carathédory map. We define a multi-valued map

$$S_\varphi : \mathbb{R}^n \to \mathcal{P}(PC)$$

by

$$S_\varphi(x) = \{y \mid y(\cdot, x) \text{ is a solution of the problem satisfying } y(0,x) = x\}.$$

Consider the operator P_t defined by $P_t = \Psi \circ S_\varphi$ where

$$P_t : \mathbb{R}^n \xrightarrow{S_\varphi} \mathcal{P}(PC) \xrightarrow{\Psi_t} \mathcal{P}(\mathbb{R}^n)$$

and

$$\Psi_t(y) = y(0) - y(t).$$

Here, P_t is called the Poincaré translation map associated with the Cauchy problem

$$y'(t) \in \varphi(t, y(t)), \quad \text{a.e. } t \in J \backslash \{t_1, \ldots, t_m\}, \tag{23}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{24}$$

$$y(0) = y_0 \in \mathbb{R}^n. \tag{25}$$

The following lemma is easily proved.

Lemma 6.9 *Let $\varphi : J \times \mathbb{R}^n \to \mathcal{P}_{cv,cp}(\mathbb{R}^n)$ be a Carathédory multfunction. Then the periodic problem (20)–(22) has a solution if and only if for some $y_0 \in \mathbb{R}^n$ we have $0 \in P_b(y_0)$, where P_b is the Poincaré map associated with (23)–(25).*

Next, we define what is meant by an upper-Scorza-Dragoni map.

Definition 6.10 *We say that a multi-valued map* $F : J\times\mathbb{R}^n \to \mathcal{P}_{cl}(\mathbb{R}^n)$ *has the upper-Scorza-Dragoni property if, given* $\delta > 0$*, there is a closed subset* $A_\delta \subset J$ *such that the measure* $\mu(A_\delta) \leq \delta$ *and the restriction* $\widetilde{F}$ *of* F *to* $A_\delta \times \mathbb{R}^n$ *is u.s.c.*

We also need the following two lemmas.

Lemma 6.11 ([16]) *Let* $\varphi\colon\ J \times \mathbb{R}^n \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ *be upper-Scorza-Dragoni. Assume that:*

($\mathcal{R}_1$) There exist functions $p \in L^1(J, \mathbb{R}_+)$ *and* $\psi : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\|\varphi(t, y)\|_{\mathcal{P}} \leq p(t)\psi(|y|) \quad \textit{for each} \quad (t, y) \in J \times \mathbb{R}^n.$$

($\mathcal{R}_2$) *There exist constants* c^*_k*,* $b^*_k \in \mathbb{R}_+$ *and* $\alpha_k \in [0, 1)$ *such that*

$$|I_k(y)| \leq c^*_k + b^*_k|y|^{\alpha_k} \quad y \in \mathbb{R}^n.$$

Then the set S_φ *is* $R_\delta-$*contractible.*

Lemma 6.12 ([16]) *Let* $\varphi : J \times \mathbb{R}^n \to \mathcal{P}_{cv,cp}(\mathbb{R}^n)$ *be upper-Scagoni-Dragoni. Let* $P_b : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ *be the Poincaré map associated with the problem (23)–(25). Assume that there exists* $r > 0$ *such that*

$$0 \notin P_b(y_0) \quad \textit{for every} \quad y_0 \in S^{n-1}(r).$$

Then,

$$P_b \in CJ(K^n(r), \mathbb{R}^n).$$

Furthermore, if $Deg(P_b) \neq 0$*, then the impulsive periodic problem (20)–(22) has a solution.*

The following Theorem due to Gorniewicz [17] is critical in the proof of the main result in this section.

Theorem 6.13 (Nonlinear Alterntive). *Assume that* $\varphi \in CJ_0(K^n(r), \mathbb{R}^n)$. *Then* φ *has at least one of the following properties:*

(i) $Fix(\varphi) \neq \emptyset$,

(ii) there is an $x \in S^{n-1}(r)$ *with* $x \in \lambda\varphi(x)$ *for some* $0 < \lambda < 1$.

The following definition and lemma can be found in [17, 23].

Definition 6.14 *A mapping* $F : X \to \mathcal{P}(Y)$ *is LL-selectionable provided there exists a measurable, locally-Lipchitzian map* $f : X \to Y$ *such that* $f \subset F$.

Lemma 6.15 *If* $\varphi : X \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ *is an u.s.c. multi-valued map, then* φ *is* $\sigma-$*LL-selectionable.*

We are now ready to give our main result in this section.

Theorem 6.16 *Let* φ: $\mathbb{R}^n \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ *be an u.s.c. multifunction. In addition to conditions* $(\mathcal{R}_1) - (\mathcal{R}_2)$, *assume that*

(R_3) *There exists* $r > 0$ *such that*

$$\frac{r}{\psi(r)\|p\|_{L^1} + \sum_{k=1}^{m}[c_k^* + b_k^* r^{\alpha_k}]} > 1.$$

Then the problem (20)–(22) has at least one solution.

Proof. From Lemma 6.15, φ is $\sigma-$LL-selectionable, so by a result of Djebali *et al.* [16], S_φ is $R_\delta-$contractible. Set $A = B = \mathbb{R}^n$ and $X = PC \in ANR$. We will prove that

$$\Psi : PC \to \mathbb{R}^n \quad \text{defined by} \quad y \to \Psi(y) = y(0) - y(\cdot)$$

is a continuous map. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in PC. Then,

$$|\Psi(y_n)(t) - \Psi(y)(t)| \ \leq \ 2\|y_n - y\|_{PC} \to 0 \ \text{ as } n \to \infty.$$

Hence,

$$P_b \in CJ_0(K^n(r), \mathbb{R}^n).$$

Let $a \in P_t(a) = \lambda(\Psi_t \circ S_\varphi)(a)$ for some $\lambda \in (0,1)$. Then, there exist $y \in PC$ such that $y \in S_\varphi(a)$. This implies $y(0) = a$ and $a = \lambda(a - y(t)),\ a \in S^{n-1}(r)$. For $t \in J$, we have

$$\begin{aligned} |a| &\leq \|y(t)\| \\ &\leq \int_0^t p(s)\psi(|y(s)|)ds + \sum_{k=1}^m [c_k^* + b_k^*|y(t_k)|^{\alpha_k}] \\ &\leq \psi(r)\int_0^b p(s)ds + \sum_{k=1}^m [c_k^* + b_k^* r^{\alpha_k}]. \end{aligned}$$

Hence,

$$\frac{|a|}{\psi(r)\|p\|_{L^1} + \sum_{k=1}^m [c_k^* + b_k^* r^{\alpha_k}]} \leq 1.$$

Next, we will show that S_φ is u.s.c. by proving that the graph

$$\Gamma_\varphi := \{(x,y) \mid y \in S_\varphi(x)\}$$

of S_φ is closed. Let $(x_n, y_n) \in \Gamma_\varphi$, i.e., $y_n \in S_\varphi(x_n)$, and let $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Since $y_n \in S_\varphi(x_n)$, there exists $v_n \in L^1(J, \mathbb{R}^n)$ such that

$$y_n(t) = x_n + \int_0^t v_n(s)ds + \sum_{0<t_k<t} I_k(y_n(t_k)),\ t \in J.$$

Since (x_n, y_n) converge to (x, y), there exists $M > 0$ such that

$$|x_n| \leq M \ \text{ for all } n \in \mathbb{N}.$$

By using $(\mathcal{R}_1) - (\mathcal{R}_2)$, we can easily prove that there exist $\overline{M} > 0$ such that

$$\|y_n\|_{PC} \leq \overline{M} \ \text{ for all } \ n \in \mathbb{N}.$$

From the definition of y_n, we have $y_n'(t) = v_n(t)$ a.e. $t \in J$, so

$$|v_n(t)| \leq p(t)\psi(M), \ t \in J.$$

Thus, $v_n(t) \in p(t)\psi(M)\overline{B}(0,1) := \chi(t)$ a.e. $t \in J$. It is clear that $\chi : J \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is a multivalued map that is integrably bounded. Since $\{v_n(\cdot) : n \geq 1\} \in \chi(\cdot)$, we may pass to a subsequence if necessary to obtain that v_n converges weakly to v in $L^1_w(J, \mathbb{R}^n)$. From Mazur's lemma, there exists

$$v \in \overline{conv}\{v_n(t) : n \geq 1\},$$

so there exists a subsequence $\{\bar{v}_n(t) : n \geq 1\}$ in $\overline{conv}\{v_n(t) : n \geq 1\}$, such that $\bar{v}_n$ converges strongly to $v \in L^1(J, \mathbb{R}^n)$. Since $F(t, .)$ is u.s.c., for every $\epsilon > 0$ there exists $n_0(\epsilon)$ such that for every $n \geq n_0(\epsilon)$, we have

$$\bar{v}_n(t) \in F(t, y_n(t)) \subseteq F(t, \widetilde{y}(t)) + \epsilon B(0,1).$$

This implies that $v(t) \in F(t, y(t))$, a.e. $t \in J$. Let

$$z(t) = x + \int_0^t v(s)ds + \sum_{0<t_k<t} I_k(y(t_k)), \ t \in J.$$

Since the functions I_k, $k = 1, \ldots, m$ are continuous, we obtain the estimates

$$||y_n - z\|_{PC} \ \leq \ |x_n - x| + \int_0^b |\bar{v}_n(s) - v(s)|ds + \sum_{k=1}^m |I_k(y_n(t_k)) - I_k(y(t_k))|.$$

The right-hand side of the above expression tends to 0 as $n \to +\infty$. Hence,

$$y(t) = x + \int_0^t v(s)ds + \sum_{0<t_k<t} I_k(y(t_k)), \ t \in J.$$

Thus, $y \in S_\varphi(x)$. Now, we show that S_φ maps bounded sets into relatively compact sets of PC. Let B be a bounded set in $\mathbb{R}^n$ and let $\{y_n\} \subset S_\varphi(B)$. Then there exist $\{x_n\} \subset B$ such that

$$y_n(t) = x_n + \int_0^t v_n(s)ds + \sum_{0<t_k<t} I_k(y_n(t_k)), \ t \in J,$$

where $v_n \in S_{\varphi,y_n},\ n \in \mathbb{N}$. Since $\{x_n\}$ is a bounded sequence, there exists a subsequence of $\{x_n\}$ converging to x, so from $(\mathcal{R}_1) - (\mathcal{R}_2)$, there exist $M_* > 0$ such that

$$\|y_n\|_{PC} \leq M_*,\ n \in \mathbb{N}.$$

As in the proof of Theorem 5.1, we can show that $\{y_n : n \in \mathbb{N}\}$ is equicontinuous in PC. As a consequence of the Arzelá-Ascoli Theorem, we conclude that there exists a subsequence of $\{y_n\}$ converging to y in PC. By a similar argument to the one above, we can prove that

$$y(t) = x + \int_0^t v(s)ds + \sum_{0<t_k<t} I_k(y(t_k)),\ t \in J,$$

where $v \in S_{F,y}$. Thus, $y \in S_\varphi(x)$. This implies that S_φ is u.s.c.

As a consequence of the nonlinear alternative of Leray Schauder type [17], we conclude that $Fix\ P_b \neq \emptyset$. This completes the proof of the theorem.

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