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Global configurations of singularities for quadratic differential systems with exactly two finite singularities of total multiplicity four

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Abstract. In this article we obtain the *geometric classification* of singularities, finite and infinite, for the three subclasses of quadratic differential systems with finite singularities with total multiplicity $m_f = 4$ possessing exactly two finite singularities, namely: (i) systems with two double complex singularities (18 configurations); (ii) systems with two double real singularities (33 configurations) and (iii) systems with one triple and one simple real singularities (123 configurations). We also give here the global bifurcation diagrams of configurations of singularities, both finite and infinite, with respect to the *geometric equivalence relation*, for these subclasses of systems. The bifurcation set of this diagram is algebraic. The bifurcation diagram is done in the 12-dimensional space of parameters and it is expressed in terms of invariant polynomials, which give an algorithm for determining the geometric configuration of singularities for any quadratic system.

Keywords: quadratic vector fields, infinite and finite singularities, affine invariant polynomials, Poincaré compactification, configuration of singularities, geometric equivalence relation.

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1 Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1.1}$$

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where $p, q \in \mathbb{R}[x,y]$, i.e. p, q are polynomials in x, y over $\mathbb{R}$. We call *degree* of a system (1.1) the integer $m = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1.1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

The study of the class **QS** has proved to be quite a challenge since hard problems formulated more than a century ago, are still open for this class. It is expected that we have a finite number of phase portraits in **QS**. We have phase portraits for several subclasses of **QS** but to obtain the complete topological classification of these systems, which occur rather often in applications, is a daunting task. This is partly due to the elusive nature of limit cycles and partly to the rather large number of parameters involved. This family of systems depends on twelve parameters but due to the group action of real affine transformations and time homothecies, the class ultimately depends on five parameters which is still a rather large number of parameters. For the moment only subclasses depending on at most three parameters were studied globally, including their global bifurcation diagrams (for example [1]). On the other hand we can restrict the study of the whole quadratic class by focusing on specific global features of the systems in this family. We may thus focus on the global study of singularities and their bifurcation diagram. The singularities are of two kinds: finite and infinite. The infinite singularities are obtained by compactifying the differential systems on the sphere, on the Poincaré disk, or on the projective plane as defined in Subsection 2 (see [16,20]).

The global study of quadratic vector fields began with the study of these systems in the neighborhood of infinity [15,22,27,28]. In [7] the authors classified topologically (adding also the distinction between nodes and foci) the whole quadratic class, according to configurations of their finite singularities.

To reduce the number of phase portraits in half in topological classification problems of quadratic systems, the *topological equivalence relation* was taken to mean the existence of a homeomorphism of the phase plane carrying orbits to orbits and *preserving or reversing* the orientation.

We use the concepts and notations introduced in [6] and [2] which we describe in Section 2. To distinguish among the foci (or saddles) we use the notion of order of the focus (or of the saddle) defined using the algebraic concept of Poincaré–Lyapunov constants. We call strong focus (or strong saddle) a focus (or a saddle) whose linearization matrix has non-zero trace. Such a focus (or saddle) will be denoted by f (respectively s). A focus (or saddle) with trace zero is called a weak focus (weak saddle). We denote by $f^{(i)}$ ($s^{(i)}$) the weak foci (weak saddles) of order i and by c and $\$$ the centers and integrable saddles. For more notations see Subsection 2.5.

In the topological classification no distinction was made among the various types of foci or saddles, strong or weak of various orders. However these distinctions of an algebraic nature are very important in the study of perturbations of systems possessing such singularities. Indeed, the maximum number of limit cycles which can be produced close to the weak foci of a system in **QS** in perturbations inside the class **QS** depends on the orders of the foci.

There are also three kinds of simple nodes: nodes with two characteristic directions (the generic nodes), nodes with one characteristic direction and nodes with an infinite number of characteristic directions (the star nodes). The three kinds of nodes are distinguished algebraically. Indeed, the linearization matrices of the two direction nodes have distinct eigenvalues, they have identical eigenvalues and they are not diagonal for the one direction nodes, and they have identical eigenvalues and they are diagonal for the star nodes (see [2,4,6]). We recall that the star nodes and the one direction nodes could produce foci in perturbations.

Furthermore a generic node at infinity may or may not have the two exceptional curves lying on the line at infinity. This leads to two different situations for the phase portraits. For this reason we split the generic nodes at infinity in two types as indicated in Subsection 2.5.

The *geometric equivalence relation* (see further below) for finite or infinite singularities, introduced in [6] and used in [2–5], takes into account such distinctions. This equivalence relation is also deeper than the *qualitative equivalence relation* introduced by Jiang and Llibre in [19] because it distinguishes among the foci (or saddles) of different orders and among the various types of nodes. This equivalence relation induces also a deeper distinction among the more complicated degenerate singularities.

In quadratic systems weak singularities could be of orders 1, 2 or 3 [12]. For details on Poincaré–Lyapunov constants and weak foci of various orders we refer to [20, 26]. As indicated before, algebraic information plays a fundamental role in the study of perturbations of systems possessing such singularities. In [31] necessary and sufficient conditions for a quadratic system to have weak foci (saddles) of orders i, $i = 1, 2, 3$ are given in invariant form.

For the purpose of classifying **QS** according to their singularities, finite or infinite, we use the *geometric equivalence relation* which involves only algebraic methods. It is conjectured that there are about 2000 distinct *geometric configurations* of singularities. The first step in this direction was done in [6] where the global classification of singularities at infinity of *the whole class* **QS**, was done according to the *geometric equivalence relation* of configurations of infinite singularities. This work was then (partially) extended to also incorporate finite singularities. We initiated this work in [2] where this classification was done for the case of singularities with a total finite multiplicity $m_f \leq 1$, continued it in [3] where the classification was done for $m_f = 2$ and in [4] and [5] where the classification was done for $m_f = 3$.

In the present article our goal is to go one step further in the *geometric classification* of global configurations of singularities by studying here the case of finite singularities with total finite multiplicity four and exactly two finite singularities.

We recall below the notion of *geometric configuration of singularities* defined in [3] for both finite and infinite singularities. We distinguish two cases:

1) Consider a system with a finite number of singularities, finite and infinite. In this case we call *geometric configuration of singularities*, finite and infinite, the set of all these singularities (real and complex) together with additional structure consisting of i) their multiplicities, ii) their local phase portraits around real singularities, each endowed with additional geometric structure involving the concepts of tangent, order and blow-up equivalence defined in Section 4 of [6] (or [2]) and Section 3 of [3].

2) If the line at infinity is filled up with singularities, in each one of the charts at infinity, the corresponding system in the Poincaré compactification (see Section 2) is degenerate and we need to do a rescaling of an appropriate degree of the system, so that the degeneracy be removed. The resulting systems have only a finite number of singularities on the line at infinity. In this case we call *geometric configuration of singularities*, finite and infinite, the set of all points at infinity (they are all singularities) in which we single out the singularities at infinity of the "reduced" system, taken together with their local phase portraits and we also take the local phase portraits of finite singularities each endowed with additional geometric structure to be described in Section 2.

Remark 1.1. We note that the geometric equivalence relation for configurations is much deeper than the topological equivalence. Indeed, for example the topological equivalence does not distinguish between the following three configurations which are geometrically non-equivalent: **1)** n, f; $\overline{\binom{1}{1}}SN$, ©, ©, **2)** n, $f^{(1)}$; $\overline{\binom{1}{1}}SN$, ©, ©, and **3)** n^d, $f^{(1)}$; SN, ©, © where n and

n^d mean singularities which are nodes, respectively two directions and one direction nodes, capital letters indicate points at infinity, ⓒ in case of a complex point and SN a saddle–node at infinity and $\overline{\binom{1}{1}}$ encodes the multiplicities of the saddle-node SN. For more details see the notation in Subsection 2.5.

The invariants and comitants of differential equations used for proving our main result are obtained following the theory of algebraic invariants of polynomial differential systems, developed by Sibirsky and his disciples (see for instance [9,14,24,30,33]).

Our results are stated in the following theorem.

Main Theorem. (A) *We consider here all configurations of singularities, finite and infinite, of quadratic vector fields with finite singularities of total multiplicity* $m_f = 4$ *possessing exactly two distinct finite singularities. These configurations are classified in the diagrams from Tables 1.1–1.3 according to the geometric equivalence relation. We have 174 geometrically distinct configurations of singularities, finite and infinite. More precisely 18 geometrically distinct configurations with two double complex finite singularities; 33 geometrically distinct configurations with two double real finite singularities, and 123 with one triple and one simple real finite singularities.*

(B) *Necessary and sufficient conditions for each one of the 174 different geometric equivalence classes can be assembled from these diagrams in terms of 20 invariant polynomials with respect to the action of the affine group and time rescaling appearing in the Tables 1.1–1.3 (see Remark 1.2 for a source of these invariants).*

(C) *The Tables 1.1–1.3 actually contain the global bifurcation diagrams in the 12-dimensional space of parameters, of the global geometric configurations of singularities, finite and infinite, of these subclasses of quadratic differential systems and provide an algorithm for finding for any given system in any of the three families considered, its respective geometric configuration of singularities.*

Remark 1.2. The diagrams are constructed using the invariant polynomials $\mu_0, \mu_1, \ldots$ which are defined in Section 5 of [5] and may be downloaded from the web page:

`http://mat.uab.es/~artes/articles/qvfinvariants/qvfinvariants.html`

together with other useful tools. In Tables 1.1–1.3 the conditions on these invariant polynomials are listed on the left side of the diagrams, while the specific geometric configurations appear on the right side of the diagrams. These configurations are expressed using the notation described in Subsection 2.5.

2 Concepts and results in the literature useful for this paper

2.1 Compactification on the sphere and on the Poincaré disk

Planar polynomial differential systems (1.1) can be compactified on the 2-dimensional sphere as follows. We first include the affine plane (x, y) in $\mathbb{R}^3$, with its origin at $(0, 0, 1)$, and we consider it as the plane $z = 1$. We then use a central projection to send the vector field to the upper and to the lower hemisphere. The vector fields thus obtained on the two hemispheres are analytic and diffeomorphic to our vector field on the (x, y) plane. By a theorem stated by Poincaré and proved in [17] there exists an analytic vector field on the whole sphere which simultaneously extends the vector fields on the two hemispheres to the whole sphere. We call *Poincaré compactification on the sphere* of the planar polynomial system, the restriction of the vector field thus obtained on the sphere, to the upper hemisphere completed with the

$\mu_0 \neq 0$, $\mathbf{D} = \mathbf{T} = 0$, $\mathbf{PR} < 0$

- $\eta < 0$
  - $\theta < 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ N^\infty,\ \copyright, \copyright$
  - $\theta > 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ N^f,\ \copyright, \copyright$
  - $\theta = 0$
    - $\theta_2 \neq 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ N^d,\ \copyright, \copyright$
    - $\theta_2 = 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ N^*,\ \copyright, \copyright$
- $\eta > 0$
  - $\theta < 0$
    - $\theta_1 < 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ S, N^\infty,\ N^\infty$
    - $\theta_1 > 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ S, N^f,\ N^f$
  - $\theta > 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ S, N^\infty,\ N^f$
  - $\theta = 0$
    - $\theta_1 < 0$
      - $\theta_2 \neq 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ S, N^\infty,\ N^d$
      - $\theta_2 = 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ S, N^\infty,\ N^*$
    - $\theta_1 > 0$
      - $\theta_2 \neq 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ S, N^f,\ N^d$
      - $\theta_2 = 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ S, N^f,\ N^*$
    - $\theta_1 = 0$
      - $\theta_4 \neq 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ S, N^d,\ N^*$
      - $\theta_4 = 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ S, N^*,\ N^*$
- $\eta = 0$
  - $\widetilde{M} \neq 0$
    - $\theta < 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ \overline{\binom{0}{2}}SN,\ N^\infty$
    - $\theta > 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ \overline{\binom{0}{2}}SN,\ N^f$
    - $\theta = 0$
      - $\theta_2 \neq 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ \overline{\binom{0}{2}}SN,\ N^d$
      - $\theta_2 = 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ \overline{\binom{0}{2}}SN,\ N^*$
  - $\widetilde{M} = 0 \rightarrow \copyright_{(2)}, \copyright_{(2)};\ \overline{\binom{0}{3}}N$

Table 1.1: **Global configurations: the case $\mu_0 \neq 0$, $\mathbf{D} = \mathbf{T} = 0$, $\mathbf{PR} < 0$.**

equator. For more details we refer to [16]. The vertical projection of this vector field defined on the upper hemisphere and completed with the equator, yields a diffeomorphic vector field on the unit disk, called the *Poincaré compactification on the disk* of the polynomial differential system. By a *singular point at infinity* of a planar polynomial vector field we mean a singular point of the vector field which is located on the equator of the sphere, also located on the boundary circle of the Poincaré disk.

2.2 Compactification on the projective plane

For a polynomial differential system (1.1) of degree m with real coefficients we associate the differential equation $\omega_1 = q(x,y)dx - p(x,y)dy = 0$. This equation defines two foliations with singularities, one on the real and one on the complex affine planes. We can compactify these foliations with singularities on the real respectively complex projective plane with homogeneous coordinates X, Y, Z. This is done as follows: Consider the pull-back of the form ω_1 via the map $r\colon K^3 \setminus \{Z = 0\} \to K^2$ defined by $r(X,Y,Z) = (X/Z, Y/Z)$. We obtain a form

- $\mu_0\neq0$, $\mathbf{D}=\mathbf{T}=0$, $\mathbf{PR}>0$
  - $\mathcal{T}_4\neq0$
    - $\eta<0$
      - $\theta<0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ N^\infty,$ ⓒ, ⓒ
      - $\theta>0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ N^f,$ ⓒ, ⓒ
      - $\theta=0$
        - $\theta_2\neq0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ N^d,$ ⓒ, ⓒ
        - $\theta_2=0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ N^*,$ ⓒ, ⓒ
    - $\eta>0$
      - $\theta<0$
        - $\theta_1<0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ S,N^\infty,\ N^\infty$
        - $\theta_1>0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ S,N^f,\ N^f$
      - $\theta>0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ S,N^\infty,\ N^f$
      - $\theta=0$
        - $\theta_1<0$
          - $\theta_2\neq0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ S,N^\infty,\ N^d$
          - $\theta_2=0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ S,N^\infty,\ N^*$
        - $\theta_1>0$
          - $\theta_2\neq0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ S,N^f,\ N^d$
          - $\theta_2=0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ S,N^f,\ N^*$
        - $\theta_1=0$
          - $\theta_4\neq0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ S,N^d,\ N^*$
          - $\theta_4=0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ S,N^*,\ N^*$
    - $\eta=0$
      - $\widetilde{M}\neq0$
        - $\theta<0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ \overline{\binom{0}{2}}SN,\ N^\infty$
        - $\theta>0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ \overline{\binom{0}{2}}SN,\ N^f$
        - $\theta=0$
          - $\theta_2\neq0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ \overline{\binom{0}{2}}SN,\ N^d$
          - $\theta_2=0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ \overline{\binom{0}{2}}SN,\ N^*$
      - $\widetilde{M}=0$ → $\overline{sn}_{(2)},\overline{sn}_{(2)};\ \overline{\binom{0}{3}}N;$
  - $\mathcal{T}_4=0$
    - $\mathcal{T}_2\neq0$
      - $\eta<0$
        - $\theta<0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ N^\infty,$ ⓒ, ⓒ
        - $\theta>0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ N^f,$ ⓒ, ⓒ
        - $\theta=0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ N^d,$ ⓒ, ⓒ
      - $\eta>0$
        - $\theta<0$
          - $\theta_1<0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ S,N^\infty,\ N^\infty$
          - $\theta_1>0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ S,N^f,\ N^f$
        - $\theta>0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ S,N^\infty,\ N^f;$
        - $\theta=0$
          - $\theta_1<0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ S,N^\infty,\ N^d$
          - $\theta_1>0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ S,N^f,\ N^d$
      - $\eta=0$
        - $\widetilde{M}\neq0$
          - $\theta<0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ \overline{\binom{0}{2}}SN,\ N^\infty$
          - $\theta>0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ \overline{\binom{0}{2}}SN,\ N^f$
          - $\theta=0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ \overline{\binom{0}{2}}SN,\ N^d$
        - $\widetilde{M}=0$ → $\overline{sn}_{(2)},\ \widehat{cp}_{(2)};\ \overline{\binom{0}{3}}N$
    - $\mathcal{T}_2=0$
      - $\eta<0$ → $\widehat{cp}_{(2)},\ \widehat{cp}_{(2)};\ N^f,$ ⓒ, ⓒ
      - $\eta>0$ → $\widehat{cp}_{(2)},\ \widehat{cp}_{(2)};\ S,N^\infty,\ N^f$
      - $\eta=0$ → $\widehat{cp}_{(2)},\ \widehat{cp}_{(2)};\ \overline{\binom{0}{2}}SN,\ N^f$

Table 1.2: **Global configurations: the case $\mu_0\neq0$, $\mathbf{D}=\mathbf{T}=0$, $\mathbf{PR}>0$.**

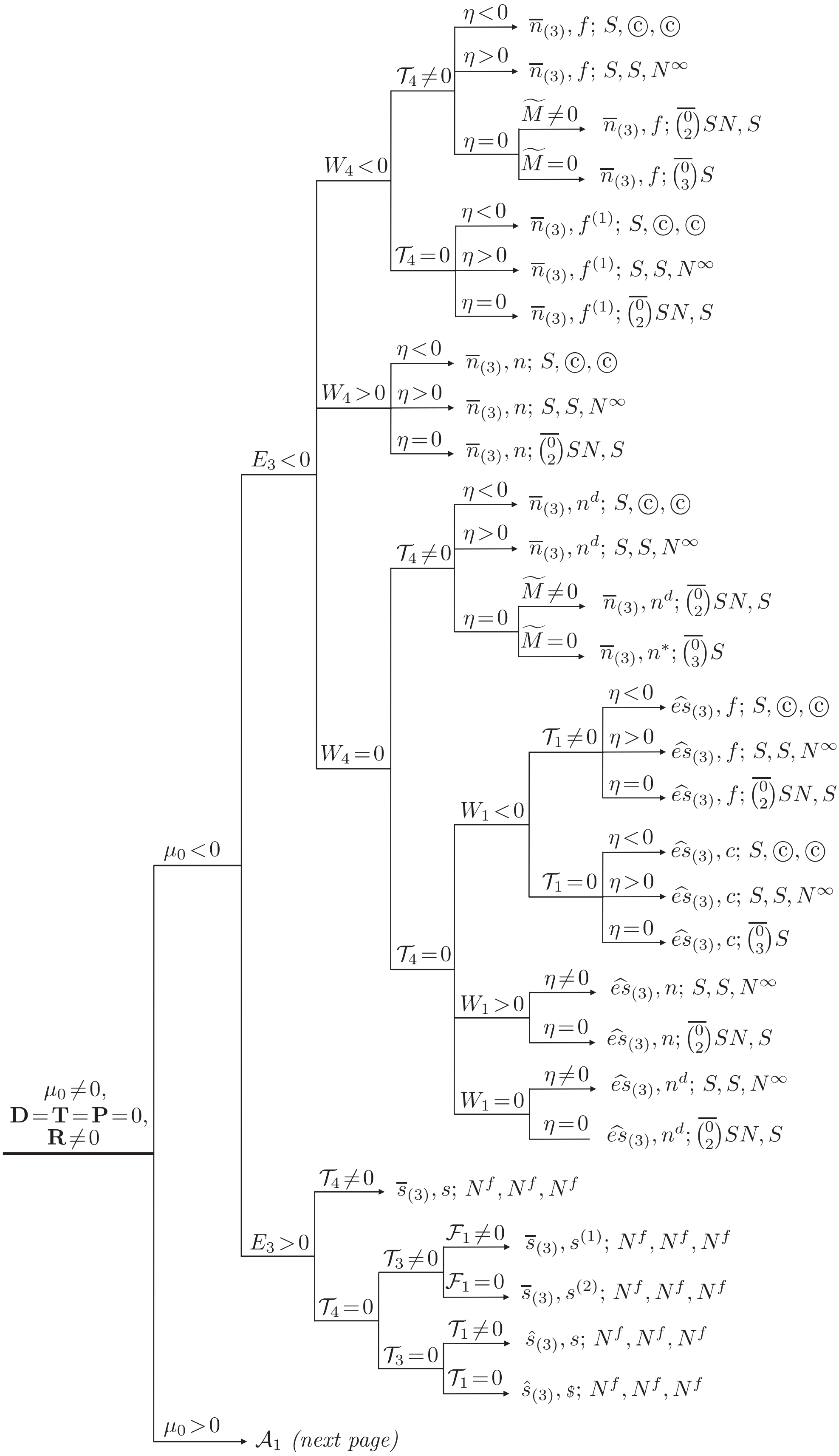

Table 1.3: **Global configurations: the case $\mu_0 \neq 0$, $\mathbf{D}=\mathbf{T}=\mathbf{P}=0$, $\mathbf{R}\neq 0$.**

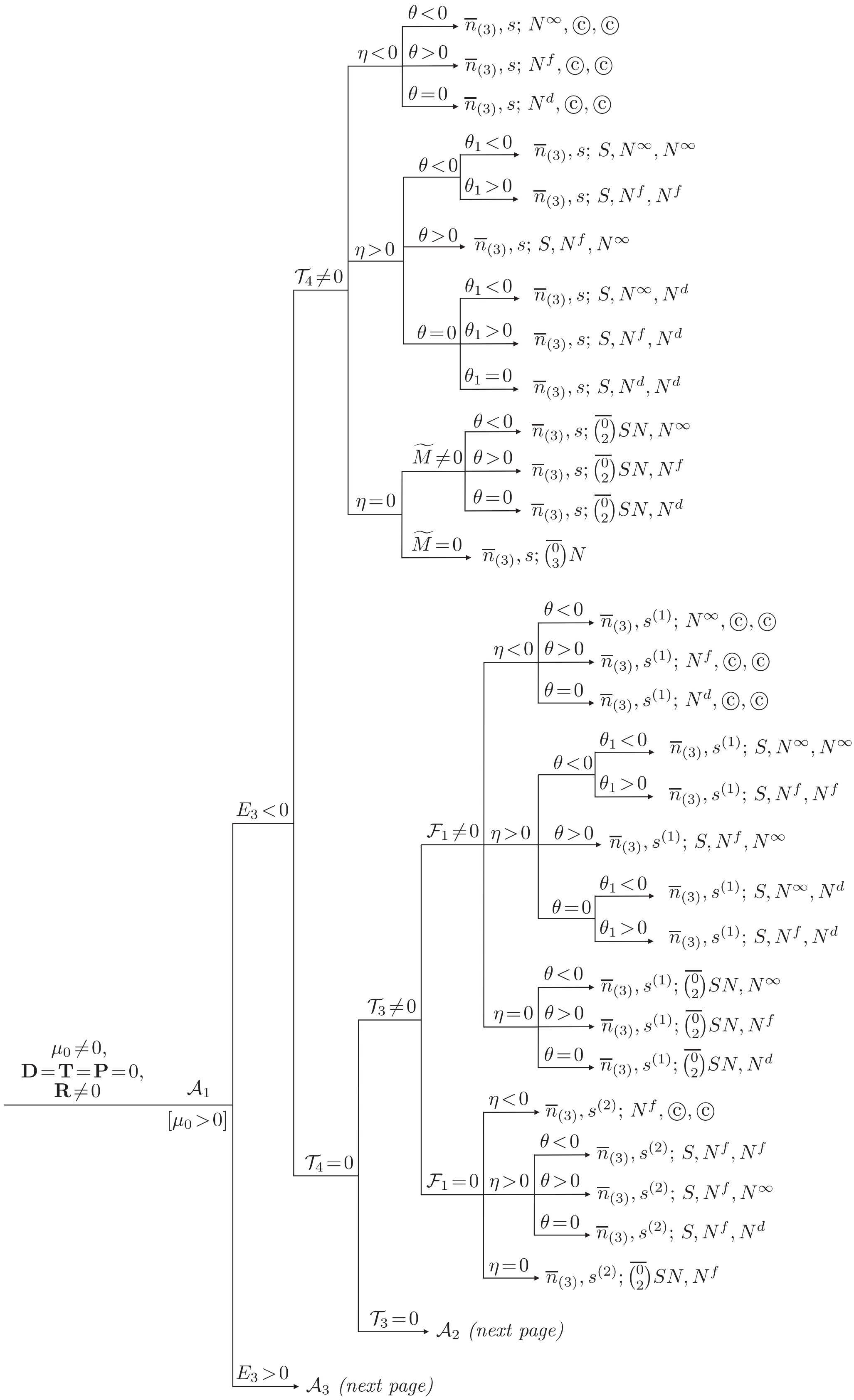

Table 1.3 (*continued*). **Global configurations: the case $\mu_0 \neq 0$, $\mathbf{D} = \mathbf{T} = \mathbf{P} = 0$, $\mathbf{R} \neq 0$.**

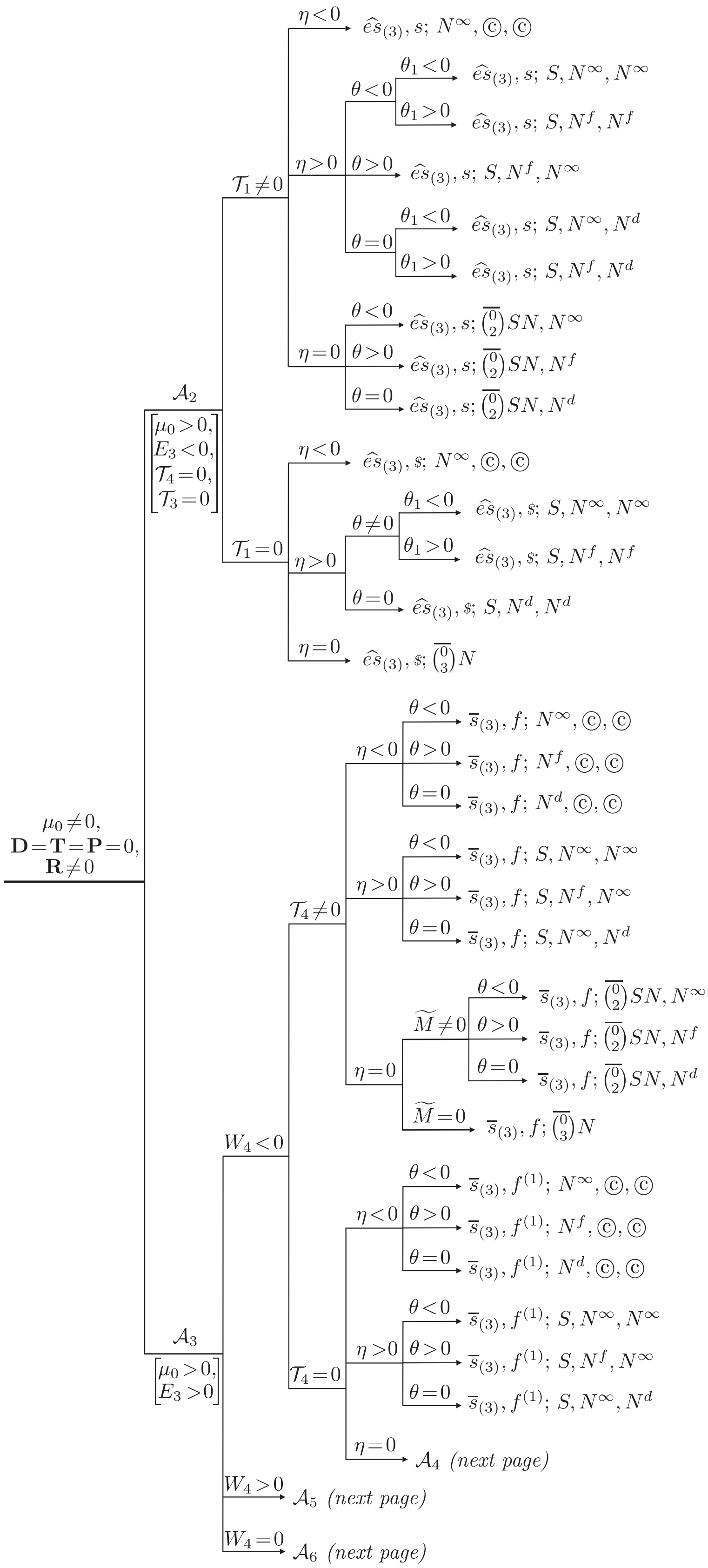

Table 1.3 (*continued*). **Global configurations: the case $\mu_0 \neq 0$, $\mathbf{D} = \mathbf{T} = \mathbf{P} = 0$, $\mathbf{R} \neq 0$.**

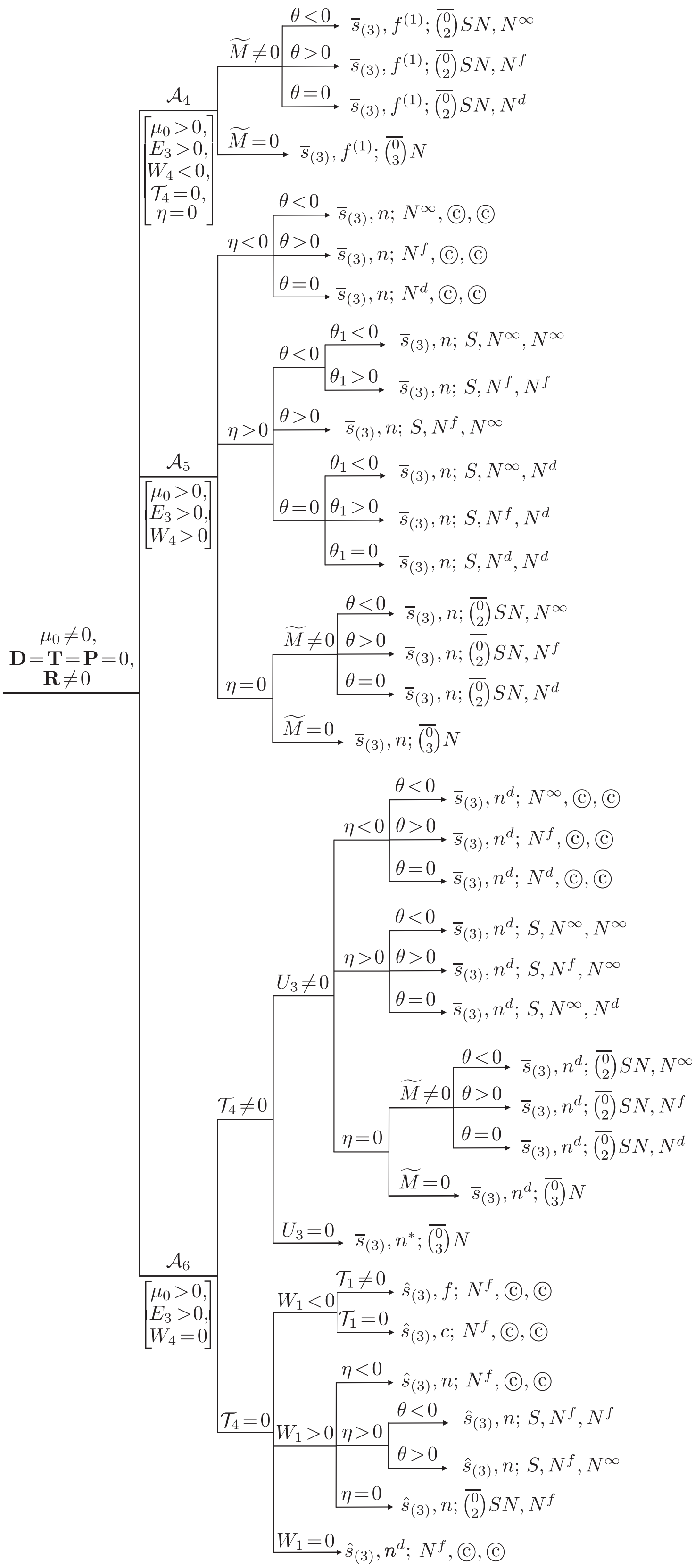

Table 1.3 (*continued*). **Global configurations: the case $\mu_0\neq0$, $\mathbf{D}=\mathbf{T}=\mathbf{P}=0$, $\mathbf{R}\neq0$.**

$r^*(\omega_1) = \tilde{\omega}$ which has poles on $Z = 0$. Eliminating the denominators in the equation $\tilde{\omega} = 0$ we obtain an equation $\omega = 0$ of the form $\omega = A(X,Y,Z)\,dX + B(X,Y,Z)\,dY + C(X,Y,Z)\,dZ = 0$ with A, B, C homogeneous polynomials of the same degree. The equation $\omega = 0$ defines a foliation with singularities on $P_2(K)$ which, via the map $(x,y) \to [x : y : 1]$, extends the foliation with singularities, given by $\omega_1 = 0$ on K^2 to a foliation with singularities on $P_2(K)$ which we call the compactification on the projective plane of the foliation with singularities defined by $\omega_1 = 0$ on the affine plane K^2 (K equal to $\mathbb{R}$ or $\mathbb{C}$). This is because A, B, C are homogeneous polynomials over K, defined by $A(X,Y,Z) = ZQ(X,Y,Z)$, $Q(X,Y,Z) = Z^m q(X/Z, Y/Z)$, $B(X,Y,Z) = ZP(X,Y,Z)$, $P(X,Y,Z) = Z^m p(X/Z, Y/Z)$ and $C(X,Y,Z) = YP(X,Y,Z) - XQ(X,Y,Z)$. The *points at infinity* of the foliation defined by $\omega_1 = 0$ on the affine plane are the singular points of the type $[X : Y : 0] \in P_2(\mathbb{K})$ and the line $Z = 0$ is called the *line at infinity* of this foliation. The singular points of the foliation on $P_2(K)$ are the solutions of the three equations $A = 0$, $B = 0$, $C = 0$. In view of the definitions of A, B, C it is clear that the singular points at infinity are the points of intersection of $Z = 0$ with $C = 0$. For more details see [20], or [6] or [2].

2.3 Assembling multiplicities of singularities in divisors of the line at infinity and in zero-cycles of the plane

An isolated singular point p at infinity of a polynomial vector field of degree n has two types of multiplicities: the maximum number m of finite singularities which can split from p, in small perturbations of the system within polynomial systems of degree n, and the maximum number m' of infinite singularities which can split from p, in small such perturbations of the system. We encode the two in the column $(m, m')^t$. We then encode the global information about all isolated singularities at infinity using formal sums called *cycles* and *divisors* as defined in [23] or in [20] and used in [2,6,20,28].

We have two formal sums (*divisors on the line at infinity* $Z = 0$ of the complex affine plane) $D_S(P,Q;Z) = \sum_w I_w(P,Q)w$ and $D_S(C,Z) = \sum_w I_w(C,Z)w$ where $w \in \{Z = 0\}$ and where by $I_w(F,G)$ we mean the intersection multiplicity at w of the curves $F(X,Y,Z) = 0$ and $G(X,Y,Z) = 0$ on the complex projective plane. For more details see [20]. Following [28] we encode the above two divisors on the line at infinity into just one but with values in the ring $\mathbb{Z}^2$:

$$D_S = \sum_{\omega \in \{Z=0\}} \begin{pmatrix} I_w(P,Q) \\ I_w(C,Z) \end{pmatrix} w.$$

For a system (1.1) with isolated finite singularities we consider the formal sum (*zero-cycle on the plane*) $D_S(p,q) = \sum_{\omega \in \mathbb{R}^2} I_w(p,q)w$ encoding the multiplicities of all finite singularities. For more details see [1,20].

2.4 Some geometrical concepts

Firstly we recall some terminology introduced in [6].

We call *elemental* a singular point with its both eigenvalues not zero.

We call *semi-elemental* a singular point with exactly one of its eigenvalues equal to zero.

We call *nilpotent* a singular point with both its eigenvalues zero but with its Jacobian matrix at this point not identically zero.

We call *intricate* a singular point with its Jacobian matrix identically zero.

The *intricate* singularities are usually called in the literature *linearly zero*. We use here the term *intricate* to indicate the rather complicated behavior of phase curves around such a singularity.

In this section we use the same concepts we considered in [2,3,5,6], such as *orbit γ tangent to a semi-line L at p, well defined angle at p, characteristic orbit at a singular point p, characteristic angle at a singular point, characteristic direction at p*. If a singular point has an infinite number of characteristic directions, we will call it a *star-like* point.

It is known that the neighborhood of any isolated singular point of a polynomial vector field, which is not a focus or a center, is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [16]). It is also known that any degenerate singular point can be desingularized by means of a finite number of changes of variables, called blowups, into elemental and semi-elemental singular points (for more details see the section on blowup in [6] or [16]).

Topologically equivalent local phase portraits can be distinguished according to the algebraic properties of their phase curves. For example they can be distinguished algebraically in the case when the singularities possess distinct numbers of characteristic directions.

The usual definition of a sector is of topological nature and it is local, defined with respect to a neighborhood around the singular point. We work with a new notion, namely of *geometric local sector*, introduced in [6], based on the notion of *borsec*, term meaning "border of a sector" (a new kind of sector, i.e. *geometric sector*) which takes into account orbits tangent to the half-lines of the characteristic directions at a singular point. For example a generic or semi–elemental node p has two characteristic directions generating four half lines at p. For each one of these half lines at p there exists at least one orbit tangent to that half line at p and we pick such an orbit (one for each half line). Removing these four orbits together with the singular point, we are left with four sectors which we call *geometric local sectors* and we call *borsecs* these four orbits. The notion of *geometric local sector* and of *borsec* was extended for nilpotent and intricate singular points using the process of desingularization as indicated in [3]. We end up with the following definition: We call *geometric local sector* of a singular point p with respect to a sufficiently small neighborhood V, a region in V delimited by two consecutive borsecs. As already mentioned, these are defined using the desingularization process.

A nilpotent or intricate singular point can be desingularized by passing to polar coordinates or by using rational changes of coordinates. The first method has the inconvenience of using trigonometrical functions, and this becomes a serious problem when a chain of blowups is needed in order to complete the desingularization of the degenerate point. The second uses rational changes of coordinates, convenient for our polynomial systems. In such a case two blowups in different directions are needed and information from both must be glued together to obtain the desired portrait.

Here for desingularization we use the second possibility, namely with rational changes of coordinates at each stage of the process. Two rational changes are needed, one for each direction of the blow-up. If at a stage the coordinates are (x, y) and we do a blow-up of a singular point in y-direction, this means that we introduce a new variable z and consider the diffeomorphism of the (x, y) plane for $x \neq 0$ defined by $\phi(x, y) = (x, y, z)$ where $y = xz$. This diffeomorphism transfers our vector field on the subset $x \neq 0$ of the plane (x, y) on the subset $x \neq 0$ of the algebraic surface $y = zx$. It can easily be checked that the projection $(x, xz, z) \mapsto (x, z)$ of this surface on the (x, z) plane is a diffeomorphism. So our vector field on the plane (x, y) for $x \neq 0$ is diffeomeorphic via the map $(x, y) \mapsto (x, y/x)$ for $x \neq 0$ to the vector field thus obtained on the (x,z) plane for $x \neq 0$. The point $p = (0,0)$ is then replaced by the straight line $x = 0 = y$ in the 3-dimensional space of coordinates x, y, z. This line is also the z-axis of the plane (x,z) and it is called the *blowup line*.

The two directional blowups can be reduced to only one 1-direction blowup but making sure that the direction in which we do a blowup is not a characteristic direction, not to lose information by blowing up in the chosen direction. This can be easily solved by a simple linear change of coordinates of the type $(x,y) \to (x+ky, y)$ where k is a constant (usually 1). It seems natural to call this linear change a *k-twist* as the y-axis gets turned with some angle depending on k. It is obvious that the phase portrait of the degenerate point which is studied cannot depend on the values of k's used in the desingularization process.

We recall that after a complete desingularization all singular points are elemental or semi–elemental. For more details and a complete example of the desingularization of an intricate singular point see [3].

Generically a *geometric local sector* is defined by two borsecs arriving at the singular point with two different well defined angles and which are consecutive. If this sector is parabolic, then the solutions can arrive at the singular point with one of the two characteristic angles, and this is a geometric information that can be revealed with the blowup.

There is also the possibility that two borsecs defining a geometric local sector at a point p are tangent to the same half-line at p. Such a sector will be called a *cusp-like sector* which can either be hyperbolic, elliptic or parabolic denoted by $H_{\curlywedge}$, $E_{\curlywedge}$ and $P_{\curlywedge}$ respectively. In the case of parabolic sectors we want to include the information about how the orbits arrive at the singular points namely tangent to one or to the other borsec. We distinguish the two cases by writing $\overset{\curvearrowleft}{P}$ if they arrive tangent to the borsec limiting the previous sector in clockwise sense, or $\overset{\curvearrowright}{P}$ if they arrive tangent to the borsec limiting the next sector. In the case of a cusp-like parabolic sector, all orbits must arrive with only one well determined angle, but the distinction between $\overset{\curvearrowleft}{P}$ and $\overset{\curvearrowright}{P}$ is still valid because it occurs at some stage of the desingularization and this can be algebraically determined. Examples of descriptions of complicated intricate singular points are $\overset{\curvearrowright}{P}E\overset{\curvearrowleft}{P}HHH$ and $E\overset{\curvearrowleft}{P}_{\curlywedge}HH\overset{\curvearrowright}{P}_{\curlywedge}E$.

A star-like point can either be a node or something much more complicated with elliptic and hyperbolic sectors included. In case there are hyperbolic sectors, they must be cusp-like. Elliptic sectors can either be cusp-like, or star-like.

2.5 Notations for singularities of polynomial differential systems

In this work we limit ourselves to the class of quadratic systems with finite singularities of total multiplicity four and exactly two singularities. In [6] we introduced convenient notations which we also used in [2–5] some of which we also need here. Because these notations are essential for understanding the bifurcation diagram, we indicate below the notations needed for this article.

The finite singularities will be denoted by small letters and the infinite ones by capital letters. In a sequence of singular points we always place the finite ones first and then the infinite ones, separating them by a semicolon ';'.

Elemental points: We use the letters 's','S' for "saddles"; $\$$ for "integrable saddles"; 'n', 'N' for "nodes"; 'f' for "foci"; 'c' for "centers" and © (respectively ©) for complex finite (respectively infinite) singularities. We distinguish the finite nodes as follows:

- 'n' for a node with two distinct eigenvalues (generic node);

- 'n^d' (a one–direction node) for a node with two identical eigenvalues whose Jacobian matrix is not diagonal;
- 'n^*' (a star node) for a node with two identical eigenvalues whose Jacobian matrix is diagonal.

The case n^d (and also n^*) corresponds to a real finite singular point with zero discriminant.

In the case of an elemental infinite generic node, we want to distinguish whether the eigenvalue associated to the eigenvector directed towards the affine plane is, in absolute value, greater or lower than the eigenvalue associated to the eigenvector tangent to the line at infinity. This is relevant because this determines if all the orbits except one on the Poincaré disk arrive at infinity tangent to the line at infinity or transversal to this line. We will denote them as 'N^∞' and 'N^f' respectively.

Finite elemental foci and saddles are classified as strong or weak foci, respectively strong or weak saddles. The strong foci or saddles are those with non-zero trace of the Jacobian matrix evaluated at them. In this case we denote them by 'f' and 's'. When the trace is zero, except for centers, and saddles of infinite order (i.e. with all their Poincaré–Lyapounov constants equal to zero), it is known that the foci and saddles, in the quadratic case, may have up to 3 orders. We denote them by '$f^{(i)}$' and '$s^{(i)}$' where $i = 1, 2, 3$ is the order. In addition we have the centers which we denote by 'c' and saddles of infinite order (integrable saddles) which we denote by '$\$$'.

Foci and centers cannot appear as singular points at infinity and hence there is no need to introduce their order in this case. In case of saddles, we can have weak saddles at infinity but the maximum order of weak singularities in cubic systems is not yet known. For this reason, a complete study of weak saddles at infinity cannot be done at this stage. Due to this, in [5,6] and here we chose not even to distinguish between a saddle and a weak saddle at infinity.

All non-elemental singular points are multiple points, in the sense that there are perturbations which have at least two elemental singular points as close as we wish to the multiple point. For finite singular points we denote with a subindex their multiplicity as in '$\overline{s}_{(5)}$' or in '$\widehat{es}_{(3)}$' (the notation '$\overline{\ }$' indicates that the saddle is semi-elemental and '$\widehat{\ }$' indicates that the singular point is nilpotent, in this case a triple *elliptic saddle*, i.e. it has two sectors, one elliptic and one hyperbolic). In order to describe the two kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [28]. Thus we denote by '$\binom{a}{b}\dots$' the maximum number a (respectively b) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example '$\overline{\binom{1}{1}}SN$' means a saddle–node at infinity produced by the collision of one finite singularity with an infinite one; '$\overline{\binom{0}{3}}S$' means a saddle produced by the collision of 3 infinite singularities.

Semi-elemental points: They can either be nodes, saddles or saddle–nodes, finite or infinite (see [16]). We denote the semi-elemental ones always with an overline, for example '$\overline{sn}$', '$\overline{s}$' and '$\overline{n}$' with the corresponding multiplicity. In the case of infinite points we put '$\overline{\ }$' on top of the parenthesis with multiplicities.

Semi-elemental nodes could never be 'n^d' or 'n^*' since their eigenvalues are always different. In case of an infinite semi-elemental node, the type of collision determines whether the point is an 'N^f' or an 'N^∞'. The point '$\overline{\binom{2}{1}}N$' is an 'N^f' and '$\overline{\binom{0}{3}}N$' is an 'N^∞'.

Nilpotent points: They can either be saddles, nodes, saddle–nodes, elliptic saddles, cusps, foci or centers (see [16]). The first four of these could be at infinity. We denote the nilpotent singular points with a hat '$\widehat{\ }$' as in $\widehat{es}_{(3)}$ for a finite nilpotent elliptic saddle of multiplicity 3, and $\widehat{cp}_{(2)}$ for a finite nilpotent cusp point of multiplicity 2.

When $m_f = 4$ and there is more than one finite singularity there are neither nilpotent singularities at infinity nor intricate singularities (finite and infinite). Also, for this class, the line at infinity cannot be filled up with singularities. For these reasons we skip the notations for these points in this paper. The interested could see these notations in [5,6].

2.6 Affine invariant polynomials and preliminary results

Consider real quadratic systems of the form

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y), \end{aligned} \tag{2.1}$$

with homogeneous polynomials p_i and q_i $(i = 0,1,2)$ in x, y which are defined as follows:

$$\begin{aligned} p_0 &= a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of systems (2.1) and denote $\mathbb{R}[\tilde{a}, x, y] = \mathbb{R}[a_{00}, \ldots, b_{02}, x, y]$.

It is known that on the set **QS** of all quadratic differential systems (2.1) acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [28]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (2.1) with a subset of $\mathbb{R}^{12}$ via the map **QS**$\longrightarrow \mathbb{R}^{12}$ which associates to each system (2.1) the 12-tuple $\tilde{a} = (a_{00}, \ldots, b_{02})$ of its coefficients. We associate to this group action polynomials in x, y and parameters which behave well with respect to this action, the GL-comitants, the T-comitants and the CT-comitants. For their constructions we refer the reader to the paper [28] (see also [30]). In the statement of our main theorem intervene invariant polynomials constructed in these articles and which could also be found on the following associated web page:

http://mat.uab.es/~artes/articles/qvfinvariants/qvfinvariants.html

3 The proof of the Main Theorem

Consider real quadratic systems (2.1). According to [31] for a quadratic system (2.1) to have finite singularities of total multiplicity four (i.e. $m_f = 4$) the condition $\mu_0 \neq 0$ must be satisfied. We consider here the three subclasses of quadratic differential systems with $m_f = 4$ possessing exactly two finite singularities, namely:

- systems with two double complex singularities ($\mu_0 \neq 0$, $\mathbf{D} = \mathbf{T} = 0$, $\mathbf{PR} < 0$);
- systems with two double real singularities ($\mu_0 \neq 0$, $\mathbf{D} = \mathbf{T} = 0$, $\mathbf{PR} > 0$);
- systems with one triple and one simple real singularities ($\mu_0 \neq 0$, $\mathbf{D} = \mathbf{T} = \mathbf{P} = 0$, $\mathbf{R} \neq 0$).

We observe that the systems from each one in the above mentioned subclasses have finite singularities of total multiplicity 4 and therefore by [6] the following lemma is valid.

Lemma 3.1. *The geometric configurations of singularities at infinity of the family of quadratic systems possessing finite singularities of total multiplicity 4 (i.e. $\mu_0 \neq 0$) are classified in the diagram from Table 3.1 according to the geometric equivalence relation. Necessary and sufficient conditions for each one of the 24 different equivalence classes can be assembled from these diagrams in terms of 9 invariant polynomials with respect to the action of the affine group and time rescaling.*

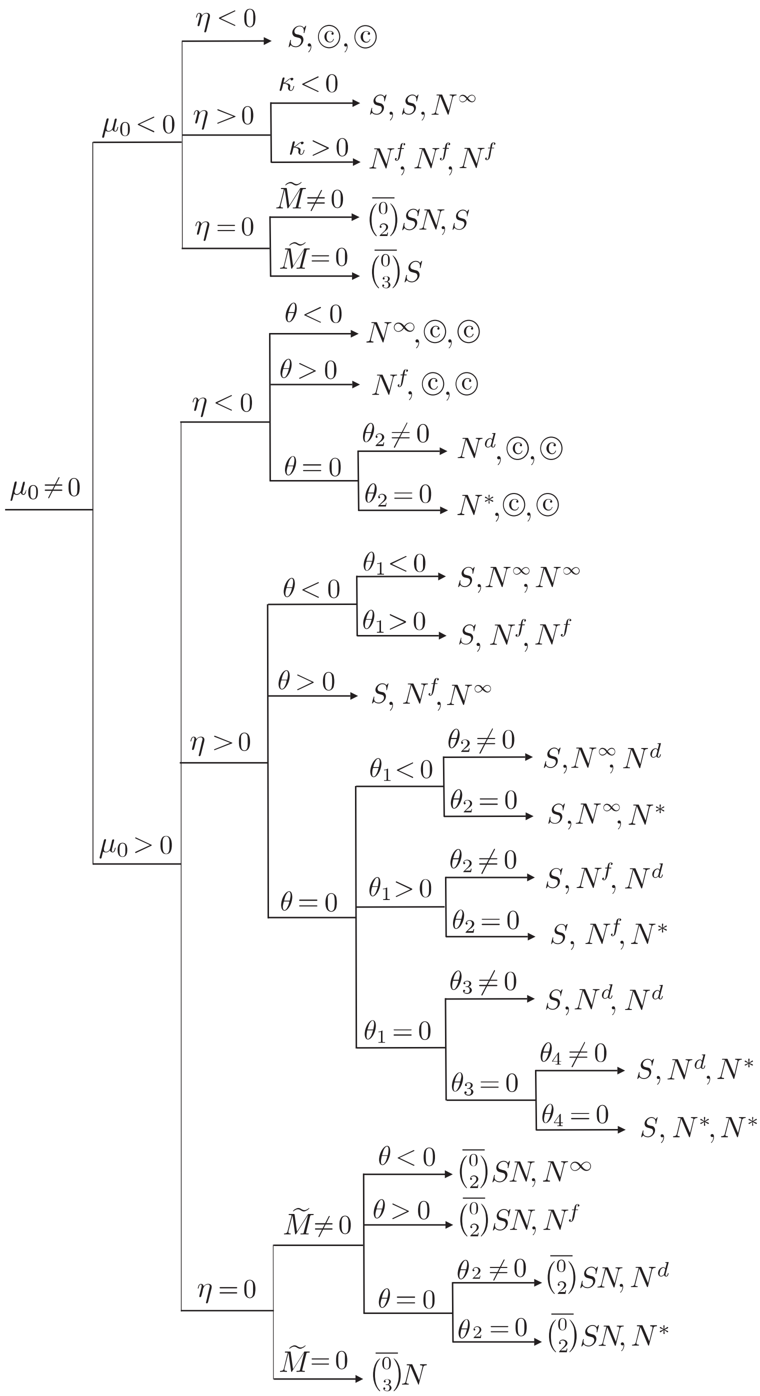

Table 3.1: **Configurations of infinite singularities: the case $\mu_0 \neq 0$.**

3.1 Systems with two double complex singularities

Assume that systems (2.1) have two double complex finite singularities. In this case according to [31] we shall consider the family of systems

$$\begin{aligned} \dot{x} &= a + aux + gx^2 + 2avxy + ay^2, \\ \dot{y} &= b + bux + lx^2 + 2bvxy + by^2, \end{aligned} \tag{3.1}$$

with $al - bg \neq 0$, possessing the following two double distinct singularities: $M_{1,2}(0, i)$, $M_{3,4}(0, -i)$.

Lemma 3.2. *The conditions $\theta = \theta_1 = 0$ imply for a system (3.1) the condition $\theta_3 = 0$.*

Proof. For systems (3.1) we have

$$\theta = 64a(al - bg)(l + gv - bv^2 - av^3), \quad \mu_0 = (al - bg)^2, \quad \theta_3 = a(al - bg)\widehat{U}(a, b, g, l, u, v),$$

where $\widehat{U}(a, b, g, l, u, v)$ is a polynomial. As $\mu_0 \neq 0$ the condition $\theta = 0$ gives $a(l + gv - bv^2 - av^3) = 0$.

If $a = 0$ then evidently we get $\theta_3 = 0$ and the statement of the lemma is valid.

Assume $a \neq 0$. Then $l = -v(g - bv - av^2)$ and calculation yield

$$\theta = 0, \qquad \theta_1 = 64a(-g + av^2)^3, \qquad \theta_3 = a(b + av)^2(g - av^2)\widehat{V}(a, b, g, u, v),$$

where $\widehat{V}(a, b, g, u, v)$ is a polynomial. Clearly in this case the condition $\theta_1 = 0$ implies again $\theta_3 = 0$, and this completes the proof of the lemma. □

3.1.1 The case $\eta < 0$

Then systems (3.1) possess one real and two complex infinite singular points and according to Lemma 3.1 there could be only 4 distinct configurations at infinity. It remains to construct corresponding examples:

- $©_{(2)}, ©_{(2)}$; N^∞, ©, ©: *Example* $\Rightarrow$ $(a = 1,\ b = 1,\ g = 3,\ l = 0,\ u = 0,\ v = 1)$ (if $\theta < 0$);
- $©_{(2)}, ©_{(2)}$; N^f, ©, ©: *Example* $\Rightarrow$ $(a = 1,\ b = -1,\ g = 1,\ l = 0,\ u = 0,\ v = 1)$ (if $\theta > 0$);
- $©_{(2)}, ©_{(2)}$; N^d, ©, ©: *Example* $\Rightarrow$ $(a = 1,\ b = 1,\ g = 1,\ l = 0,\ u = 1,\ v = 0)$ (if $\theta = 0, \theta_2 \neq 0$);
- $©_{(2)}, ©_{(2)}$; N^*, ©, ©: *Example* $\Rightarrow$ $(a = 0,\ b = 1,\ g = 1,\ l = 1,\ u = 0,\ v = 0)$ (if $\theta = 0, \theta_2 = 0$).

3.1.2 The case $\eta > 0$

In this case systems (3.1) possess three real infinite singular points and taking into consideration Lemma 3.3 and the condition $\mu_0 > 0$, by Lemma 3.1 we could have at infinity only 9 distinct configurations. Corresponding examples are:

- $©_{(2)}, ©_{(2)}$; S, N^∞, N^∞: *Example* $\Rightarrow$ $(a = 1,\ b = 1,\ g = 35/16,\ l = 0,\ u = 0,\ v = 1)$ (if $\theta < 0, \theta_1 < 0$);
- $©_{(2)}, ©_{(2)}$; S, N^f, N^f: *Example* $\Rightarrow$ $(a = 1,\ b = 1,\ g = 5/4,\ l = 0,\ u = 0,\ v = -2)$ (if $\theta < 0, \theta_1 > 0$);
- $©_{(2)}, ©_{(2)}$; S, N^∞, N^f: *Example* $\Rightarrow$ $(a = 1,\ b = 1,\ g = 5/4,\ l = 0,\ u = 0,\ v = 1)$ (if $\theta > 0$);
- $©_{(2)}, ©_{(2)}$; S, N^∞, N^d: *Example* $\Rightarrow$ $(a = 1,\ b = 1,\ g = 6,\ l = 0,\ u = 1,\ v = 2)$ (if $\theta = 0$, $\theta_1 < 0, \theta_2 \neq 0$);

- $©_{(2)}, ©_{(2)}$; S, N^∞, N^*: *Example* $\Rightarrow$ $(a=0,\ b=1,\ g=3,\ l=0,\ u=0,\ v=1)$ (if $\theta=0$, $\theta_1<0, \theta_2=0$);
- $©_{(2)}, ©_{(2)}$; S, N^f, N^d: *Example* $\Rightarrow$ $(a=1,\ b=1,\ g=2,\ l=0,\ u=1,\ v=-2)$ (if $\theta=0, \theta_1>0, \theta_2\neq 0$);
- $©_{(2)}, ©_{(2)}$; S, N^f, N^*: *Example* $\Rightarrow$ $(a=0,\ b=1,\ g=-1,\ l=0,\ u=0,\ v=1)$ (if $\theta=0, \theta_1>0, \theta_2=0$);
- $©_{(2)}, ©_{(2)}$; S, N^d, N^*: *Example* $\Rightarrow$ $(a=0,\ b=1,\ g=1,\ l=0,\ u=1,\ v=1)$ (if $\theta=0, \theta_1=0,\ \theta_4\neq 0$);
- $©_{(2)}, ©_{(2)}$; S, N^*, N^*: *Example* $\Rightarrow$ $(a=0,\ b=1,\ g=1,\ l=0,\ u=0,\ v=1)$ (if $\theta=\theta_1=\theta_4=0$).

3.1.3 The case $\eta=0$

In this case systems (3.1) possess at infinity either one double and one simple real singular points (if $\widetilde{M}\neq 0$), or one triple real singularity (if $\widetilde{M}=0$). So by Lemma 3.1 we could have at infinity exactly 5 distinct configurations. We have the following 4 configurations:

- $©_{(2)}, ©_{(2)}$; $\overline{\binom{0}{2}}SN, N^\infty$: *Example* $\Rightarrow$ $(a=1,\ b=3,\ g=6,\ l=0,\ u=0,\ v=1)$ (if $\theta<0$);
- $©_{(2)}, ©_{(2)}$; $\overline{\binom{0}{2}}SN, N^f$: *Example* $\Rightarrow$ $(a=-1,\ b=1,\ g=2,\ l=0,\ u=0,\ v=1)$ (if $\theta>0$);
- $©_{(2)}, ©_{(2)}$; $\overline{\binom{0}{2}}SN, N^d$: *Example* $\Rightarrow$ $(a=1,\ b=1,\ g=2,\ l=0,\ u=1,\ v=1)$ (if $\theta=0, \theta_2\neq 0$);
- $©_{(2)}, ©_{(2)}$; $\overline{\binom{0}{2}}SN, N^*$: *Example* $\Rightarrow$ $(a=1,\ b=1,\ g=2,\ l=0,\ u=0,\ v=1)$ (if $\theta=\theta_2=0$),

if $\widetilde{M}\neq 0$; and one configuration

- $©_{(2)}, ©_{(2)}$; $\overline{\binom{0}{3}}N$: *Example* $\Rightarrow$ $(a=1,\ b=2,\ g=4,\ l=0,\ u=0,\ v=1)$

if $\widetilde{M}=0$.

3.2 Systems with two double real singularities

Assume that systems (2.1) possess two double real finite singularities. In this case according to [31] we shall consider the family of systems

$$\begin{aligned}\dot{x}&=cx+cuy-cx^2+2cvxy+ky^2,\\ \dot{y}&=ex+euy-ex^2+2evxy+ny^2,\end{aligned}\tag{3.2}$$

with $cn-ek\neq 0$, possessing the following two double distinct singularities: $M_{1,2}(0,0)$, $M_{3,4}(1,0)$.

Following [7] for this family of systems we calculate

$$\mu_0=(cn-ek)^2,\quad G_1=(cn-ek)^2(c+eu)^2(c-eu-2ev)^2,\quad E_2=-e(ek-cn)^2(u+v)\tag{3.3}$$

and hence $\mu_0>0$. Moreover, according to [7] systems (3.2) possess: two saddle–nodes if $G_1\neq 0$; one saddle–node and one cusp if $G_1=0$ and $E_2\neq 0$ and two cusps if $G_1=E_2=0$.

Lemma 3.3. *The conditions $\theta=\theta_1=0$ imply for a system (3.2) the condition $\theta_3=0$.*

Proof. For systems (3.2) we have

$$\theta=64e(ek-cn)(k-nv+cv^2-ev^3),\qquad \theta_3=e(ek-cn)\widehat{U}(c,e,k,n,u,v),$$

where $\widehat{U}(c,e,k,n,u,v)$ is a polynomial. As $\mu_0 \neq 0$ the condition $\theta = 0$ gives $e(k - nv + cv^2 - ev^3) = 0$.

If $e = 0$ then evidently we get $\theta_3 = 0$ and the statement of the lemma holds.

Assume $e \neq 0$. Then $k = nv - cv^2 + ev^3$ and calculations yield

$$\theta = 0, \qquad \theta_1 = 64e(n+ev^2)^3, \qquad \theta_3 = e(c-ev)^2(n+ev^2)\widehat{V}(c,e,n,u,v),$$

where $\widehat{V}(c,e,n,u,v)$ is a polynomial. Clearly in this case again the condition $\theta_1 = 0$ implies $\theta_3 = 0$ and this completes the proof of the lemma. $\square$

Lemma 3.4. *Assume that for a system (3.2) the condition $E_2 \neq 0$ holds. Then for this system we have $\theta_2 \neq 0$. Moreover if in addition the condition $\theta = 0$ is satisfied, then the condition $\theta_1 \neq 0$ holds.*

Proof. For systems (3.2) we have

$$\theta = 64e(ek-cn)(k-nv+cv^2-ev^3), \quad E_2 = -e(ek-cn)^2(u+v), \quad \theta_2 = e(ek-cn)(u+v),$$

and evidently the condition $E_2 \neq 0$ implies $\theta_2 \neq 0$.

Assume now $\theta = 0$. As $E_2 \neq 0$ this yields $k = nv - cv^2 + ev^3$. Then we calculate

$$\theta_1 = 64e(n+ev^2)^3, \qquad E_2 = -e(u+v)(c-ev)^2(n+ev^2)^2,$$

and clearly the condition $E_2 \neq 0$ gives $\theta_1 \neq 0$. $\square$

Lemma 3.5. *The conditions $G_1 = E_2 = 0$ imply for systems (3.2) $\theta > 0$ and $\widetilde{M} \neq 0$.*

Proof. Considering (3.3) the conditions $E_2 = 0$ and $\mu_0 \neq 0$ imply $e(u+v) = 0$. We observe that the condition $u = -v$ has to be satisfied, otherwise in the case $e = 0$ we obtain $G_1 = c^6n^2$ and $\mu_0 = c^2n^2$, and hence the condition $\mu_0 \neq 0$ implies $G_1 \neq 0$. So $u = -v$ and then calculations yield

$$G_1 = (ek-cn)^2(c-ev)^4, \qquad \mu_0 = (ek-cn)^2.$$

Therefore as $\mu_0 \neq 0$ the condition $G_1 = 0$ gives $c = ev$ and we obtain

$$G_1 = E_2 = 0, \qquad \theta = 64e^2(k-nv)^2, \qquad \mu_0 = e^2(k-nv)^2,$$
$$\widetilde{M}/8 = -3e(n+ev^2)x^2 + 3e(3k-nv+2ev^3)xy + (4env^2 - n^2 - 9ekv - 4e^2v^4)y^2.$$

Hence the condition $\mu_0 \neq 0$ implies $\theta > 0$. On the other hand the condition $\widetilde{M} = 0$ gives $n = -ev^2$ and then we obtain:

$$\widetilde{M} = 72e(k+ev^3)y(x-vy), \qquad \mu_0 = e^2(k+ev^3)^2.$$

So the condition $\mu_0 \neq 0$ implies $\widetilde{M} \neq 0$, and this completes the proof of the lemma. $\square$

Lemma 3.6. *For systems (3.2) the condition $G_1 = 0$ is equivalent to $\mathcal{T}_4 = 0$, and if $G_1 = 0$ then the condition $E_2 = 0$ is equivalent to $\mathcal{T}_2 = 0$.*

Proof. For systems (3.2) we have $\mathcal{T}_4 = G_1$. Assuming $G_1 = 0$ (i.e. $\mathcal{T}_4 = 0$) due to $\mu_0 \neq 0$ we obtain either

$$E_2 = -e^3(k+nu)^2(u+v), \qquad \mathcal{T}_2 = 4e^4(k+nu)^2(u+v)^2$$

if $c = -eu$, or

$$E_2 = -e^3(u+v)(k-nu-2nv)^2, \qquad \mathcal{T}_2 = 4e^4(u+v)^2(k-nu-2nv)^2$$

if $c = e(u+2v)$. In both cases we obtain that the condition $E_2 = 0$ is equivalent to $\mathcal{T}_2 = 0$. $\square$

Considering [7] and the above lemma we get the next remark.

Remark 3.7. Systems (3.2) possess two saddle–nodes if $\mathcal{T}_4 \neq 0$; one saddle–node and one cusp if $\mathcal{T}_4 = 0$ and $\mathcal{T}_2 \neq 0$, and two cusps if $\mathcal{T}_4 = \mathcal{T}_2 = 0$.

3.2.1 The case $\mathcal{T}_4 \neq 0$

In this case both double finite singular points are saddle-nodes.

The subcase $\eta < 0$. Then systems (3.2) possess one real and two complex infinite singular points and according to Lemma 3.1 there could be only 4 distinct configurations at infinity. It remains to construct corresponding examples:

- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; N^∞, ©, ©: *Example* $\Rightarrow$ $(c = -1, e = 1, k = 0, n = -3, u = 0, v = 1)$ (if $\theta < 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; N^f, ©, ©: *Example* $\Rightarrow$ $(c = 1, e = 1, k = 0, n = -3, u = 0, v = 1)$ (if $\theta > 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; N^d, ©, ©: *Example* $\Rightarrow$ $(c = 1, e = -1, k = 0, n = 1, u = 2, v = 0)$ (if $\theta = 0, \theta_2 \neq 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; N^*, ©, ©: *Example* $\Rightarrow$ $(c = 1, e = 0, k = -1, n = 1, u = 0, v = 0)$ (if $\theta = 0, \theta_2 = 0$).

The subcase $\eta > 0$. In this case systems (3.2) possess three real infinite singular points and taking into consideration Lemma 3.3 and the condition $\mu_0 > 0$, by Lemma 3.1 we could have at infinity only 9 distinct configurations. Corresponding examples are:

- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; S, N^∞, N^∞: *Example* $\Rightarrow$ $(c = -4, e = 1, k = -1, n = -8, u = 0, v = 1)$ (if $\theta < 0, \theta_1 < 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; S, N^f, N^f: *Example* $\Rightarrow$ $(c = -1, e = 1, k = 0, n = 1, u = 0, v = 1)$ (if $\theta < 0, \theta_1 > 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; S, N^∞, N^f: *Example* $\Rightarrow$ $(c = 1, e = 1, k = 0, n = 1, u = 0, v = 1)$ (if $\theta > 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; S, N^∞, N^d: *Example* $\Rightarrow$ $(c = 3, e = 1, k = 0, n = -1, u = 1, v = 0)$ (if $\theta = 0, \theta_1 < 0, \theta_2 \neq 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; S, N^∞, N^*: *Example* $\Rightarrow$ $(c = 1, e = 0, k = 1, n = 3, u = 0, v = 1)$ (if $\theta = 0, \theta_1 < 0, \theta_2 = 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; S, N^f, N^d: *Example* $\Rightarrow$ $(c = 2, e = 1, k = 0, n = 1, u = 1, v = 1)$ (if $\theta = 0, \theta_1 > 0, \theta_2 \neq 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; S, N^f, N^*: *Example* $\Rightarrow$ $(c = 1, e = 0, k = 1, n = 1, u = 0, v = 1)$ (if $\theta = 0, \theta_1 > 0, \theta_2 = 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; S, N^d, N^*: *Example* $\Rightarrow$ $(c = 1, e = 0, k = 0, n = 1, u = 1, v = 0)$ (if $\theta = 0, \theta_1 = 0, \theta_4 \neq 0$);
- $\overline{sn}_{(2)}, \overline{sn}_{(2)}$; S, N^*, N^*: *Example* $\Rightarrow$ $(c = 1, e = 0, k = -2, n = 1, u = 1, v = -1)$ (if $\theta = \theta_1 = \theta_4 = 0$).

The subcase $\eta = 0$. In this case systems (3.2) possess at infinity either one double and one simple real singular points (if $\widetilde{M} \neq 0$), or one triple real singularity (if $\widetilde{M} = 0$). So by Lemma 3.1 we could have at infinity exactly 5 distinct configurations. We have the following 4 configurations:

• $\overline{sn}_{(2)}, \overline{sn}_{(2)}; \overline{\binom{0}{2}}SN,\ N^{\infty}$: *Example* $\Rightarrow$ $(c=-3,\ e=1,\ k=0,\ n=-6,\ u=0,\ v=1)$ (if $\theta<0$);

• $\overline{sn}_{(2)}, \overline{sn}_{(2)}; \overline{\binom{0}{2}}SN,\ N^{f}$: *Example* $\Rightarrow$ $(c=1,\ e=1,\ k=0,\ n=2,\ u=0,\ v=1)$ (if $\theta>0$);

• $\overline{sn}_{(2)}, \overline{sn}_{(2)}; \overline{\binom{0}{2}}SN,\ N^{d}$: *Example* $\Rightarrow$ $(c=-1,\ e=1,\ k=0,\ n=-2,\ u=0,\ v=1)$ (if $\theta=0, \theta_2\neq 0$);

• $\overline{sn}_{(2)}, \overline{sn}_{(2)}; \overline{\binom{0}{2}}SN,\ N^{*}$: *Example* $\Rightarrow$ $(c=-1,\ e=1,\ k=0,\ n=-2,\ u=-1,\ v=1)$ (if $\theta=\theta_2=0$),

in the case $\widetilde{M}\neq 0$; and one configuration

• $\overline{sn}_{(2)}, \overline{sn}_{(2)}; \overline{\binom{0}{3}}N$: *Example* $\Rightarrow$ $(c=3,\ e=3,\ k=-1/9,\ n=-1,\ u=0,\ v=0)$

if $\widetilde{M}=0$.

3.2.2 The case $\mathcal{T}_4=0$.

In this case we have at least one cusp.

The subcase $\mathcal{T}_2\neq 0$. Then by Remark 3.7 systems (3.2) possess one saddle–node and one cusp.

The possibility $\eta<0$. In this case systems (3.2) possess one real and two complex infinite singular points. Considering Lemmas 3.4, 3.6 and the condition $\mu_0>0$, by Lemma 3.1 there could be only 3 distinct configurations at infinity. It remains to construct corresponding examples:

• $\overline{sn}_{(2)}, \widehat{cp}_{(2)};\ N^{\infty}$, ©, ©: *Example* $\Rightarrow$ $(c=1,\ e=3,\ k=0,\ n=-3,\ u=1,\ v=-1/3)$ (if $\theta<0$);

• $\overline{sn}_{(2)}, \widehat{cp}_{(2)};\ N^{f}$, ©, ©: *Example* $\Rightarrow$ $(c=-2,\ e=-1,\ k=0,\ n=1,\ u=0,\ v=1)$ (if $\theta>0$);

• $\overline{sn}_{(2)}, \widehat{cp}_{(2)};\ N^{d}$, ©, ©: *Example* $\Rightarrow$ $(c=-1,\ e=-1,\ k=0,\ n=1,\ u=1,\ v=0)$ (if $\theta=0$).

The possibility $\eta>0$. In this case systems (3.2) possess three real infinite singular points and taking into consideration Lemmas 3.4, 3.6 and the condition $\mu_0>0$, by Lemma 3.1 there could be only 5 distinct configurations at infinity. Corresponding examples are:

• $\overline{sn}_{(2)}, \widehat{cp}_{(2)};\ S, N^{\infty}, N^{\infty}$: *Example* $\Rightarrow$ $(c=-5,\ e=1,\ k=0,\ n=-11,\ u=5,\ v=1)$ (if $\theta<0, \theta_1<0$);

• $\overline{sn}_{(2)}, \widehat{cp}_{(2)};\ S, N^{f}, N^{f}$: *Example* $\Rightarrow$ $(c=3,\ e=1,\ k=0,\ n=1,\ u=-3,\ v=1)$ (if $\theta<0, \theta_1>0$);

• $\overline{sn}_{(2)}, \widehat{cp}_{(2)};\ S, N^{\infty}, N^{f}$: *Example* $\Rightarrow$ $(c=1,\ e=1,\ k=0,\ n=1,\ u=-1,\ v=2)$ (if $\theta>0$);

• $\overline{sn}_{(2)}, \widehat{cp}_{(2)};\ S, N^{\infty}, N^{d}$: *Example* $\Rightarrow$ $(c=-3,\ e=1,\ k=0,\ n=-1,\ u=3,\ v=0)$ (if $\theta=0, \theta_1<0$);

• $\overline{sn}_{(2)}, \widehat{cp}_{(2)};\ S, N^{f}, N^{d}$: *Example* $\Rightarrow$ $(c=1,\ e=1,\ k=0,\ n=1,\ u=-1,\ v=0)$ (if $\theta=0, \theta_1>0$).

The possibility $\eta=0$. In this case systems (3.2) possess at infinity either one double and one simple real singular points (if $\widetilde{M}\neq 0$), or one triple real singularity (if $\widetilde{M}=0$). So by Lemmas 3.4, 3.6 and 3.1 we could have at infinity exactly 4 distinct configurations. We have the following 3 configurations and corresponding examples:

• $\overline{sn}_{(2)}$, $\widehat{cp}_{(2)}$; $\overline{\binom{0}{2}}SN$, N^{∞}: *Example* $\Rightarrow$ $(c=-3,\, e=1,\, k=0,\, n=-6,\, u=3,\, v=1)$ (if $\theta<0$);

• $\overline{sn}_{(2)}$, $\widehat{cp}_{(2)}$; $\overline{\binom{0}{2}}SN$, N^{f}: *Example* $\Rightarrow$ $(c=2,\, e=1,\, k=0,\, n=4,\, u=-2,\, v=1)$ (if $\theta>0$);

• $\overline{sn}_{(2)}$, $\widehat{cp}_{(2)}$; $\overline{\binom{0}{2}}SN$, N^{d}: *Example* $\Rightarrow$ $(c=-1,\, e=1,\, k=0,\, n=-2,\, u=1,\, v=1)$ (if $\theta=0$)

in the case $\widetilde{M}\neq 0$; and one configuration

• $\overline{sn}_{(2)}$, $\widehat{cp}_{(2)}$; $\overline{\binom{0}{3}}N$: *Example* $\Rightarrow$ $(c=3,\, e=3,\, k=-1/9,\, n=-1,\, u=-1,\, v=0)$

if $\widetilde{M}=0$.

The subcase $\mathcal{T}_2=0$. Then by Remark 3.7 systems (3.2) possess two nilpotent cusps. On the other hand by Lemma 3.5 the conditions $\theta>0$ and $\widetilde{M}\neq 0$ are satisfied. Therefore according to Lemma 3.1 in this case we could have at infinity only 3 distinct configurations:

• $\widehat{cp}_{(2)}$, $\widehat{cp}_{(2)}$; N^f, ©, ©: *Example* $\Rightarrow$ $(c=0,\, e=1,\, k=1,\, n=0,\, u=0,\, v=0)$ (if $\eta<0$);

• $\widehat{cp}_{(2)}$, $\widehat{cp}_{(2)}$; S, N^{∞}, N^f: *Example* $\Rightarrow$ $(c=-1,\, e=1,\, k=0,\, n=1,\, u=1,\, v=-1)$ (if $\eta>0$);

• $\widehat{cp}_{(2)}$, $\widehat{cp}_{(2)}$; $\overline{\binom{0}{2}}SN$, N^f: *Example* $\Rightarrow$ $(c=1,\, e=1,\, k=0,\, n=2,\, u=-1,\, v=1)$ (if $\eta=0$).

3.3 Systems with one triple and one simple real singularities

Assume that systems (2.1) possess one triple and one simple real finite singularities.

Then via an affine transformation we may assume that these systems possess as singularities the points $M_1(0,0)$ and $M_2(1,0)$ and the singular point M_0 is triple. Moreover, as in this case we have $C_2\neq 0$ (otherwise we have $m_f\leq 3$) we may consider that there exists at least one isolated real infinite singularity. We shall consider two possibilities: $a)$ there exists a real infinite singular point which does not coincide with $N(1,0,0)$ (i.e. the end of the axis $y=0$), and $b)$ there exists a unique real infinite singularity and it is located at the point $N(1,0,0)$.

In the second case we get the systems

$$\dot{x}=cx+dy-cx^2+2hxy+ky^2,\qquad \dot{y}=fy+2mxy+ny^2, \tag{3.4}$$

for which we calculate:

$$\mu_0=c(cn^2-4km^2+4hmn),\qquad \Delta_1=cf,$$

and since we have to force the singular point $M_1(0,0)$ to be triple we get $\Delta_1=0$, where Δ_1 is the corresponding determinant. Since $\mu_0\neq 0$ (i.e. $c\neq 0$) we have $f=0$ and then we calculate:

$$\mu_4=\mu_3=0,\qquad \mu_2=c(cn-2dm)y(2mx+ny).$$

So according to [11] the singular point $M_1(0,0)$ of systems (3.4) is of multiplicity three if and only if $\mu_2=0$, which is equivalent to $c(cn-2dm)=0$. As $c\neq 0$ we may assume $c=1$ due to a time rescaling and then we obtain $n=2dm$. Thus we arrive at the family of systems

$$\dot{x}=x+dy-x^2+2hxy+ky^2,\qquad \dot{y}=2my(x+dy),\ \ d\in\{0,1\}, \tag{3.5}$$

since in the case $d\neq 0$ we apply the rescaling $y\to y/d$.

Remark 3.8. We remark that since we have assumed that the singularity $N(1,0,0)$ is the unique real infinite singularity of this family of systems, then the condition either $\eta<0$, or $\widetilde{M}=0$ must hold for these systems.

We shall construct the normal form of systems in the case $a)$, i.e. when there exists a real infinite singular point which does not coincide with $N(1,0,0)$. Then via a linear transformation (which keeps the singularities $M_1(0,0)$ and $M_2(1,0)$) we may assume that this real point is located at the point $N_1(0,1,0)$ (i.e. on the end of the axis $x=0$) and we obtain the systems

$$\dot{x} = cx + dy - cx^2 + 2hxy, \qquad \dot{y} = ex + fy - ex^2 + 2mxy + ny^2. \tag{3.6}$$

We want $M_1(0,0)$ to be triple so for its associated determinant we have $\Delta_1 = cf - de = 0$ and we shall consider two cases: $e \neq 0$ and $e = 0$.

***A*. The case $e \neq 0$.** Then we assume $e = 1$ (due to a time rescaling) and therefore the condition $\Delta_1 = 0$ gives $d = cf$. Herein we calculate

$$\mu_4 = \mu_3 = 0, \qquad \mu_2 = (2fh - 2cfm + cn)y(2cmx - 2hx + cny),$$

and we split the examination in two subcases: $f \neq 0$ and $f = 0$.

A_1. *The subcase $f \neq 0$.* In this case via the rescaling $(x,y,t) \mapsto (x, y/f, t/f)$ we may assume $f = 1$, and then the condition $\mu_2 = 0$ is equivalent to $h = c(2m-n)/2$. So we arrive at the family of systems:

$$\dot{x} = cx + cy - cx^2 + c(2m-n)xy, \qquad \dot{y} = x + y - x^2 + 2mxy + ny^2. \tag{3.7}$$

A_2. *The subcase $f = 0$.* Then the condition $\mu_2 = 0$ implies $cn = 0$ and as $\mu_0 = n(4chm + c^2n - 4h^2) \neq 0$ (i.e. $n \neq 0$) we get $c = 0$. This leads to the family of systems

$$\dot{x} = 2hxy, \qquad \dot{y} = x - x^2 + 2mxy + ny^2, \;\; m \in \{0,1\} \tag{3.8}$$

since in the case $m \neq 0$ we apply the rescaling $(x,y,t) \mapsto (x, y/m, t/m)$.

***B*. The case $e = 0$.** Then the condition $\Delta_1 = 0$ implies $cf = 0$, and as $\mu_0 = cn(4hm + cn) \neq 0$ (i.e. $c \neq 0$) we get $f = 0$. In this case we calculate

$$\mu_4 = \mu_3 = 0, \qquad \mu_2 = c(cn - 2dm)y(2mx + ny), \qquad \mu_0 = cn(4hm + cn) \neq 0.$$

As $c \neq 0$ we may assume $c = 1$ due to a time rescaling and then the condition $\mu_2 = 0$ gives $n = 2dm \neq 0$, and we may assume $d = 1$ due to the rescaling $y \to y/d$. So we arrive at the 2-parameter family of systems

$$\dot{x} = x + y - x^2 + 2hxy, \qquad \dot{y} = 2my(x+y). \tag{3.9}$$

3.3.1 The family of systems (3.7)

This family of systems possesses the triple singular point $M_{1,2,3}(0,0)$ and the elemental one $M_4(1,0)$, for which we have the following values for the traces ρ_i, for the determinants Δ_i, discriminants τ_i and for the linearization matrices $\mathcal{M}_{1,2,3}$ and $\mathcal{M}_4$:

$$\begin{gathered}
\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3 = \begin{pmatrix} c & c \\ 1 & 1 \end{pmatrix}, \quad \mathcal{M}_4 = \begin{pmatrix} -c & c(1+2m-n) \\ -1 & 1+2m \end{pmatrix}, \\
\rho_1 = \rho_2 = \rho_3 = 1 + c, \;\; \Delta_1 = 0; \quad \rho_4 = 1 - c + 2m, \;\; \Delta_4 = -cn; \\
\tau_i = \rho_i^2 - 4\Delta_i, \;\; i = 1,4.
\end{gathered} \tag{3.10}$$

For this family we calculate

$$
\begin{gathered}
\mu_0 = \alpha\Delta_4^2, \;\; \eta = (2cm - n - cn)^2\big[(c-2m)^2 + 4n(c+1)\big],\\
E_3 = \alpha\Delta_4^3/4, \;\; G_{10} = -\alpha\rho_1^3\Delta_4^3/8, \;\; \mathcal{F}_1 = 3\alpha\rho_1\Delta_4;\\
\mathcal{T}_4 = \alpha\Delta_4^2\rho_1^3\rho_4, \;\; \mathcal{T}_3 = \alpha\Delta_4^2\rho_1^2(\rho_1 + 3\rho_4), \;\; \mathcal{T}_2 = 3\alpha\Delta_4^2\rho_1(\rho_1+\rho_4),\\
\mathcal{T}_1 = \alpha\Delta_4^2(3\rho_1+\rho_4), \;\; W_4 = \alpha^2\rho_1^6\Delta_4^4\tau_4,\\
\widetilde{M} = -8(c^2 - 2cm + 4m^2 + 3n\rho_1)x^2 + 8(c+2m)(2cm - n\rho_1)xy - 8(2cm - n\rho_1)^2y^2,
\end{gathered}
\tag{3.11}
$$

where $\alpha = (1+2m-n)$.

Lemma 3.9. *If for a system (3.7):*

(i) the condition $\widetilde{M} = 0$ holds, then $\rho_1 \neq 0$ and:

(i_1) the condition $\mu_0 < 0$ implies $E_3 < 0$, $W_4 < 0$ and $\mathcal{T}_4 \neq 0$;

(i_2) the condition $\mathcal{T}_4 = 0$ implies $E_3 > 0$;

(ii) the conditions $\eta < 0$ and $\rho_1 \neq 0$ are satisfied, then $\operatorname{sign}(\mu_0) = \operatorname{sign}(c+1)$ *and* $\operatorname{sign}(E_3) = \operatorname{sign}(c)$*;*

(iii) the conditions $\theta = \theta_2 = 0$ hold, then for this system we have $\mu_0 = \eta$. Moreover in this case the condition $\mu_0 > 0$ implies $W_4 > 0$;

(iv) the condition $\theta = 0$ is fulfilled, then the condition $\theta_1 = 0$ is equivalent to $\theta_2 = 0$. Moreover if $\theta_1 = 0$ then $\theta_3 \neq 0$.

Proof. (i) First we observe that in the case $\widetilde{M} = 0$ the condition $\rho_1 = c+1 \neq 0$ must hold. Indeed, admitting $c = -1$ we have Coefficient$[\widetilde{M}, y^2] = -32m^2 = 0$ and then $\widetilde{M} = -8x^2 \neq 0$. So $c+1 \neq 0$ and considering (3.11) the condition $\widetilde{M} = 0$ yields $m = -c/2$ and $n = -c^2/(c+1)$. Then we calculate

$$
\mu_0 = \frac{c^6}{(c+1)^3}, \qquad E_3 = \frac{c^9}{4(c+1)^4}, \qquad W_4 = \frac{c^{12}(1-3c)}{c+1}, \qquad \mathcal{T}_4 = c^6(1-2c),
$$

and hence the condition $\mu_0 < 0$ gives $c < -1$ which obviously implies the validity of the statement (i_1).

On the other hand we observe that the condition $\mathcal{T}_4 = 0$ implies $c = 1/2$ and then $E_3 > 0$. This completes the proof of the statement (i).

(ii) Assume $\eta < 0$. Then considering (3.11) we conclude that the second factor of η is negative and due to the condition $\rho_1 = c+1 \neq 0$ we may assume

$$
(c-2m)^2 + 4n(c+1) = -v^2 \;\Rightarrow\; n = -\big[(c-2m)^2 + v^2\big]/(4(1+c)).
$$

Therefore we calculate

$$
\begin{aligned}
\mu_0 &= \frac{c^2\big[(c-2m)^2+v^2\big]^2\big[(c+2m+2)^2+v^2\big]}{64(1+c)^3},\\
E_3 &= \frac{c^3\big[(c-2m)^2+v^2\big]^3\big[(c+2m+2)^2+v^2\big]}{1024(1+c)^4},
\end{aligned}
$$

and obviously the statement *(ii)* of the lemma is proved.

(iii) Assume now that for systems (3.7) the conditions $\theta = \theta_2 = 0$ are satisfied. For these systems we have

$$
\begin{gathered}
\theta = -8c(2m-n)n(4cm + 2c^2m - 4m^2 - 4n - 4cn - c^2n),\\
\theta_2 = -cn(4 + 2c + 6m + 2cm - 2n - cn)/4,
\end{gathered}
$$

and as $\mu_0 \neq 0$ (i.e. $cn \neq 0$) we get the relations

$$(2m-n)\big[2m(2c+c^2-2m)-n(c+2)^2\big]=0=(2+c)(2+2m-n)+2m.$$

We consider two subcases: $c+2\neq 0$ and $c+2=0$.

Assume first $c+2\neq 0$. Then from the second equation we get $n=2(2+c+3m+cm)/(2+c)$, and we have

$$\theta_2=0,\qquad \theta=-\frac{64c(2+c+m)^2(2+c+2m)(2+c+3m+cm)}{(2+c)^2},$$
$$\mu_0=-\frac{4c^2(2+c+2m)(2+c+3m+cm)^2}{(2+c)^3},$$

and as $\mu_0\neq 0$ the condition $\theta=0$ gives $m=-2-c$. Therefore we obtain

$$\theta_2=\theta=0,\qquad \mu_0=4c^2(2+c)^2=\eta,\qquad W_4=16c^4(1+c)^6(2+c)^4(9+2c+c^2),$$

and we observe that $W_4>0$.

Suppose now $c=-2$. Then we calculate

$$\theta=-64m^2(2m-n)n,\qquad \theta_2=mn,\qquad \mu_0=4(1+2m-n)n^2,$$

and as $\mu_0\neq 0$ (i.e. $n\neq 0$) the condition $\theta_2=0$ yields $m=0$. Then we obtain:

$$\theta_2=\theta=0,\qquad \mu_0=4(1-n)n^2=\eta,\qquad W_4=16(1-n)^2n^4(9-8n),$$

and it is clear that the condition $\mu_0>0$ implies $W_4>0$. This completes the proof of the statement *(iii)* of the lemma.

(iv) Assume $\theta=0$. As $\mu_0\neq 0$ the condition $\theta=0$ implies one of the following three relations: (a) $n=2m$; (b) $c+2\neq 0$ and $n=2\big[cm(c+2)-2m^2\big]/(2+c)^2$; ($c$) $c=-2$ and $m=0$.

In the case (a) we calculate:

$$\mu_0=4c^2m^2,\qquad \theta_1=256m^3(2+c+m),\qquad \theta_2=-cm(2+c+m),$$

and clearly due to $\mu_0\neq 0$ the condition $\theta_1=0$ is equivalent to $\theta_2=0$, and this yields $m=-(2+c)$. However in this case we get $\theta_3=-2c^3(2+c)^3\neq 0$ since $\mu_0=4c^2(2+c)^2\neq 0$.

Considering the case (b) we have

$$\mu_0=\frac{4c^2(2c+c^2-2m)^2m^2(2+c+2m)^2}{(2+c)^6},\qquad \theta_1=\frac{256c^3m^3(2+c+m)(2+c+2m)^2}{(2+c)^5},$$
$$\theta_2=-\frac{c(2c+c^2-2m)m(2+c+m)(2+c+2m)}{(2+c)^3},$$

and since $\mu_0\neq 0$ we obtain that the condition $\theta_1=0$ is equivalent to $\theta_2=0$, and this again implies $m=-(2+c)$. And then we calculate $\theta_3=-2c^3(2+c)^3\neq 0$ because $\mu_0=4c^2(2+c)^2\neq 0$.

In the case (c) (i.e. when $c=-2$ and $m=0$) we obtain

$$\theta_1=\theta_2=0,\qquad \theta_3=2(n-1)n^3,\qquad \mu_0=-4(n-1)n^2\neq 0.$$

This completes the proof of the lemma. □

Since $\mu_0\neq 0$ (then $E_3\neq 0$) we consider two cases: $\mu_0<0$ and $\mu_0>0$.

The case $\mu_0 < 0$. Then $\alpha < 0$ and by (3.11) this implies $\operatorname{sign}(E_3) = -\operatorname{sign}(\Delta_4)$.

The subcase $E_3 < 0$. In this case the elemental singularity M_4 is an anti-saddle (i.e. a node, a focus or a center). On the other hand by [7] the type of this point is governed by the invariant polynomials W_4 or W_1.

1) *The possibility* $W_4 < 0$. Then from (3.11) it follows $G_{10} \neq 0$ and according to [7, Table 1, line 96] we have a focus and a semi-elemental triple node. And the focus could be weak only if $\mathcal{T}_4 = 0$.

a) *The case* $\mathcal{T}_4 \neq 0$. In this case we have a strong focus and considering Lemma 3.1 and the condition $C_2 \neq 0$ we arrive at the following four configurations of singularities:

- $\overline{n}_{(3)}, f;\ S, ©, ©:$ *Example* $\Rightarrow (c = -2,\ m = -1,\ n = 1)$ (if $\eta < 0$);
- $\overline{n}_{(3)}, f;\ S, S, N^\infty:$ *Example* $\Rightarrow (c = -6/5,\ m = -1/10,\ n = 1)$ (if $\eta > 0$);
- $\overline{n}_{(3)}, f;\ \overline{\binom{0}{2}}SN, S:$ *Example* $\Rightarrow (c = -5/4,\ m = -1/8,\ n = 1)$ (if $\eta = 0$, $\widetilde{M} \neq 0$);
- $\overline{n}_{(3)}, f;\ \overline{\binom{0}{3}}S:$ *Example* $\Rightarrow (c = -2,\ m = 1,\ n = 4)$ (if $\eta = 0$, $\widetilde{M} = 0$).

b) *The case* $\mathcal{T}_4 = 0$. Since by (3.11) the condition $W_4 \neq 0$ implies $\mathcal{F}_1 \neq 0$ and $\rho_1 \neq 0$ we obtain $\rho_4 = 0$, i.e. we have a first order weak focus. According to Lemma 3.9 in this case the condition $\widetilde{M} \neq 0$ must hold. So considering Lemma 3.1 we get the following three configurations of singularities:

- $\overline{n}_{(3)}, f^{(1)};\ S, ©, ©:$ *Example* $\Rightarrow (c = -3,\ m = -2,\ n = 2)$ (if $\eta < 0$);
- $\overline{n}_{(3)}, f^{(1)};\ S, S, N^\infty:$ *Example* $\Rightarrow (c = -2,\ m = -3/2,\ n = 1/10)$ (if $\eta > 0$);
- $\overline{n}_{(3)}, f^{(1)};\ \overline{\binom{0}{2}}SN, S:$ *Example* $\Rightarrow (c = -3,\ m = -2,\ n = 1/8)$ (if $\eta = 0$).

2) *The possibility* $W_4 > 0$. Since $G_{10} \neq 0$ according to [7, Table 1, line 95] we have a node (which is generic due to $W_4 \neq 0$) and a semi-elemental triple node.

Since by Lemma 3.9 the condition $\widetilde{M} \neq 0$ has to be satisfied, by Lemma 3.1 we arrive at the following three configurations of singularities:

- $\overline{n}_{(3)}, n;\ S, ©, ©:$ *Example* $\Rightarrow (c = -2,\ m = -2/3,\ n = 1/4)$ (if $\eta < 0$);
- $\overline{n}_{(3)}, n;\ S, S, N^\infty:$ *Example* $\Rightarrow (c = 1,\ m = -4,\ n = -2)$ (if $\eta > 0$);
- $\overline{n}_{(3)}, n;\ \overline{\binom{0}{2}}SN, S:$ *Example* $\Rightarrow (c = 2,\ m = -2,\ n = -8/3)$ (if $\eta = 0$).

3) *The possibility* $W_4 = 0$. In this case due to $\mu_0 \neq 0$ we get $\rho_1 \tau_4 = 0$ and we consider two cases: $\mathcal{T}_4 \neq 0$ and $\mathcal{T}_4 = 0$.

a) *The case* $\mathcal{T}_4 \neq 0$. Then $\rho_1 \neq 0$ and hence we have $\tau_4 = 0$, i.e. we have a node with coinciding eigenvalues. Considering the corresponding linear matrix $\mathcal{M}_4$ from (3.10) we conclude that this node is not a star node. Therefore considering Lemmas 3.1 and 3.9 we arrive at the following three configurations of singularities:

- $\overline{n}_{(3)}, n^d;\ S, ©, ©:$ *Example* $\Rightarrow (c = -4,\ m = 0,\ n = 25/16)$ (if $\eta < 0$);
- $\overline{n}_{(3)}, n^d;\ S, S, N^\infty:$ *Example* $\Rightarrow (c = -1/4,\ m = 0,\ n = 25/16)$ (if $\eta > 0$);
- $\overline{n}_{(3)}, n^d;\ \overline{\binom{0}{2}}SN, S:$ *Example* $\Rightarrow (c = -2,\ m = 9/2,\ n = 18)$ (if $\eta = 0$).

b) *The case* $\mathcal{T}_4 = 0$. Then $\rho_1 \rho_4 = 0$ and considering the condition $\rho_1 \tau_4 = 0$ we obtain $\rho_1 = 0$ (otherwise we get $\rho_4 = \tau_4 = 0$ which is impossible for an elemental singular point). Thus $\rho_1 = 0$ (i.e. $c = -1$) and we calculate

$$\begin{gathered}
\mu_0 = (1 + 2m - n)n^2, \quad \eta = 4m^2(1 + 2m)^2, \quad G_{10} = 0, \\
E_3 = (1 + 2m - n)n^3/4, \quad \mathcal{F}_1 = \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = 0, \\
\mathcal{T}_1 = (1 + 2m - n)n^2\rho_4, \quad W_4 = 0, \quad W_1 = 4(1 + 2m - n)^2 n^4 \tau_4.
\end{gathered} \tag{3.12}$$

As $G_{10}=0$ by [7] the triple point is an elliptic saddle and the elemental one is an anti-saddle, the type of which is governed by W_1.

***b*$_1$) *The subcase* $W_1<0$.** Then M_4 is a focus which is strong if $\mathcal{T}_1\neq 0$ and it is weak if $\mathcal{T}_1=0$. Moreover in the second case, i.e. when $\rho_1=\rho_4=0$ (which implies $c=-1=m$) we calculate

$$\begin{gathered}\mu_0=-n^2(1+n),\;\; E_3=-n^3(1+n)/4,\;\; \eta=4,\\ \theta=8n(2+n)^2,\;\; \theta_1=16(2+n)(2-3n),\\ \mathcal{T}_4=\mathcal{T}_3=\mathcal{T}_2=\mathcal{T}_1=\mathcal{F}=0,\;\; \sigma=(2+3n)y,\\ \mathcal{H}=-n(2+3n)^2/2,\;\; \mathcal{B}=-(2+3n)^4/8.\end{gathered}\tag{3.13}$$

So the conditions $\mu_0<0$ and $E_3<0$ implies $n>0$, and this gives $\sigma\neq 0$, $\mathcal{H}<0$ and $\mathcal{B}<0$. By [31, Main Theorem, $(e_4),\alpha$] we obtain a center.

Thus considering Lemma 3.1 we get the following configurations of singularities:

- $\widehat{es}_{(3)}, f;\ S, S, N^\infty:$ *Example* $\Rightarrow$ $(c=-1,\ m=-1/3,\ n=1)$ (if $\eta>0$, $\mathcal{T}_1\neq 0$);
- $\widehat{es}_{(3)}, c;\ S, S, N^\infty:$ *Example* $\Rightarrow$ $(c=-1,\ m=-1,\ n=1)$ (if $\eta>0$, $\mathcal{T}_1=0$);
- $\widehat{es}_{(3)}, f;\ \overline{\binom{0}{2}}SN, S$: *Example* $\Rightarrow$ $(c=-1,\ m=-1/2,\ n=1)$ (if $\eta=0$).

***b*$_2$) *The subcase* $W_1>0$.** Then M_4 is a generic node and as $\eta\geq 0$ we get the next two configurations:

- $\widehat{es}_{(3)}, n;\ S, S, N^\infty:$ *Example* $\Rightarrow$ $(c=-1,\ m=-3,\ n=2)$ (if $\eta>0$);
- $\widehat{es}_{(3)}, n;\ \overline{\binom{0}{2}}SN, S$: *Example* $\Rightarrow$ $(c=-1,\ m=-1/2,\ n=1/5)$ (if $\eta=0$).

***b*$_3$) *The subcase* $W_1=0$.** Then M_4 is a one–direction node and as $\eta\geq 0$ we get the next two configurations:

- $\widehat{es}_{(3)}, n^d;\ S, S, N^\infty:$ *Example* $\Rightarrow$ $(c=-1,\ m=-1/3,\ n=4/9)$ (if $\eta>0$);
- $\widehat{es}_{(3)}, n^d;\ \overline{\binom{0}{2}}SN, S$: *Example* $\Rightarrow$ $(c=-1,\ m=-1/2,\ n=1/4)$ (if $\eta=0$).

The subcase $E_3>0$ In this case the elemental singularity M_4 is a saddle, whereas M_1 is a triple saddle, which could be semi-elemental or nilpotent. So at infinity we have 3 nodes.

1) *The possibility $\mathcal{T}_4\neq 0$.* Then $\rho_1\rho_4\neq 0$ and both saddles have non zero traces. In this case we arrive at the configuration

- $\overline{s}_{(3)}, s;\ N^f, N^f, N^f:$ *Example* $\Rightarrow$ $(c=-2,\ m=-2,\ n=-2)$;

2) *The possibility $\mathcal{T}_4=0$.* We consider two cases: $\mathcal{T}_3\neq 0$ and $\mathcal{T}_3=0$.

***a*)** *The case $\mathcal{T}_3\neq 0$.* According to (3.11) this implies $\rho_1\neq 0$, and then $\mathcal{F}_1\neq 0$ and $\rho_4=0$, i.e. $c=2m+1$. Then we have a first order weak saddle and we get the configuration

- $\overline{s}_{(3)}, s^{(1)};\ N^f, N^f, N^f:$ *Example* $\Rightarrow$ $(c=-3,\ m=-2,\ n=-2)$.

***b*)** *The case $\mathcal{T}_3=0$.* This implies $\rho_1=0$ (i.e. $c=-1$) and the triple saddle is nilpotent. So we get the expressions (3.12), and in this case the conditions $\mu_0<0$ and $E_3>0$ imply $1+2m<n<0$. Therefore depending on the value of the invariant polynomial $\mathcal{T}_1$ we arrive at the following configurations:

- $\hat{s}_{(3)}, s;\ N^f, N^f, N^f:$ *Example* $\Rightarrow$ $(c=-1,\ m=-2,\ n=-2)$ (if $\mathcal{T}_1\neq 0$);
- $\hat{s}_{(3)}, \$;\ N^f, N^f, N^f:$ *Example* $\Rightarrow$ $(c=-1,\ m=-1,\ n=-1/2)$ (if $\mathcal{T}_1=0$).

The case $\mu_0>0$. Then considering (3.11) we have $\alpha>0$ and this implies $\operatorname{sign}(E_3)=\operatorname{sign}(\Delta_4)$.

The subcase $E_3 < 0$. In this case the elemental singularity M_4 is a saddle. On the other hand by [7] the triple point could be either a semi-elemental triple node, or a nilpotent elliptic saddle.

1) *The possibility $\mathcal{T}_4 \neq 0$.* Then by (3.11) we get $\rho_1\rho_4 \neq 0$ and $G_{10} \neq 0$. In this case by [7, Table 1, line 87] the triple point is a semi-elemental triple node and the saddle is strong. Therefore considering Lemmas 3.1 and 3.9 we examine three cases: $\eta < 0$, $\eta > 0$ and $\eta = 0$.

a) *The case $\eta < 0$.* By Lemma 3.1 the configurations of infinite singularities are governed by the invariant polynomials θ and θ_2. According to Lemma 3.9 in this case the condition $\theta = 0$ implies $\theta_2 \neq 0$, otherwise we get $\mu_0 = \eta$ which contradicts $\eta < 0$ and $\mu_0 > 0$. So considering Lemma 3.1 we arrive at the following three configurations of singularities:

- $\overline{n}_{(3)}, s;\ N^{\infty}, ©, ©$: *Example* $\Rightarrow (c = -1/2,\ m = 1,\ n = -29/8)$ (if $\theta < 0$);
- $\overline{n}_{(3)}, s;\ N^{f}, ©, ©$: *Example* $\Rightarrow (c = -1/2,\ m = -1/2,\ n = -1/4)$ (if $\theta > 0$);
- $\overline{n}_{(3)}, s;\ N^{d}, ©, ©$: *Example* $\Rightarrow (c = -1/2,\ m = -5/4,\ n = -5/2)$ (if $\theta = 0$).

b) *The case $\eta > 0$.* From Lemmas 3.9 and 3.1 we get the following 6 configurations:

- $\overline{n}_{(3)}, s;\ S, N^{\infty}, N^{\infty}$: *Example* $\Rightarrow (c = -3/2,\ m = 0,\ n = -3)$ (if $\theta < 0$, $\theta_1 < 0$);
- $\overline{n}_{(3)}, s;\ S, N^{f}, N^{f}$: *Example* $\Rightarrow (c = 2,\ m = 7/4,\ n = 1/2)$ (if $\theta < 0$, $\theta_1 > 0$);
- $\overline{n}_{(3)}, s;\ S, N^{f}, N^{\infty}$: *Example* $\Rightarrow (c = 2,\ m = 5/2,\ n = 2)$ (if $\theta > 0$);
- $\overline{n}_{(3)}, s;\ S, N^{\infty}, N^{d}$: *Example* $\Rightarrow (c = -3/2,\ m = -1/12,\ n = -1/6)$ (if $\theta = 0$, $\theta_1 < 0$);
- $\overline{n}_{(3)}, s;\ S, N^{f}, N^{d}$: *Example* $\Rightarrow (c = -4,\ m = -1/2,\ n = -1)$ (if $\theta = 0$, $\theta_1 > 0$);
- $\overline{n}_{(3)}, s;\ S, N^{d}, N^{d}$: *Example* $\Rightarrow (c = -3/2,\ m = -1/2,\ n = -1)$ (if $\theta = 0$, $\theta_1 = 0$).

c) *The case $\eta = 0$.* According to Lemma 3.9 in this case the conditions $\theta = 0$ and $\theta_2 = 0$ are incompatible (otherwise we get $\eta = \mu_0 \neq 0$). So considering Lemma 3.1 we get the following 4 configurations:

- $\overline{n}_{(3)}, s;\ \overline{\binom{0}{2}}SN, N^{\infty}$: *Example* $\Rightarrow (c = -3/2,\ m = -1/6,\ n = -1)$ (if $\widetilde{M} \neq 0$, $\theta < 0$);
- $\overline{n}_{(3)}, s;\ \overline{\binom{0}{2}}SN, N^{f}$: *Example* $\Rightarrow (c = 1,\ m = 1,\ n = 1)$ (if $\widetilde{M} \neq 0$, $\theta > 0$);
- $\overline{n}_{(3)}, s;\ \overline{\binom{0}{2}}SN, N^{d}$: *Example* $\Rightarrow (c = -1/2,\ m = 3/4,\ n = -3/2)$ (if $\widetilde{M} \neq 0$, $\theta = 0$);
- $\overline{n}_{(3)}, s;\ \overline{\binom{0}{3}}N$: *Example* $\Rightarrow (c = -1/2,\ m = 1/4,\ n = -1/2)$ (if $\widetilde{M} = 0$).

2) *The possibility $\mathcal{T}_4 = 0$.* Then by (3.11) we get $\rho_1\rho_4 = 0$ and we consider two cases: $\mathcal{T}_3 \neq 0$ and $\mathcal{T}_3 = 0$.

a) *The case $\mathcal{T}_3 \neq 0$.* Then $\rho_1 \neq 0$ and by (3.11) this implies $\mathcal{F}_1 G_{10} \neq 0$. Therefore we get $\rho_4 = 1 - c + 2m = 0$, i.e. the finite saddle is weak of order one (due to $\mathcal{F}_1 \neq 0$).

a$_1$) *The subcase $\eta < 0$.* So considering Lemmas 3.1 and 3.9 we arrive at the following three configurations of singularities:

- $\overline{n}_{(3)}, s^{(1)};\ N^{\infty}, ©, ©$: *Example* $\Rightarrow (c = -1/2,\ m = -3/4,\ n = -2)$ (if $\theta < 0$);
- $\overline{n}_{(3)}, s^{(1)};\ N^{f}, ©, ©$: *Example* $\Rightarrow (c = -1/2,\ m = -3/4,\ n = -1)$ (if $\theta > 0$);
- $\overline{n}_{(3)}, s^{(1)};\ N^{d}, ©, ©$: *Example* $\Rightarrow (c = -1/2,\ m = -3/4,\ n = -3/2)$ (if $\theta = 0$).

a$_2$) *The subcase $\eta > 0$.* From Lemmas 3.9 and 3.1 we can get up to 6 configurations. Examples for five of them are:

- $\overline{n}_{(3)}, s^{(1)};\ S, N^{\infty}, N^{\infty}$: *Example* $\Rightarrow (c = -21/20,\ m = -41/40,\ n = -3)$ (if $\theta < 0$, $\theta_1 < 0$);
- $\overline{n}_{(3)}, s^{(1)};\ S, N^{f}, N^{f}$: *Example* $\Rightarrow (c = 1/2,\ m = -1/4,\ n = 1/3)$ (if $\theta < 0$, $\theta_1 > 0$);
- $\overline{n}_{(3)}, s^{(1)};\ S, N^{f}, N^{\infty}$: *Example* $\Rightarrow (c = -2,\ m = -3/2,\ n = -4)$ (if $\theta > 0$);
- $\overline{n}_{(3)}, s^{(1)};\ S, N^{\infty}, N^{d}$: *Example* $\Rightarrow (c = -9/10,\ m = -19/20,\ n = -19/10)$ (if $\theta = 0$, $\theta_1 < 0$);

• $\overline{n}_{(3)}, s^{(1)}; S, N^f, N^d$: *Example* $\Rightarrow$ $(c=-2,\ m=-3/2,\ n=-3)$ (if $\theta=0,\ \theta_1>0$).

The sixth configuration corresponding to the case $\theta=\theta_1=0$ can not be realizable. Indeed setting $\rho_4=1-c+2m=0$ (i.e. $m=(c-1)/2$) we found that the common solutions (c,n) of the equations $\theta=0$ and $\theta_1=0$ are the following ones: $\{(-1,2),\ (-1/2,-1/2),\ (0,-1/4),\ (0,0),\ (1,0)\}$. However all these solutions contradict the condition $\mu_0=c^2n^2(c-n)>0$.

***a*$_3$) *The subcase* $\eta=0$.** According to Lemma 3.9 in this case the conditions $\theta=0$ and $\theta_2=0$ are incompatible (otherwise we get $\eta=\mu_0\neq 0$). Moreover since $\mathcal{T}_4=0$, and $E_3<0$, by Lemma 3.9 in this case we have $\widetilde{M}\neq 0$. So considering Lemma 3.1 we get the following 3 configurations:

• $\overline{n}_{(3)}, s^{(1)}; \overline{\binom{0}{2}}SN, N^\infty$: *Example* $\Rightarrow$ $(c=-22/25,\ m=-47/50,\ n=-25/12)$ (if $\theta<0$);

• $\overline{n}_{(3)}, s^{(1)}; \overline{\binom{0}{2}}SN, N^f$: *Example* $\Rightarrow$ $(c=-4/5,\ m=-9/10,\ n=-5/4)$ (if $\theta>0$);

• $\overline{n}_{(3)}, s^{(1)}; \overline{\binom{0}{2}}SN, N^d$: *Example* $\Rightarrow$ $(c=-\sqrt{3}/2,\ m=-(2+\sqrt{3})/4,\ n=-1-\sqrt{3}/2)$ (if $\theta=0$).

***b*) *The case* $\mathcal{T}_3=0$.** This implies $\rho_1=0$ (i.e. $c=-1$) and then $G_{10}=0$). Therefore by [7, Table 1, line 88] the triple singularity is an elliptic saddle (nilpotent). On the other hand the elemental saddle could be a weak one and we shall consider two subcases: $\mathcal{T}_1\neq 0$ and $\mathcal{T}_1=0$.

***b*$_1$) *The subcase* $\mathcal{T}_1\neq 0$.** Then we have a strong saddle and by (3.12) the condition $\eta\geq 0$ holds.

***α*) *The possibility* $\eta>0$.** Considering Lemmas 3.1 and 3.9 we can get up to 6 configurations. Examples for five of them are:

• $\widehat{es}_{(3)}, s; S, N^\infty, N^\infty$: *Example* $\Rightarrow$ $(c=-1,\ m=-1/3,\ n=-1)$ (if $\theta<0,\ \theta_1<0$);

• $\widehat{es}_{(3)}, s; S, N^f, N^f$: *Example* $\Rightarrow$ $(c=-1,\ m=-21/25,\ n=-1)$ (if $\theta<0,\ \theta_1>0$);

• $\widehat{es}_{(3)}, s; S, N^f, N^\infty$: *Example* $\Rightarrow$ $(c=-1,\ m=-4/5,\ n=-1)$ (if $\theta>0$);

• $\widehat{es}_{(3)}, s; S, N^\infty, N^d$: *Example* $\Rightarrow$ $(c=-1,\ m=-1/4,\ n=-1/2)$ (if $\theta=0,\ \theta_1<0$);

• $\widehat{es}_{(3)}, s; S, N^f, N^d$: *Example* $\Rightarrow$ $(c=-1,\ m=-3/2,\ n=-3)$ (if $\theta=0,\ \theta_1>0$).

The sixth configuration corresponding to the case $\theta=\theta_1=0$ can not be realized. Indeed, considering (3.12) the conditions $\mu_0>0$ and $E_3<0$ imply $n<0$. On the other hand supposing $\theta=0=\theta_1$ we get the unique solution with $n<0$ and namely $(m,n)=(-1,-2)$. However in this case we get $\mathcal{T}_1=0$ which enters in contradiction with the case under examination.

***β*) *The possibility* $\eta=0$.** In this case according to Lemma 3.9 the condition $\theta=0$ implies $\theta_2\neq 0$. Moreover since $\rho_1=0$, by Lemma 3.9 we have $\widetilde{M}\neq 0$. So considering Lemma 3.1 we get the following 3 configurations:

• $\widehat{es}_{(3)}, s; \overline{\binom{0}{2}}SN, N^\infty$: *Example* $\Rightarrow$ $(c=-1,\ m=0,\ n=-1)$ (if $\theta<0$);

• $\widehat{es}_{(3)}, s; \overline{\binom{0}{2}}SN, N^f$: *Example* $\Rightarrow$ $(c=-1,\ m=-1/2,\ n=-1/2)$ (if $\theta>0$);

• $\widehat{es}_{(3)}, s; \overline{\binom{0}{2}}SN, N^d$: *Example* $\Rightarrow$ $(c=-1,\ m=-1/2,\ n=-1)$ (if $\theta=0$).

***b*$_2$) *The subcase* $\mathcal{T}_1=0$.** In this case the saddle is weak and the conditions $\rho_1=\rho_4=0$ yield $c=-1=m$. Then we arrive at the formulas (3.13) and clearly the condition $\mu_0>0$ implies $n<-1$ (then $E_3<0$), and therefore we obtain $\sigma\neq 0$, $\mathcal{H}>0$ and $\mathcal{B}<0$. By [31, Main Theorem, $(e_3), \alpha$] we obtain an integrable saddle.

On the other hand from formulas (3.13) we observe that the condition $\theta\leq 0$ holds. Since $\eta>0$ considering Lemmas 3.1 and 3.9 we arrive at the following three configurations:

• $\widehat{es}_{(3)}, \$; S, N^\infty, N^\infty$: *Example* $\Rightarrow$ $(c=-1,\ m=-1,\ n=-3)$ (if $\theta\neq 0,\ \theta_1<0$);

• $\widehat{es}_{(3)}, \$; S, N^f, N^f$: *Example* $\Rightarrow$ $(c=-1,\ m=-1,\ n=-3/2)$ (if $\theta\neq 0,\ \theta_1>0$);

- $\widehat{es}_{(3)}, \$; \ S, N^d, N^d$: *Example* $\Rightarrow (c=-1,\ m=-1,\ n=-2)$ (if $\theta=0$).

The subcase $E_3 > 0$. In this case the elemental singularity M_4 is an anti-saddle. On the other hand by [7] the singular point $M_1(0,0)$ is a triple saddle (semi-elemental or nilpotent).

1) *The possibility* $W_4 < 0$. Then from (3.11) it follows $G_{10} \neq 0$ and according to [7, Table 1, line 91] we have a focus and a semi-elemental triple saddle. And the focus could be weak only if $\mathcal{T}_4 = 0$.

***a*)** *The case* $\mathcal{T}_4 \neq 0$. In this case we have a strong focus.

***a*$_1$)** *The subcase* $\eta < 0$. By Lemma 3.1 the configurations of infinite singularities are governed by the invariant polynomials θ and θ_2. According to Lemma 3.9 in this case the condition $\theta = 0$ implies $\theta_2 \neq 0$ and considering Lemma 3.1 we arrive at the following three configurations of singularities:

- $\overline{s}_{(3)}, f; \ N^\infty, ©, ©$: *Example* $\Rightarrow (c=2,\ m=-11/8,\ n=-2)$ (if $\theta<0$);
- $\overline{s}_{(3)}, f; \ N^f, ©, ©$: *Example* $\Rightarrow (c=2,\ m=-1/2,\ n=-2)$ (if $\theta>0$);
- $\overline{s}_{(3)}, f; \ N^d, ©, ©$: *Example* $\Rightarrow (c=2,\ m=-1,\ n=-2)$ (if $\theta=0$).

***a*$_2$)** *The subcase* $\eta > 0$. We claim that in this case instead of 10 possible configurations indicated in Table 3.1 there could be realized only 3. More precisely we claim that if $\mu_0 > 0$, $W_4 < 0$ and $\eta > 0$, then the condition $\theta \leq 0$ implies $\theta_1 < 0$, and in the case $\theta = 0$ we have $\theta_2 \neq 0$.

Indeed, it can be proved directly that if the conditions $\mu_0 > 0$, $W_4 < 0$ (i.e. $\tau_4 < 0$) and $\eta > 0$ hold, then the surfaces $\theta = 0$ and $\theta_1 = 0$ do not intersect in real points. So it is sufficient to take any slice in the 3-dimensional space of the parameters (c, m, n) which intersects the region where the conditions hold, and to check that at a point where $\theta_1 = 0$ the polynomial θ is positive.

It remains to observe, that in the case $\theta = 0$ by Lemma 3.9 (see statement *(iii)*) the condition $\theta_2 \neq 0$ must hold. This completes the proof of our claim.

Remark 3.10. We point out that the above arguments hold regardless of the value of $\mathcal{T}_4$. So the same reasons could be used when $\mathcal{T}_4 = 0$.

Thus by Lemma 3.1 the remaining three configurations of singularities are:

- $\overline{s}_{(3)}, f; \ S, N^\infty, N^\infty$: *Example* $\Rightarrow (c=20,\ m=-17/2,\ n=-65/4)$ (if $\theta<0 \Rightarrow \theta_1<0$);
- $\overline{s}_{(3)}, f; \ S, N^f, N^\infty$: *Example* $\Rightarrow (c=545/64,\ m=127/128,\ n=-65/64)$ (if $\theta>0$);
- $\overline{s}_{(3)}, f; \ S, N^\infty, N^d$: *Example* $\Rightarrow (c=137/16,\ m=-65/32,\ n=-65/16)$ (if $\theta=0$).

***a*$_3$)** *The subcase* $\eta = 0$. According to Lemma 3.9 in this case the condition $\theta = 0$ implies $\theta_2 \neq 0$ and considering Lemma 3.1 we arrive at the following four configurations of singularities:

- $\overline{s}_{(3)}, f; \ \overline{\binom{0}{2}}SN, N^\infty$: *Example* $\Rightarrow (c=5/4,\ m=-7/8,\ n=-1)$ (if $\widetilde{M}\neq 0,\ \theta<0$);
- $\overline{s}_{(3)}, f; \ \overline{\binom{0}{2}}SN, N^f$: *Example* $\Rightarrow (c=65/16,\ m=47/32,\ n=-1/16)$ (if $\widetilde{M}\neq 0,\ \theta>0$);
- $\overline{s}_{(3)}, f; \ \overline{\binom{0}{2}}SN, N^d$: *Example* $\Rightarrow (c=5/4,\ m=-1/8,\ n=-1/4)$ (if $\widetilde{M}\neq 0,\ \theta=0$);
- $\overline{s}_{(3)}, f; \ \overline{\binom{0}{3}}N$: *Example* $\Rightarrow (c=5/11,\ m=-5/22,\ n=-25/176)$ (if $\widetilde{M}=0$).

***b*)** *The case* $\mathcal{T}_4 = 0$. Then by (3.11) we get $\rho_1\rho_4 = 0$ and as $W_4 \neq 0$ implies $\rho_1\mathcal{F}_1 \neq 0$ we obtain $\rho_4 = 0$ (i.e. $1 - c + 2m = 0$). So we have a first order weak focus.

Considering Remark 3.10 it remains to present the examples for the realizations of the next 10 configurations.

***b*$_1$)** *The subcase* $\eta < 0$.

• $\overline{s}_{(3)}, f^{(1)}; N^{\infty}, ©, ©$: *Example* $\Rightarrow$ $(c = 1/2,\ m = -1/4,\ n = -1/3)$ (if $\theta < 0$);
• $\overline{s}_{(3)}, f^{(1)}; N^{f}, ©, ©$: *Example* $\Rightarrow$ $(c = 1/2,\ m = -1/4,\ n = -1)$ (if $\theta > 0$);
• $\overline{s}_{(3)}, f^{(1)}; N^{d}, ©, ©$: *Example* $\Rightarrow$ $(c = 1/2,\ m = -1/4,\ n = -1/2)$ (if $\theta = 0$).

***b*$_2$) *The subcase* $\eta > 0$.** Considering Lemmas 3.1 and 3.9 we arrive at the following 3 configurations of singularities:
• $\overline{s}_{(3)}, f^{(1)}; S, N^{\infty}, N^{\infty}$: *Example* $\Rightarrow$ $(c = 1/2,\ m = -1/4,\ n = -8/50)$ (if $\theta < 0$);
• $\overline{s}_{(3)}, f^{(1)}; S, N^{f}, N^{\infty}$: *Example* $\Rightarrow$ $(c = 1/2,\ m = -1/4,\ n = -6/50)$ (if $\theta > 0$);
• $\overline{s}_{(3)}, f^{(1)}; S, N^{\infty}, N^{d}$: *Example* $\Rightarrow$ $(c = 1/2,\ m = -1/4,\ n = -7/50)$ (if $\theta = 0$).

Considering the graphic $\theta_1 = 0$ we observe that we can not have any branch inside the region defined by the conditions $\mu_0 > 0$, $E_3 > 0$ and $\theta < 0$.

***b*$_3$) *The subcase* $\eta = 0$.**
• $\overline{s}_{(3)}, f^{(1)}; \overline{\binom{0}{2}}SN, N^{\infty}$: *Example* $\Rightarrow$ $(c = 1/3,\ m = -1/3,\ n = -3/16)$ (if $\widetilde{M} \neq 0$, $\theta < 0$);
• $\overline{s}_{(3)}, f^{(1)}; \overline{\binom{0}{2}}SN, N^{f}$: *Example* $\Rightarrow$ $(c = 1,\ m = 0,\ n = -1/8)$ (if $\widetilde{M} \neq 0$, $\theta > 0$);
• $\overline{s}_{(3)}, f^{(1)}; \overline{\binom{0}{2}}SN, N^{d}$: *Example* $\Rightarrow$ $(c = \sqrt{3}/2,\ m = (\sqrt{3}-2)/4,\ n = (\sqrt{3}-2)/2)$ (if $\widetilde{M} \neq 0$, $\theta = 0$);
• $\overline{s}_{(3)}, f^{(1)}; \overline{\binom{0}{3}}N$: *Example* $\Rightarrow$ $(c = 1/2,\ m = -1/4,\ n = -1/6)$ (if $\widetilde{M} = 0$).

2) *The possibility* $W_4 > 0$. In this case $G_{10} \neq 0$ and according to [7, Table 1, line 89] we have a generic node and a semi-elemental triple saddle.

***a*) *The case* $\eta < 0$.** Therefore considering Lemmas 3.1 and 3.9 (the statement *(iii)*) we arrive at the following three configurations of singularities:
• $\overline{s}_{(3)}, n; N^{\infty}, ©, ©$: *Example* $\Rightarrow$ $(c = 5/2,\ m = -23/8,\ n = -5)$ (if $\theta < 0$);
• $\overline{s}_{(3)}, n; N^{f}, ©, ©$: *Example* $\Rightarrow$ $(c = 1,\ m = 5/4,\ n = -1/2)$ (if $\theta > 0$);
• $\overline{s}_{(3)}, n; N^{d}, ©, ©$: *Example* $\Rightarrow$ $(c = 1,\ m = -5/2,\ n = -5)$ (if $\theta = 0$).

***b*) *The case* $\eta > 0$.** So by Lemmas 3.1 and 3.9 (the statement *(iv)*) we obtain the following six configurations:
• $\overline{s}_{(3)}, n; S, N^{\infty}, N^{\infty}$: *Example* $\Rightarrow$ $(c = 1,\ m = -\dfrac{14849}{5000},\ n = -59/10)$ (if $\theta < 0$, $\theta_1 < 0$);
• $\overline{s}_{(3)}, n; S, N^{f}, N^{f}$: *Example* $\Rightarrow$ $(c = 1,\ m = -65/18,\ n = -8)$ (if $\theta < 0$, $\theta_1 > 0$);
• $\overline{s}_{(3)}, n; S, N^{f}, N^{\infty}$: *Example* $\Rightarrow$ $(c = 1,\ m = -29/50,\ n = -1/5)$ (if $\theta > 0$);
• $\overline{s}_{(3)}, n; S, N^{\infty}, N^{d}$: *Example* $\Rightarrow$ $(c = 1,\ m = -59/20,\ n = -59/10)$ (if $\theta = 0$, $\theta_1 < 0$);
• $\overline{s}_{(3)}, n; S, N^{f}, N^{d}$: *Example* $\Rightarrow$ $(c = 1,\ m = -61/20,\ n = -61/10)$ (if $\theta = 0$, $\theta_1 > 0$);
• $\overline{s}_{(3)}, n; S, N^{d}, N^{d}$: *Example* $\Rightarrow$ $(c = 1,\ m = -3,\ n = -6)$ (if $\theta = 0$, $\theta_1 = 0$).

***c*) *The case* $\eta = 0$.** In this case considering Lemmas 3.1 and 3.9 we get the following four configurations:
• $\overline{s}_{(3)}, n; \overline{\binom{0}{2}}SN, N^{\infty}$: *Example* $\Rightarrow$ $(c = 1/5,\ m = -1/30,\ n = -1/90)$ (if $\widetilde{M} \neq 0$, $\theta < 0$);
• $\overline{s}_{(3)}, n; \overline{\binom{0}{2}}SN, N^{f}$: *Example* $\Rightarrow$ $(c = 1/5,\ m = -31/120,\ n = -31/360)$ (if $\widetilde{M} \neq 0$, $\theta > 0$);
• $\overline{s}_{(3)}, n; \overline{\binom{0}{2}}SN, N^{d}$: *Example* $\Rightarrow$ $(c = 1/5,\ m = -11/60,\ n = -11/180)$ (if $\widetilde{M} \neq 0$, $\theta = 0$);
• $\overline{s}_{(3)}, n; \overline{\binom{0}{3}}N$: *Example* $\Rightarrow$ $(c = 1/4,\ m = -1/8,\ n = -1/20)$ (if $\widetilde{M} = 0$).

3) *The possibility* $W_4 = 0$. In this case due to $\mu_0 \neq 0$ we get $\rho_1 \tau_4 = 0$ and we consider two cases: $\mathcal{T}_4 \neq 0$ and $\mathcal{T}_4 = 0$.

***a*) *The case* $\mathcal{T}_4 \neq 0$.** Then $\rho_1 \neq 0$ and hence we obtain $\tau_4 = 0$, i.e. we have a node with coinciding eigenvalues which by (3.10) is not a star node.

***a*₁) *The subcase* $\eta < 0$. In this case considering Lemmas 3.1 and 3.9 (the statement *(iii)*) we arrive at the following three configurations of singularities:

- $\overline{s}_{(3)}, n^d; N^\infty, ©, ©$: *Example* $\Rightarrow (c = 1,\ m = -3/2,\ n = -9/4)$ (if $\theta < 0$);
- $\overline{s}_{(3)}, n^d; N^f, ©, ©$: *Example* $\Rightarrow (c = 1,\ m = -3,\ n = -9)$ (if $\theta > 0$);
- $\overline{s}_{(3)}, n^d; N^d, ©, ©$: *Example* $\Rightarrow (c = 1,\ m = -2,\ n = -4)$ (if $\theta = 0$).

***a*₂) *The subcase* $\eta > 0$. Then by Lemmas 3.1 and 3.9 (the statements *(iii)* and *(iv)*) we could have 5 possible configurations of singularities. For three of them we give corresponding examples:

- $\overline{s}_{(3)}, n^d; S, N^\infty, N^\infty$: *Example* $\Rightarrow (c = 5/2,\ m = -1/2,\ n = -5/8)$ (if $\theta < 0, \theta_1 < 0$);
- $\overline{s}_{(3)}, n^d; S, N^f, N^\infty$: *Example* $\Rightarrow (c = 1,\ m = -1/2,\ n = -1/4)$ (if $\theta > 0$);
- $\overline{s}_{(3)}, n^d; S, N^\infty, N^d$: *Example* $\Rightarrow (c = 9/4,\ m = -1/8,\ n = -1/4)$ (if $\theta = 0, \theta_1 < 0$).

We claim that the remaining two configurations (defined respectively by the conditions $\theta < 0, \theta_1 > 0$ and $\theta = 0, \theta_1 > 0$) could not be realizable. More exactly we prove below that the condition $\theta \le 0$ implies in this case $\theta_1 < 0$.

Indeed, as $\Delta_4 \neq 0$ (i.e. $cn \neq 0$) the condition $\tau_4 = 0$ yields $n = -(1 - c + 2m)^2/(4c)$ and then the invariant polynomials $\theta = \theta(c, m)$ and $\theta_1 = \theta_1(c, m)$ depend on 2 parameters c and m. Considering the system of the equations $\theta(c, m) = 0$ and $\theta_1(c, m) = 0$ we detect that the only real solutions of this system are $(-2, 0)$ and $(1, 0)$. However in these points we have $\mu_0 = -81/128 < 0$ and $\mu_0 = 0$, respectively.

On the other hand in this case we calculate

$$\mu_0 = \frac{(1 - c + 2m)^4 (1 + c + 2m)^2}{64c},$$
$$\eta = \frac{(1 + c + 2m)^2 \left[(c-1)^2 + 2(1+c)m\right]^2 \left[c^2 + c - (1+2m)^2\right]}{16c^3},$$

and as $\mu_0 > 0$ we get $c > 0$. So it is enough to check in the region defined by $c > 0$ and $c^2 + c - (1 + 2m)^2 > 0$ the signs of θ and θ_1, and to verify that in any point of this region where $\theta \le 0$ and $\theta_1 \neq 0$ we have $\theta_1 < 0$.

In order to check the conditions we need to find out the intersections of the graphic $\theta_1 = 0$ with the component $c^2 + c - 1 - 4m - 4m^2 = 0$. We obtain only one point (c, m) with $c > 0$, namely $(1/3, -1/6)$, and this is a contact point of $\theta_1 = 0$ and $\eta = 0$ which does not produce any new region inside $\eta > 0$.

***a*₃) *The subcase* $\eta = 0$. In this case considering Lemmas 3.1 and 3.9 we get the following four configurations of singularities:

- $\overline{s}_{(3)}, n^d; \overline{\binom{0}{2}}SN, N^\infty$: *Example* $\Rightarrow (c = 9/16,\ m = -31/32,\ n = -1)$ (if $\widetilde{M} \neq 0, \theta < 0$);
- $\overline{s}_{(3)}, n^d; \overline{\binom{0}{2}}SN, N^f$: *Example* $\Rightarrow (c = 49/72,\ m = 5/144,\ n = -1/18)$ (if $\widetilde{M} \neq 0, \theta > 0$);
- $\overline{s}_{(3)}, n^d; \overline{\binom{0}{2}}SN, N^d$: *Example* $\Rightarrow (c = 9/16,\ m = -1/32,\ n = -1/16)$ (if $\widetilde{M} \neq 0, \theta = 0$);
- $\overline{s}_{(3)}, n^d; \overline{\binom{0}{3}}N$: *Example* $\Rightarrow (c = 1/3,\ m = -1/6,\ n = -1/12)$ (if $\widetilde{M} = 0$).

***b*) *The case* $\mathcal{T}_4 = 0$. Then $\rho_1\rho_4 = 0$ and considering the condition $\rho_1\tau_4 = 0$ (as $W_4 = 0$) we obtain $\rho_1 = 0$ (otherwise we get $\rho_4 = \tau_4 = 0$ which is impossible for an elemental singular point). Thus $\rho_1 = 0$ (i.e. $c = -1$) and we arrive at the relations (3.11). As $G_{10} = 0$ by [7] the triple point is a nilpotent saddle and the elemental one is an anti-saddle. We claim that the condition $\mu_0 > 0$ implies $\tau_4 > 0$. Indeed considering (3.11) the condition $\mu_0 > 0$ gives $1 + 2m - n > 0$ and therefore $\tau_4 = (1 + 2m - n) + m^2 > 0$. So our claim is proved and hence the elemental singular point is a generic node.

We observe also that the conditions $\mu_0 > 0$ and $E_3 > 0$ imply $0 < n < 2m+1$. Then considering the graphics $\theta = 0$ and $\theta_1 = 0$ it is easy to detect that they do not intersection for $n > 0$ and that the condition $\theta < 0$ implies $\theta_1 > 0$.

On the other hand by (3.11) we have $\eta \geq 0$. Moreover due to the condition $E_3 > 0$ we could have $\eta = 0$ only if $m = 0$ (otherwise in the case $m = -1/2$ we get $E_3 = -n^4/4 < 0$). And in the case $m = 0$ we obtain $\widetilde{M} = -8x^2 \neq 0$ and $\theta = 8n^3 > 0$ as $n > 0$. Thus considering Lemma 3.1 we arrive at the following configurations of singularities:

- $\hat{s}_{(3)}, n;\ S, N^f, N^f$: *Example* $\Rightarrow$ $(c = -1,\ m = 2,\ n = 1)$ (if $\eta > 0, \theta < 0$);
- $\hat{s}_{(3)}, n;\ S, N^f, N^\infty$: *Example* $\Rightarrow$ $(c = -1,\ m = 1,\ n = 9/4)$ (if $\eta > 0, \theta > 0$);
- $\hat{s}_{(3)}, n;\ \overline{\binom{0}{2}}SN, N^f$: *Example* $\Rightarrow$ $(c = -1,\ m = 0,\ n = 1/2)$ (if $\eta = 0$).

3.3.2 The family of systems (3.8)

So we consider the family of systems

$$\dot{x} = 2hxy, \qquad \dot{y} = x - x^2 + 2mxy + ny^2, \quad m \in \{0, 1\}, \tag{3.14}$$

which possess the triple nilpotent singular point $M_1(0,0)$ and the elemental one $M_4(1,0)$. For the second singularity we have

$$\mathcal{M}_4 = \begin{pmatrix} 0 & 2h \\ -1 & 2m \end{pmatrix}, \qquad \rho_4 = 2m, \qquad \Delta_4 = 2h; \qquad \tau_4 = 4(m^2 - 2h). \tag{3.15}$$

For this family we calculate

$$\begin{gathered} \mu_0 = -n\Delta_4^2, \ \ \eta = 4(2h-n)^2(m^2+n-2h), \ \ E_3 = -n\Delta_4^3/4, \\ G_{10} = 0, \ \ \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = 0, \ \ \mathcal{T}_1 = -n\Delta_4^2\rho_4, \ \ \mathcal{F}_1 = 0, \\ \mathcal{F} = m(h+n)^2\Delta_4 \ \ W_4 = 0, \ \ W_1 = n^2\Delta_4^4\tau_4, \\ \widetilde{M} = 8(6h - 4m^2 - 3n)x^2 + 16m(2h-n)xy - 8(2h-n)^2y^2. \end{gathered} \tag{3.16}$$

Lemma 3.11. *If for a system* (3.14)*:*

(i) the condition $\widetilde{M} = 0$ holds, then $\mathcal{T}_1 = \theta_1 = 0$, $E_3 < 0$ and $\operatorname{sign}(\theta) = -\operatorname{sign}(\mu_0)$*;*

(ii) the conditions $\theta = 0$ and $\mathcal{T}_1 \neq 0$ hold, then for this system we have $\eta \geq 0$, $\theta_1\theta_2 \neq 0$.

Proof. (i) By (3.16) we obtain that the condition $\widetilde{M} = 0$ is equivalent to $n - 2h = m = 0$ and we obtain:

$$\mathcal{T}_1 = \theta_1 = 0, \qquad E_3 = -4h^4, \qquad \mu_0 = -8h^3, \qquad \theta = 64h^3.$$

Evidently statement (i) follows immediately.

(ii) Setting a new parameter $u = h - n$ we have $\theta = 64(n+u)(m^2n + u^2)$ and $\mu_0 = -4n(n+u)^2 \neq 0$. So due to $m \neq 0$ (as $\mathcal{T}_1 \neq 0$) the condition $\theta = 0$ gives $n = -u^2/m^2$. Then we calculate

$$\eta = \frac{4(m^2-u)^2(2m^2-u)^2u^2}{m^6}, \ \ \theta_1 = \frac{256(m^2-u)^3u^2}{m^4}, \ \ \theta_2 = \frac{(m^2-u)u^2}{m^2}, \ \ \mu_0 = \frac{4(m^2-u)^2u^4}{m^6},$$

and this completes the proof of the lemma as $\mu_0 \neq 0$. □

We observe that for systems (3.8) the parameter $m \in \{0, 1\}$, and as the condition $m = 0$ is equivalent to $\mathcal{T}_1 = 0$ we consider two cases.

The case $\mathcal{T}_1 \neq 0$. Then $m = 1$ and the condition $\mu_0 \neq 0$ implies $E_3 \neq 0$.

The subcase $\mu_0 < 0$. Therefore $n > 0$ and we obtain $\text{sign}\,(E_3) = -\text{sign}\,(\Delta_4)$.

1) *The possibility* $E_3 < 0$. In this case by [7, Table 1, lines 97–99] the nilpotent point is an elliptic saddle whereas the elemental singularity M_4 is an anti-saddle. Therefore since the sum of the indexes of the finite singularities equals $+2$ we conclude that the sum of the indices of the infinite singularities must be -1.

On the other hand by [7] the type of the elemental singularity is governed by the invariant polynomial W_1.

a) *The case* $W_1 < 0$. Then the elemental singular point is a strong focus and considering Lemmas 3.1 and 3.11 we arrive at the next three configurations

• $\widehat{es}_{(3)}, f;\ S, ©, ©$: *Example* $\Rightarrow$ $(h = 2,\ m = 1,\ n = 1)$ (if $\eta < 0$);

$\widehat{es}_{(3)}, f;\ S, S, N^\infty$ (if $\eta > 0$), or $\widehat{es}_{(3)}, f;\ \overline{\binom{0}{2}}SN, S$ (if $\eta = 0$).

We observe that only the configuration corresponding to $\eta < 0$ is a new one, other two being realizable for the family (3.7) in the case $\mu_0 < 0$, $E_3 < 0$, $W_4 = 0$, $\mathcal{T}_4 = 0$, $W_1 < 0$, $\mathcal{T}_1 \neq 0$ and $\eta \geq 0$. It remains to remark that according to (3.12) for the family (3.7) in the case we consider we could not have $\eta < 0$.

b) *The case* $W_1 > 0$. Under this condition we have a generic node. Moreover the above conditions imply $n > 0$, $h > 0$ and $h < 1/2$ and this gives $\eta > 0$. So by Lemma 3.1 we get the unique configuration $\widehat{es}_{(3)}, n;\ S, S, N^\infty$ which already is obtained for the previous family.

c) *The case* $W_1 = 0$. Then $\tau_4 = 0$ and considering the matrix $\mathcal{M}_4$ from (3.15) we have a node n^d. We get $h = 1/2$ and this implies $\eta > 0$ and we arrive at the configuration $\widehat{es}_{(3)}, n^d;\ S, S, N^\infty$, obtained earlier.

2) *The possibility* $E_3 > 0$. In this case by [7] we have a nilpotent triple saddle and a saddle. So this implies at infinity the existence of three nodes and according to Lemma 3.1 we get the configuration $\hat{s}_{(3)}, s;\ N^f, N^f, N^f$, which is already obtained.

The subcase $\mu_0 > 0$. Therefore $n < 0$ and we obtain $\text{sign}\,(E_3) = \text{sign}\,(\Delta_4)$.

1) *The possibility* $E_3 < 0$. Then $h < 0$ and in this case the elemental singularity M_4 is a saddle (which is strong due to $m \neq 0$) and by [7, Table 1, line 88] the triple point is a nilpotent elliptic saddle.

a) *The case* $\eta < 0$. According to Lemma 3.11 the condition $\theta \neq 0$ holds, otherwise we get $\eta \geq 0$. Moreover we observe that the condition $\eta < 0$ implies $\theta < 0$. Indeed, as $\eta < 0$ (i.e. $n - 2h + 1 < 0$) we set $n - 2h + 1 = -u^2$ (i.e. $n = 2h - 1 - u^2$) and then we calculate $\theta = 64h\big[(u^2 - h)^2 + u^2\big] < 0$ due to $h < 0$. Thus $\theta < 0$ and by Lemma 3.1 we get the unique configuration, which is a new one:

• $\widehat{es}_{(3)}, s;\ N^\infty, ©, ©$: *Example* $\Rightarrow$ $(h = -1/4,\ m = 1,\ n = -2)$.

b) *The case* $\eta > 0$. It was proved for the previous family (see the subsection defined by the conditions $\mu_0 > 0$, $E_3 > 0$, $\mathcal{T}_4 = \mathcal{T}_3 = 0$ and $\mathcal{T}_1 \neq 0$) that there could exist only 6 configurations. More precisely the possibilities with $\theta = 0$, and either $\theta_1 \neq 0$ and $\theta_2 = 0$ or $\theta_1 = \theta_3 = 0$ can not be realizable. However according to Lemma 3.11 none of these cases can be realizable for the family (3.14).

c) *The case* $\eta = 0$. Since $\widetilde{M} \neq 0$, in this case according to Lemma 3.1 at infinity we can have in this case four configurations. We have proved (see the same subsection above mentioned) that for the previous family (3.7) all cases are realizable, with the exception of $\theta = \theta_2 = 0$

because the condition $\theta = 0$ implies $\theta_2 \neq 0$. We observe that for systems (3.14) by Lemma 3.11 the condition $\theta = 0$ also implies $\theta_2 \neq 0$ and hence we can not obtain new configurations of singularities.

2) *The possibility* $E_3 > 0$. As $\mu_0 > 0$ considering (3.16) we get $n < 0$ and $h > 0$. In this case by [7, Table 1, lines 90, 92, 93] the triple singular point is a saddle, whereas the elemental singularity M_4 is an anti-saddle, the type of which is governed by the invariant polynomial W_1.

a) *The case* $W_1 < 0$. Then $\tau_4 = 4(1-2h) < 0$ and the elemental singular point is a strong focus. On the other hand considering (3.16) the conditions $n < 0$ and $h > 1/2$ imply $\eta < 0$ and $\theta > 0$. So by Lemma 3.1 we arrive at the next new configuration:

• $\hat{s}_{(3)}, f;\ N^f, ©, ©$: *Example* $\Rightarrow$ $(h = 1,\ m = 1,\ n = -2)$.

b) *The case* $W_1 > 0$. We have a generic node and considering the conditions $n < 0$, $0 < h < 1/2$ and the graphics $2h - 1 - n = 0$ (for η) and $(h-n)^2 + n = 0$ (for θ) we obviously obtain that the condition $\eta < 0$ implies $\theta > 0$. Moreover, the conditions $\eta > 0$ and $\theta < 0$ give $\theta_1 > 0$, and the condition $\eta = 0$ (i.e. $n = 2h - 1$) yields $\theta = 64h^3 > 0$. So considering Lemma 3.1 we get the next configurations:

• $\hat{s}_{(3)}, n;\ N^f, ©, ©$: *Example* $\Rightarrow$ $(h = 1/5,\ m = 1,\ n = -1)$ (if $\eta < 0$);
$\hat{s}_{(3)}, n;\ S, N^f, N^f$ (if $\eta > 0,\ \theta < 0$); $\hat{s}_{(3)}, n;\ S, N^f, N^\infty$ (if $\eta > 0,\ \theta > 0$) and
$\hat{s}_{(3)}, n;\ \overline{\binom{0}{2}}SN, N^f$ (if $\eta = 0$).

In such a way we get a unique new configuration which is realizable if the conditions $\mu_0 > 0$, $E_3 > 0$, $W_4 = 0$, $\mathcal{T}_4 = 0$, $W_1 > 0$, $\mathcal{T}_1 \neq 0$ and $\eta < 0$ hold and this allows to include it in the diagram in the corresponding place.

c) *The case* $W_1 = 0$. Then $h = 1/2$ (in this case we have a node with one direction) and we calculate

$$\mu_0 = -n = 4E_3, \qquad \eta = 4n(n-1)^2, \qquad \theta = 8(1+4n^2) > 0.$$

Therefore as the condition $\mu_0 > 0$ implies $\eta < 0$ we only get one configuration it is a new one:

• $\hat{s}_{(3)}, n^d;\ N^f, ©, ©$: *Example* $\Rightarrow$ $(h = 1/2,\ m = 1,\ n = -1)$.

The case $\mathcal{T}_1 = 0$. Then $m = 0$ and since $n \neq 0$ due to the rescaling $(x, y, t) \mapsto (x, |n|^{-1/2}y,\ |n|^{-1/2}t)$ we may assume $n \in \{-1, 1\}$.

The subcase $\mu_0 < 0$. Then $n > 0$ and assuming $n = 1$ we get the family of systems

$$\dot{x} = 2hxy, \qquad \dot{y} = x - x^2 + y^2, \tag{3.17}$$

for which we have

$$\begin{gathered}\mu_0 = -4h^2,\ \ E_3 = -2h^3,\ \ \eta = 4(1-2h)^3,\ \ W_1 = -128h^5,\\ \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = \mathcal{T}_1 = \mathcal{F} = \mathcal{F}_1 = 0,\ \ \sigma = 2(h+1)y,\\ \mathcal{H} = -4h(h+1)^2,\ \ \mathcal{B} = -2(h+1)^4,\ \ \widetilde{M} = 8(2h-1)(3x^2 - 2hy^2 + y^2).\end{gathered} \tag{3.18}$$

1) *The possibility* $E_3 < 0$. Then $h > 0$ and this implies $\sigma \neq 0$, $\mathcal{H} < 0$, $\mathcal{B} < 0$. We observe that the condition $\eta = 0$ implies $\widetilde{M} = 0$. So by [31] and [7] we have a center and an elliptic saddle and considering Lemma 3.1 we get the following configurations:

• $\widehat{es}_{(3)}, c;\ S, ©, ©$: *Example* $\Rightarrow$ $(h = 2,\ m = 0,\ n = 1)$ (if $\eta < 0$);
$\widehat{es}_{(3)}, c;\ S, S, N^\infty$: (if $\eta > 0$);

• $\widehat{es}_{(3)}, c; \overline{\binom{0}{3}}S$: *Example* $\Rightarrow$ $(h = 1/2,\ m = 0,\ n = 1)$ (if $\eta = 0$). Thus we obtain two new configurations, which are defined by the conditions $\mu_0 < 0$, $E_3 < 0$, $W_4 = 0$, $\mathcal{T}_4 = 0$, $W_1 < 0$, $\mathcal{T}_1 = 0$ and either $\eta < 0$ or $\eta = 0$.

2) *The possibility* $E_3 > 0$. In this case by [7] we have a triple nilpotent saddle and a weak saddle which is integrable. Since the condition $h < 0$ implies $\eta > 0$ we only get one configuration $\hat{s}_{(3)}, \$; N^f, N^f, N^f$ which is already obtained for the previous family.

The subcase $\mu_0 > 0$. Then $n < 0$ and assuming $n = -1$ we get the family of systems

$$\dot{x} = 2hxy, \qquad \dot{y} = x - x^2 - y^2, \tag{3.19}$$

for which we have

$$\begin{gathered}\mu_0 = 4h^2,\ \ E_3 = 2h^3,\ \ \eta = -4(1+2h)^3,\ \ W_1 = -128h^5,\\ \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = \mathcal{T}_1 = \mathcal{F} = \mathcal{F}_1 = 0,\ \ \sigma = 2(h-1)y,\\ \theta = 64h(1+h)^2,\ \ \theta_1 = -64(1+h)(1+2h)(1+5h),\ \ \theta_3 = -h(5h^2+2h-1),\\ \mathcal{H} = -4h(h-1)^2,\ \ \mathcal{B} = -2(h-1)^4,\ \ \widetilde{M} = 8(2h+1)(3x^2-2hy^2-y^2).\end{gathered} \tag{3.20}$$

1) *The possibility* $E_3 < 0$. Then $h < 0$ and the elemental singularity is an integrable saddle, whereas the triple point is an elliptic saddle. We observe that in this case we have $\theta \le 0$ and the condition $\eta = 0$ implies $\widetilde{M} = 0$. So we arrive at the following configurations (two of which are new ones):

• $\widehat{es}_{(3)}, \$; N^\infty, ©, ©$: *Example* $\Rightarrow$ $(h = -1/3,\ m = 0,\ n = -1)$ (if $\eta < 0$);
$\widehat{es}_{(3)}, \$; S, N^\infty, N^\infty$: (if $\eta > 0$, $\theta \neq 0$, $\theta_1 < 0$);
$\widehat{es}_{(3)}, \$; S, N^f, N^f$: (if $\eta > 0$, $\theta \neq 0$, $\theta_1 > 0$);
$\widehat{es}_{(3)}, \$; S, N^d, N^d$: (if $\eta > 0$, $\theta = 0$);
• $\widehat{es}_{(3)}, \$; \overline{\binom{0}{3}}N$: *Example* $\Rightarrow$ $(h = -1/2,\ m = 0,\ n = -1)$ (if $\eta = 0$).

2) *The possibility* $E_3 > 0$. In this case $h > 0$ and the elemental singular point is a center, which exists besides a nilpotent triple saddle. We observe that the condition $h > 0$ implies $\eta < 0$, $\theta > 0$ and we get the unique configuration, which is a new one:
• $\widehat{s}_{(3)}, c; N^f, ©, ©$: *Example* $\Rightarrow$ $(h = 1,\ m = 0,\ n = -1)$.

3.3.3 The family of systems (3.9)

So we consider the family of systems

$$\dot{x} = x + y - x^2 + 2hxy, \qquad \dot{y} = 2my(x+y), \tag{3.21}$$

which possess the triple semi-elemental singular point $M_{1,2,3}(0,0)$ and the elemental one $M_4(1,0)$. For the second singularity we have

$$\mathcal{M}_4 = \begin{pmatrix} -1 & 1+2h \\ 0 & 2m \end{pmatrix}, \qquad \rho_4 = 2m-1, \quad \Delta_4 = -2m; \quad \tau_4 = (2m+1)^2. \tag{3.22}$$

For this family we calculate

$$\begin{gathered}
\mu_0=(1+2h)\Delta_4^2,\ \ \eta=4(h-m)^2(2m+1)^2,\\
E_3=(1+2h)\Delta_4^3/4,\ \ G_{10}=-(1+2h)\Delta_4^3/8,\ \ \mathcal{F}_1=0;\\
\mathcal{T}_4=(1+2h)\Delta_4^2\rho_4,\ \ \mathcal{T}_3=2(1+2h)\Delta_4^2(3m-1),\\
\mathcal{T}_2=-3(1+2h)\Delta_4^3,\ \ W_4=(1+2h)^2\Delta_4^4\tau_4,\\
\widetilde{M}=-8(1+2m)^2x^2+16(h-m)(1+2m)xy-32(h-m)^2y^2,\\
\theta=32h(h-2m-2m^2)\Delta_4,\ \ \theta_2=(1+h+m)\Delta_4/2,\\
\theta_1=64(h^2-2hm+6h^2m-14hm^2+4h^2m^2+4m^3-12hm^3+4m^4),\\
\theta_3=(1+2h)(m^2+2m-1)\Delta_4/2.
\end{gathered}\tag{3.23}$$

Lemma 3.12. *For a system (3.21) with $\mu_0\neq 0$ the following statements hold: (i) the conditions $E_3G_{10}\widetilde{M}\neq 0$, $\eta\geq 0$ and $W_4\geq 0$ are fulfilled; (ii) if $\theta=0$ then the condition $\theta_1=0$ is equivalent to $\theta_2=0$ and in the last case $\theta_3\neq 0$.*

Proof. (i) The conditions $E_3G_{10}\neq 0$, $\eta\geq 0$ and $W_4\geq 0$ follow directly from (3.23). Assume $\widetilde{M}=0$. By (3.23) this implies $m=-1/2$ and then we calculate $\widetilde{M}=-8(1+2h)^2y^2\neq 0$ due to $\mu_0\neq 0$.

(ii) Assume $\theta=0$. Then we obtain either $\alpha)$ $h=0$, or $\beta)$ $h=2m(m+1)$. Then the calculations yield

$$\theta_1=256m^3(1+m),\qquad \theta_2=-m(1+m),\qquad \theta_3=-m(m^2+2m-1),\qquad \mu_0=4m^2,$$

in the case $\alpha)$ and

$$\begin{gathered}
\theta_1=256m^3(1+m)(1+2m)^2,\qquad \theta_2=-m(1+m)(1+2m),\\
\theta_3=-m(m^2+2m-1)(1+2m)^2,\qquad \mu_0=4m^2(1+2m)^2,
\end{gathered}$$

in the case $\beta)$. Clearly in both cases due to $\mu_0\neq 0$ the condition $\theta_1=0$ is equivalent to $\theta_2=0$ and in the case $\theta_1=0$ we get $\theta_3\neq 0$. □

The case $\mu_0<0$.

The subcase $E_3<0$. Since by Lemma 3.12 we have $G_{10}\neq 0$ and $W_4\geq 0$, according to [7] the triple singularity is a semi-elemental node and the elemental one is a node.

1) *The possibility $W_4\neq 0$.* Then we have a generic node. As by Lemma 3.12 we have $\eta\geq 0$, according to Lemma 3.1 this leads to the two configurations which were previously detected for the family (3.7).

2) *The possibility $W_4=0$.* In this case $\tau_4=0$ which implies $m=-1/2$. So we have a node with two coinciding eigenvalues which could not be a star node because $\mu_0\neq 0$ (i.e. $2h+1\neq 0$ and this appears in the matrix $\mathcal{M}_4$ from (3.22)). In this case we have $\eta=0$ and since by Lemma 3.12 the condition $\widetilde{M}\neq 0$ holds, we get the unique configuration $\overline{n}_{(3)}, n^d; \overline{\binom{0}{2}}S$, which was previously detected by the same invariant conditions.

The subcase $E_3 > 0$. By [7] we have a semi-elemental triple saddle and a saddle which could be a weak one. At infinity we have three nodes (i.e. $\eta > 0$), and then in the case $\mathcal{T}_4 \neq 0$ we obtain the same configuration previously found. Assuming $\mathcal{T}_4 = 0$ we get $m = 1/2$ and then we calculate

$$\mathcal{T}_4 = \mathcal{F}_1 = 0, \qquad \mathcal{F}\mathcal{T}_3 = (1+2h)^2/8, \qquad \mathcal{F}_2 = -3(1+2h)^2/2. \tag{3.24}$$

Therefore due to $\mu_0 \neq 0$ we have $\mathcal{F}\mathcal{T}_3 > 0$ and $\mathcal{F}_2 \neq 0$ and according to [31] we have a weak saddle of order two. Considering the condition $\eta > 0$ and Lemma 3.1 we get only one configuration and it is a new one:

• $\overline{s}_{(3)}, s^{(2)}; N^f, N^f, N^f$: *Example* $\Rightarrow$ $(h = -1,\ m = 1/2)$.

The case $\mu_0 > 0$.

The subcase $E_3 < 0$. By [7] we have a triple semi-elemental node and an elemental saddle.

1) *The possibility* $\mathcal{T}_4 \neq 0$. In this case the saddle is strong. It was shown early (see page 28) that in the case $\mu_0 > 0$, $E_3 < 0$, $\mathcal{T}_4 \neq 0$ and $\eta \geq 0$, for the family (3.7) all the possible configurations for the infinite singularities are realizable except the cases $\theta = \theta_2 = 0$, $\theta_1 \neq 0$ and $\theta = \theta_1 = \theta_3 = 0$. However by Lemma 3.12 none of cases can be realizable for the family (3.21).

2) *The possibility* $\mathcal{T}_4 = 0$. We obtain $m = 1/2$ and we arrive at the relations (3.24). According to [31, Main Theorem, (b_2)] we have a weak saddle of order two. In this case we have

$$\theta = 16h(3-2h), \;\; \theta_1 = 16(3-24h+20h^2), \;\; \theta_2 = -(3+2h)/4,$$
$$E_3 = -(1+2h)/4, \;\; \eta = 4(2h-1)^2.$$

We observe that the condition $\theta \leq 0$ implies $\theta_1 > 0$ and in the case $\theta = 0$ we have $\theta_2 \neq 0$. So considering Lemmas 3.12 and 3.1 we get the following new configurations of singularities in the case $\eta \neq 0$ (i.e. $\eta > 0$):

• $\overline{n}_{(3)}, s^{(2)}; S, N^f, N^f$: *Example* $\Rightarrow$ $(h = 2,\ m = 1/2)$ (if $\theta < 0$);
• $\overline{n}_{(3)}, s^{(2)}; S, N^f, N^\infty$: *Example* $\Rightarrow$ $(h = 1,\ m = 1/2)$ (if $\theta > 0$);
• $\overline{n}_{(3)}, s^{(2)}; S, N^f, N^d$: *Example* $\Rightarrow$ $(h = 0,\ m = 1/2)$ (if $\theta = 0$).

In the case $\eta = 0$ we have $h = 1/2$ and this implies $\theta > 0$. So we get the new configuration

• $\overline{n}_{(3)}, s^{(2)}; \binom{0}{2}SN, N^f$: *Example* $\Rightarrow$ $(h = 1/2,\ m = 1/2)$.

The subcase $E_3 > 0$. By [7] the triple point is a semi-elemental saddle and the elemental singular point is an anti-saddle, which must be a node due to $\tau_4 \geq 0$.

It was shown earlier (see page 31) that in the case $\mu_0 > 0$, $E_3 > 0$, $\eta \geq 0$ and $W_4 > 0$ for the family (3.7) all the possible configurations for the infinite singularities are realizable except the cases $\theta = \theta_2 = 0$ and $\theta = \theta_1 = \theta_3 = 0$. However by Lemma 3.12 none of cases can be realizable for the family (3.21).

In the case $W_4 = 0$ we get $m = -1/2$ and then $\eta = 0$ and $\theta^2 + \theta_2^2 \neq 0$ and again no new configurations appear.

3.3.4 The family of systems (3.5)

So we consider the family of systems

$$\dot{x} = x + dy - x^2 + 2hxy + ky^2, \qquad \dot{y} = 2my(x + dy), \quad d \in \{0, 1\}, \tag{3.25}$$

which possess the triple semi-elemental singular point $M_{1,2,3}(0,0)$ and the elemental one $M_4(1,0)$. For the second singularity we have

$$\mathcal{M}_4 = \begin{pmatrix} -1 & d+2h \\ 0 & 2m \end{pmatrix}, \qquad \rho_4 = 2m - 1, \quad \Delta_4 = -2m; \quad \tau_4 = (2m+1)^2. \tag{3.26}$$

For this family we calculate

$$\begin{gathered} \mu_0 = \alpha\Delta_4^2, \;\; \eta = 4(1+2m)^2\big[(h-dm)^2 + k(2m+1)\big], \\ E_3 = \alpha\Delta_4^3/4, \;\; G_{10} = -\alpha\Delta_4^3/8, \;\; \mathcal{F}_1 = 0; \\ \mathcal{T}_4 = \alpha\Delta_4^2\rho_4, \;\; \mathcal{T}_3 = 2\alpha\Delta_4^2(3m-1), \;\; W_4 = \alpha^2\Delta_4^4\tau_4, \\ \widetilde{M} = -8(1+2m)^2x^2 + 16(1+2m)(h-dm)xy - 8(4h^2 + 3k - 8dhm + 6km + 4d^2m^2)y^2, \end{gathered} \tag{3.27}$$

where $\alpha = (d^2 + 2dh - k)$. We observe that due to $\mu_0 \neq 0$ for the above systems the conditions $E_3G_{10} \neq 0$ and $W_4 \geq 0$ are satisfied.

According to Remark 3.8 we have to consider only two cases: $\eta < 0$ and $\widetilde{M} = 0$.

The case $\eta < 0$. Then $1 + 2m \neq 0$ (i.e. $\tau_4 > 0$) and the elemental singular point is either a generic node or a saddle.

The subcase $\mu_0 < 0$. Then $E_3 < 0$ otherwise by [7] the systems possess two finite saddles (one triple and one elemental) and this implies the existence of three nodes at infinity, i.e. $\eta > 0$. So $E_3 < 0$ and according to [7] systems (3.25) possess a semi-elemental triple node and a generic node. Considering the condition $\eta < 0$ by Lemma 3.1 we arrive at the unique configuration $\overline{n}_{(3)}, n; S, ⓒ, ⓒ$, previously detected.

The subcase $\mu_0 > 0$. As we are interested in the case $\eta < 0$, according to Lemma 3.1 in this case the invariant polynomial θ is necessary to be taken into consideration. We calculate

$$\theta = -64m\big[k(m+1)^2 - 2hdm(m+1) + h^2\big],$$

and we claim that the conditions $\mu_0 \neq 0$ and $\eta < 0$ imply $\theta \neq 0$. Indeed, if $m + 1 \neq 0$ then setting $\theta = 0$ we obtain $k = \big[2hdm(m+1) - h^2\big]/(m+1)^2$. However in this case we get

$$\eta = 4m^2(1+2m)^2(d+h+dm)^2/(1+m)^2 \geq 0,$$

which contradicts to $\eta < 0$.

Assume now $m = -1$. Then the condition $\theta = 64h^2 = 0$ gives $h = 0$ and we obtain $\eta = 4(d^2 - k) = \mu_0$, and clearly the condition $\mu_0 > 0$ imply $\eta > 0$. The contradiction obtained completes the proof of our claim and in what follows we assume $\theta \neq 0$.

1) *The possibility $E_3 < 0$.* According to [7] systems (3.25) have a semi-elemental triple node and an elemental saddle.

a) *The case* $\mathcal{T}_4 \neq 0$. Then the saddle is strong and by [7] we obtain two configurations: $\overline{n}_{(3)}, s;\ N^\infty, ©, ©$ if $\theta < 0$) and $\overline{n}_{(3)}, s;\ N^f, ©, ©$ if $\theta > 0$. However both configurations were previously constructed.

b) *The case* $\mathcal{T}_4 = 0$. Then $\rho_4 = 0$ (i.e. $m = 1/2$) and we obtain:

$$\mathcal{T}_4 = \mathcal{F}_1 = 0,\ \ \mathcal{T}_3\mathcal{F} = (d^2 + 2dh - k)^2/8,\ \ \mathcal{F}_2 = -3(d^2 + 2dh - k)^2/2,$$
$$\mu_0 = d^2 + 2dh - k,\ \ \eta = 4\big[(d - 2h)^2 + 8k\big],\ \ \theta = 8(6dh - 4h^2 - 9k).$$

So the condition $\mu_0 \neq 0$ implies $\mathcal{T}_3\mathcal{F} > 0$ and $\mathcal{F}_2 \neq 0$ and by [31, Main Theorem, (b_2)] we have a weak saddle of the second order.

Since for (3.25) we have $d \in \{0, 1\}$, we observe that in both cases the functions $\eta = 0$ and $\theta = 0$ represent two parabolas having a contact point of multiplicity two. Moreover the region $\{\eta = 0\} \subset \{\theta \geq 0\}$ and the region $\{\theta = 0\} \subset \{\eta \geq 0\}$ and hence the condition $\eta < 0$ implies $\theta > 0$. So considering [7] we arrive at the following new configuration:

- $\overline{n}_{(3)}, s^{(2)};\ N^f, ©, ©$: *Example* $\Rightarrow$ $(d = 0,\ h = 0,\ k = -1, m = 1/2)$.

2) *The possibility* $E_3 > 0$. By [7] systems (3.25) have a semi-elemental triple saddle and an elemental node. We observe that due to $\eta < 0$ the condition $\tau_4 \neq 0$ (i.e. $W_4 \neq 0$) holds. Then the node is generic and we can have only the following two configurations: $\overline{s}_{(3)}, n;\ N^\infty, ©, ©$ if $\theta < 0$ and $\overline{s}_{(3)}, s;\ N^f, ©, ©$ if $\theta > 0$. However these configurations were previously detected.

The case $\widetilde{M} = 0$. Considering (3.27) the condition $\widetilde{M} = 0$ is equivalent to $m = -1/2$ and $d = -2h$ and by (3.26) the elemental singular point is a star node. Since in this case we have $\mu_0 = -k$ and $E_3 = -k/4$, according to [7] and Lemma 3.1 we get the following two new configurations:

- $\overline{n}_{(3)}, n^*;\ \overline{\binom{0}{3}}S$: *Example* $\Rightarrow$ $(d = 0,\ h = 0,\ k = 1, m = -1/2)$ (if $\mu_0 < 0$);
- $\overline{s}_{(3)}, n^*;\ \overline{\binom{0}{3}}N$: *Example* $\Rightarrow$ $(d = 0,\ h = 0,\ k = -1, m = -1/2)$ (if $\mu_0 > 0$).

In both cases we have $U_3 = 0$, whereas for the family (3.7) with $\widetilde{M} = 0$ (which is equivalent to $m = -c/2$ and $n = -c^2/(1 + c)$) we have Coefficient$[U_3, y^5] = 3c^7/(1 + c)^3 \neq 0$. Therefore the invariant polynomial U_3 will distinguish in both cases the existence of a star node from the existence of a node n^d.

We remark that the condition $U_3 = 0$ is equivalent to $\widetilde{M} = 0$ for systems (3.25). Moreover it is not difficult to check that for the families of systems (3.7), (3.8) and (3.21) the condition $U_3 = 0$ can not be satisfied.

As all the cases are examined, we have constructed all 174 possible configurations for the family of quadratic systems with $m_f = 4$ possessing exactly two finite singularities. Therefore our main theorem is completely proved.

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References

[1] J. C. Artés, J. Llibre, D. Schlomiuk, The geometry of quadratic differential systems with a weak focus of second order, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **16**(2006), 3127–3194. MR2288574

[2] J. C. Artés, J. Llibre, D. Schlomiuk, N. Vulpe, Configurations of singularities for quadratic differential systems with total finite multiplicity lower than 2, *Bul. Acad. Stiinte Repub. Mold. Mat.* **71**(2013), 72–124. MR3155835

[3] J. C. Artés, J. Llibre, D. Schlomiuk, N. Vulpe Geometric configurations of singularities for quadratic differential systems with total finite multiplicity $m_f = 2$, *Electron. J. Differential Equations* **2014**, No. 159, 1–79. MR3239402

[4] J. C. Artés, J. Llibre, D. Schlomiuk, N. Vulpe Global configurations of singularities for quadratic differential systems with total finite multiplicity three and at most two real singularities, *Qual. Theory Dyn. Syst.*, **13**(2014) 305-351. MR3265338

[5] J. C. Artés, J. Llibre, D. Schlomiuk, N. Vulpe, Geometric configurations of singularities for quadratic differential systems with three distinct real simple finite singularities, *J. Fixed Point Theory Appl.* **14**(2014), 555–618. MR3248570

[6] J. C. Artés, J. Llibre, D. Schlomiuk, N. Vulpe, From topological to geometric equivalence in the classification of singularities at infinity for quadratic vector fields, *Rocky Mountain J. of Math.*, accepted.

[7] J. C. Artés, J. Llibre, N. Vulpe, Singular points of quadratic systems: A complete classification in the coefficient space $\mathbb{R}^{12}$, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **18**(2008), 313–362. MR2404306

[8] J. C. Artés, J. Llibre, N. Vulpe, Complete geometric invariant study of two classes of quadratic systems, *Electron. J. Differential Equations* **2012**(2012) No. 9, 1–35. MR2889615

[9] V. Baltag, Algebraic equations with invariant coefficients in qualitative study of the polynomial homogeneous differential systems, *Bul. Acad. Stiinte Repub. Mold. Mat.* **2**(2003), 31–46. MR1995424

[10] V. Baltag, N. Vulpe, Affine-invariant conditions for determining the number and multiplicity of singular points of quadratic differential systems, *Izv. Akad. Nauk Respub. Moldova Mat.* **1**(1993), 39–48. MR1227460

[11] V. Baltag, N. Vulpe, Total multiplicity of all finite critical points of the polynomial differential system, Planar nonlinear dynamical systems (Delft, 1995), *Differ. Equ. Dyn. Syst.* **5**(1997), 455–471. MR1660238

[12] N. N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, *Mat. Sb.* (N.S) **30(72)**(1952), 181–196 (in Russian); *American Math. Soc. Translation* **1954**, No. 100, 397–413. MR0059426

[13] S. Cai, The weak saddle and separatrix cycle of a quadratic system (in Chinese), *Acta Math. Sinica*, **30**(1987), No. 4, 553–559. MR0917117

[14] Iu. Calin, On rational bases of $GL(2, \mathbb{R})$-comitants of planar polynomial systems of differential equations, *Bul. Acad. Stiinte Repub. Mold. Mat.* **2**(2003), 69–86. MR1995429

[15] B. Coll, *Qualitative study of some classes of vector fields in the plane*, Ph.D. thesis, Universitat Autónoma de Barcelona, 1987.

[16] F. Dumortier, J. Llibre, J. C. Artés, *Qualitative theory of planar differential systems*, Universitext, Springer-Verlag, New York–Berlin, 2006. MR2256001

[17] E. A. González Velasco, Generic properties of polynomial vector fields at infinity, *Trans. Amer. Math. Soc.* **143**(1969), 201–222. MR0252788

[18] J. H. Grace, A. Young, *The algebra of invariants*, Stechert, New York, 1941. MR2850282

[19] Q. Jiang, J. Llibre, Qualitative classification of singular points, *Qual. Theory Dyn. Syst.* **6**(2005), 87–167. MR2273491

[20] J. Llibre, D. Schlomiuk, Geometry of quadratic differential systems with a weak focus of third order, *Canad. J. Math.* **56**(2004), 310–343. MR2040918

[21] P. J. Olver, *Classical invariant theory*, London Math. Soc. Student Texts, Vol. 44, Cambridge University Press, 1999. MR1694364

[22] I. Nikolaev, N. Vulpe, Topological classification of quadratic systems at infinity, *J. London Math. Soc.* **2**(1997), 473–488. MR1452259

[23] J. Pal, D. Schlomiuk, Summing up the dynamics of quadratic Hamiltonian systems with a center, *Canad. J. Math.* **56**(1997), 583–599. MR1451262

[24] M. N. Popa, *Applications of algebraic methods to differential systems* (in Romanian), Piteşti University, The Flower Power Edit., 2004. MR2111006

[25] D. Schlomiuk, Algebraic particular integrals, integrability and the problem of the center, *Trans. Amer. Math. Soc.* **338**(1993), 799–841. MR1106193

[26] D. Schlomiuk, Algebraic and geometric aspects of the theory of polynomial vector fields, in: D. Schlomiuk (editor), *Bifurcations and periodic orbits of vector fields*, NATO ASI Series, Series C: Vol. 408, Kluwer Academic Publishers, (1993), 429–467. MR1258526S

[27] D. Schlomiuk, J. Pal, On the geometry in the neighborhood of infinity of quadratic differential phase portraits with a weak focus, *Qual. Theory Dyn. Syst.* **2**(2001), 1–43. MR1844975

[28] D. Schlomiuk, N. I. Vulpe, Geometry of quadratic differential systems in the neighborhood of infinity, *J. Differential Equations* **215**(2005), 357–400. MR2147466

[29] D. Schlomiuk, N. I. Vulpe, The full study of planar quadratic differential systems possessing a line of singularities at infinity, *J. Dynam. Differential Equations* **20**(2008), 737–775. MR2448210

[30] K. S. Sibirskii, *Introduction to the algebraic theory of invariants of differential equations*, translated from the Russian, Nonlinear Science: Theory and Applications, Manchester University Press, Manchester, 1988. MR0981596

[31] N. Vulpe, Characterization of the finite weak singularities of quadratic systems via invariant theory, *Nonlinear Anal.* **74**(2011), 6553–6582. MR2834062

[32] N. I. Vulpe, Affine-invariant conditions for the topological distinction of quadratic systems with a center, *Differentsial'nye Uravneniya* **19**(1983), 273–280. MR0696089

[33] N. I. Vulpe, *Polynomial bases of comitants of differential systems and their applications in qualitative theory (in Russian)*, "Ştiinţa", Kishinev, 1986. MR0854675

[34] N. Vulpe, M. Lupan, Quadratic systems with dicritical points, *Bul. Acad. Stiinte Repub. Mold. Mat.* **3**(1994), 52–60. MR1375560