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Modified Riccati technique for half-linear differential equations with delay

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Abstract. We study the half-linear differential equation

$$(r(t)\Phi(x'(t)))' + c(t)\Phi(x(\tau(t))) = 0, \quad \Phi(x) := |x|^{p-2}x, \ p > 1.$$

We formulate new oscillation criteria for this equation by comparing it with a certain ordinary linear or half-linear differential equation. Our proofs are based on a suitable estimate for the solution of the equation studied and on the modified Riccati technique, which, in ordinary case, appeared to be an effective replacement of the well known linear transformation formula.

Keywords: half-linear differential equation, delay equation, oscillation criteria, modified Riccati technique.

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1 Introduction

In this paper we study the half-linear differential equation with delay in the form

$$(r(t)\Phi(x'(t)))' + c(t)\Phi(x(\tau(t))) = 0, \quad \Phi(x) := |x|^{p-2}x, \ p > 1, \tag{1.1}$$

where r, c, τ are continuous functions such that $r(t) > 0$, $c(t) \geq 0$ for large t, $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$. Through the paper we denote by q the conjugate number to p, i.e., $q = \frac{p}{p-1}$.

Note that the name "half-linear equation" arises from the fact that a constant multiple of every solution of (1.1) is also a solution of this equation, but the sum of two solutions in general fails to be a solution of (1.1). Note also, that (1.1) is a natural generalization of the half-linear ordinary differential equation

$$(r(t)\Phi(x'(t)))' + c(t)\Phi(x(t)) = 0. \tag{1.2}$$

Equation (1.1) and its generalizations (including neutral equations, dynamic equations on timescale, higher order equations) attracted broad attention in the last years (see for example

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[8, 11, 13, 15, 16]) and one of the crucial problems is to find conditions which ensure that all nonsingular solutions of this equation have infinitely many zeros. This is motivation for the following definitions.

Definition 1.1. Under the *solution* of (1.1) we understand every differentiable function $x(t)$ which does not identically equal zero eventually, such that $r(t)\Phi(x'(t))$ is differentiable and (1.1) holds for large t.

Definition 1.2. The solution of equation (1.1) is said to be *oscillatory* if it has infinitely many zeros tending to infinity. Equation (1.1) is said to be *oscillatory* if all its solutions are oscillatory. In the opposite case, i.e., if there exists an eventually positive solution of (1.1), equation (1.1) is said to be *nonoscillatory*. A solution of (1.1) is called *weakly oscillatory* if it is either oscillatory or the derivative of this solution is oscillatory.

It is well known that the behavior of delay equations is very different from the behavior of ordinary differential equations. Among others, the Sturm theorem on interlacing property of zeros fails and oscillatory solutions may coexist with nonoscillatory solutions. Despite this fact, many results and methods (including the methods which allow to detect oscillation of all solutions) can be extended from the theory of ordinary differential equations also to the theory of delay differential equations.

One of the techniques in oscillation theory of (1.1) which produces reasonably sharp results is the transformation to the first order Riccati type equation. This method is in the qualitative theory of the linear ordinary differential equation

$$(r(t)x')' + c(t)x = 0 \tag{1.3}$$

(which is a special case of (1.1) for $p = 2$) often combined with suitable transformation technique – more precisely, the transformation

$$x = f(t)y$$

which transforms (1.3) into

$$\left(rf^2y'\right)' + f\left[(rf')' + cf\right]y = 0.$$

It is well known that the method of Riccati equation has a direct extension also to the half-linear case, but the same is not true for the transformation method. Fortunately, in the last years a new method appeared, which seems to be convenient half-linear substitution for the linear transformation theory – method of modified Riccati equation, introduced and elaborated in [2, 3, 4]. For examples of applications of modified Riccati technique which allowed to obtain half-linear versions of the results proved originally for the linear equation using transformation technique see [6] and [7].

One of the possible extensions of the modified Riccati equation method is the application of this method to the (undelayed) equation which arises from (1.1) using results of the paper [11]. More precisely, under certain additional assumptions we can detect oscillation of (1.1) by oscillation of a certain ordinary differential equation, for which the method of modified Riccati equation is well elaborated. In this paper we provide an alternative extension of the modified Riccati equation method for (1.1), which does not need the intermediate step based on conversion of equation (1.1) into an equation of the type (1.2) and thus we can drop the assumptions of paper [11].

The paper is organized as follows. In the next section we introduce the modified Riccati equation for equation (1.1) and in the third section we use these results to obtain explicit comparison theorems which compare (1.1) with a certain (linear or half-linear) *ordinary differential equation.*

2 Riccati technique and modified Riccati technique

The main idea of the direct modified Riccati technique for half-linear delay differential equation lies in the following steps:

(i) transformation of the positive solution of the second order differential equation (1.1) into a solution of a certain first order equation (equation (2.2) below),

(ii) transformation of the first order equation from the previous step into a certain first order inequality (inequality (2.6) below),

(iii) employing quadratic (or in general a power-like) estimate for the nonlinear term in the inequality obtained in the previous step (see, e.g., Lemma 2.3 below),

(iv) transformation of the inequality obtained in previous steps into second order equation (in order to compare (1.1) with a similar object).

First of all we utilize the well-known method of transformation into Riccati type equation, see, e.g., [8]. Suppose that (1.1) is nonoscillatory and let x be a solution of (1.1) such that $x(t) \neq 0$, $x'(t) \neq 0$. Then, by a direct computation, one can verify that the function

$$w(t) = r(t)\frac{\Phi(x'(t))}{\Phi(x(\tau(t)))} \tag{2.1}$$

satisfies the Riccati type equation

$$\mathcal{R}[w](t) := w'(t) + c(t) + (p-1)r^{1-q}(t)\tau'(t)\frac{x'(\tau(t))}{x'(t)}|w(t)|^q = 0. \tag{2.2}$$

In the ordinary case $\tau(t) = t$ the solvability of (2.2) in the neighborhood of infinity is sufficient for nonoscillation of (1.2), see the following lemma.

Lemma 2.1 ([1, Theorem 2.2.1]). *The following statements are equivalent:*

(i) Equation (1.2) is nonoscillatory.

(ii) Equation

$$w'(t) + c(t) + (p-1)r^{1-q}(t)|w(t)|^q = 0$$

has a solution defined in a neighborhood of infinity.

(iii) Inequality

$$w'(t) + c(t) + (p-1)r^{1-q}(t)|w(t)|^q \leq 0$$

has a solution defined in a neighborhood of infinity.

In the following lemma we derive a modified Riccati type inequality. Note that to obtain this inequality we need to eliminate the dependence on the quotient $x'(\tau(t))/x'(t)$ which appears in (2.2). More precisely, we suppose that we have an a priori estimate for this term. Note that under suitable conditions such an estimate exists, as shown in Lemma 2.6.

Lemma 2.2. *Let x be a solution of (1.1) such that $x(t) \neq 0$, $x'(t) \neq 0$ and let f be a positive function satisfying*

$$\frac{x'(\tau(t))}{x'(t)} \geq f(t). \tag{2.3}$$

Let w be the solution of (2.2) given by (2.1) and h be a positive differentiable function. Define

$$G(t) = r(t)h(\tau(t))\Phi\left(\frac{h'(\tau(t))}{f(t)}\right) \tag{2.4}$$

and put

$$v(t) = h^p(\tau(t))w(t) - G(t). \tag{2.5}$$

Then

$$v'(t) + C(t) + (p-1)r^{1-q}(t)\tau'(t)f(t)h^{-q}(\tau(t))H(v(t), G(t)) \leq h^p(\tau(t))\mathcal{R}[w(t)] = 0, \tag{2.6}$$

where

$$C(t) = h(\tau(t))\left[\left(r(t)\Phi\left(\frac{h'(\tau(t))}{f(t)}\right)\right)' + c(t)\Phi(h(\tau(t)))\right] \tag{2.7}$$

and

$$H(v, G) = |v+G|^q - q\Phi^{-1}(G)v - |G|^q. \tag{2.8}$$

Proof. Differentiating the function $v(t)$ and using $w(t) = h^{-p}(\tau(t))(v(t) + G(t))$ we get

$$\begin{aligned} v'(t) &= ph^{p-1}(\tau(t))h'(\tau(t))\tau'(t)w(t) + h^p(\tau(t))w'(t) - G'(t) \\ &= ph'(\tau(t))\tau'(t)h^{-1}(\tau(t))(v(t) + G(t)) + h^p(\tau(t))w'(t) - G'(t). \end{aligned}$$

Then

$$\begin{aligned} h^p(\tau(t))\mathcal{R}[w(t)] &= h^p(\tau(t))\left[w'(t) + c(t) + (p-1)r^{1-q}(t)\tau'(t)\frac{x'(\tau(t))}{x'(t)}|w(t)|^q\right] \\ &\geq v'(t) - ph'(\tau(t))\tau'(t)h^{-1}(\tau(t))(v(t) + G(t)) + G'(t) + h^p(\tau(t))c(t) \\ &\quad + (p-1)r^{1-q}(t)\tau'(t)f(t)h^{-q}(\tau(t))|(v(t) + G(t))|^q \\ &= v'(t) + \tilde{C}(t) + (p-1)r^{1-q}(t)\tau'(t)f(t)h^{-q}(\tau(t))H(v(t), G(t)), \end{aligned}$$

where

$$\begin{aligned} \tilde{C}(t) &= h^p(\tau(t))c(t) + G'(t) - ph^{-1}(\tau(t))h'(\tau(t))\tau'(t)G(t) \\ &\quad + (p-1)r^{1-q}(t)\tau'(t)f(t)h^{-q}(\tau(t))|G(t)|^q \\ &\quad - ph'(\tau(t))\tau'(t)h^{-1}(\tau(t))v + pr^{1-q}(t)\tau'(t)f(t)h^{-q}(\tau(t))\Phi^{-1}(G(t))v \\ &= h^p(\tau(t))c(t) + G'(t) - pr(t)|h'(\tau(t))|^p\tau'(t)f^{1-p}(t) \\ &\quad + (p-1)r(t)|h'(\tau(t))|^p\tau'(t)f^{1-p}(t) \\ &= h^p(\tau(t))c(t) + G'(t) - r(t)|h'(\tau(t))|^p\tau'(t)f^{1-p}(t). \end{aligned}$$

Since

$$G'(t) = \left(r(t)\Phi\left(\frac{h'(\tau(t))}{f(t)}\right)\right)' h(\tau(t)) + r(t)|h'(\tau(t))|^p\tau'(t)f^{1-p}(t),$$

we have

$$\tilde{C}(t) = h(\tau(t))\left[\left(r(t)\Phi\left(\frac{h'(\tau(t))}{f(t)}\right)\right)' + c(t)\Phi(h(\tau(t)))\right]$$

and hence $\tilde{C}(t) = C(t)$. Inequality (2.6) is proved. □

The previous lemma suggests a method how to obtain oscillation criteria for the delay half-linear differential equation (1.1) by comparing it with a certain ordinary differential equation. The crucial tools are estimate (2.3) and an appropriate estimate for a function $H(v,G)$. Having a suitable function f for which (2.3) holds and if $H(v,G)$ is estimated by a function of the form c_1v^2 or $c_2|v|^q$, then (2.6) relates the Riccati type equation (2.2) associated with (1.1) and the Riccati type equation associated to a certain ordinary linear of half-linear equation with the different power in nonlinearity than in (1.1).

Lemma 2.3 ([2, Lemma 5 and Lemma 6])**.** *The function* (2.8) *has the following properties:*

(i) $H(v,G) \geq 0$ *with the equality if and only if* $v = 0$.

(ii) If $p \leq 2$, *then* $H(v,G) \geq \frac{q}{2}|G|^{q-2}v^2$.

(iii) For every $T > 0$ *there exists a constant* $K > 0$ *such that*

$$H(v(t),G(t)) \geq K|G(t)|v^2(t)$$

for any t *and* v *satisfying* $|v(t)/G(t)| \leq T$.

(iv) If $\liminf_{t\to\infty}|G(t)| > 0$ *and* $v(t) \to 0$ *for* $t \to \infty$, *then*

$$H(v(t),G(t)) = \frac{q(q-1)}{2}|G(t)|^{q-2}v^2(t)(1+o(1)), \quad \textit{as } t \to \infty.$$

The next statement gives sufficient conditions for nonnegativity of the solutions to the modified Riccati equation. The proof of this statement (in its special form) can be found in [3, Corollary 1] and [4, Theorem 3.5].

Lemma 2.4. *Suppose that* $A(t) \geq 0$ *and either*

$$\limsup_{t\to\infty}|G(t)| < \infty \quad \textit{and} \quad \int^\infty B(t)\,\mathrm{d}t = \infty$$

or

$$\lim_{t\to\infty}|G(t)| = \infty \quad \textit{and} \quad \int^\infty B(t)|G(t)|^{q-2}\,\mathrm{d}t = \infty.$$

Then all proper solutions (i.e., solutions which exist in a neighborhood of infinity) of the equation

$$v' + A(t) + B(t)H(v,G(t)) = 0$$

are nonnegative eventually.

A consequence of Lemma 2.4 applied to inequality (2.6) reads as follows.

Lemma 2.5. *Let* h *be a positive continuously differentiable function such that* $h'(t) \neq 0$ *for large* t *and* $C(t) \geq 0$ *for large* t. *Moreover, let either*

$$\limsup_{t\to\infty}|G(t)| < \infty \tag{2.9}$$

and

$$\int^\infty \frac{f(t)\tau'(t)}{r^{q-1}(t)h^q(\tau(t))}\,\mathrm{d}t = \infty, \tag{2.10}$$

or

$$\lim_{t\to\infty} |G(t)| = \infty \tag{2.11}$$

and

$$\int^{\infty} \frac{\Phi(f(t))\tau'(t)}{r(t)h^2(\tau(t))|h'(\tau(t))|^{p-2}}\,\mathrm{d}t = \infty. \tag{2.12}$$

Then all possible proper solutions (i.e., solutions which exist in a neighborhood of infinity) of (2.6) *are nonnegative eventually.*

Proof. If v is a proper solution of (2.6), then there exists $D(t) \geq 0$ such that v is a proper solution of the equation

$$v'(t) + C(t) + D(t) + (p-1)r^{1-q}(t)\tau'(t)f(t)h^{-q}(\tau(t))H(v(t), G(t)) = 0.$$

The nonnegativity of all proper solutions of this equation follows from Lemma 2.4 with $A(t) := C(t) + D(t)$ and $B(t) := (p-1)r^{1-q}(t)\tau'(t)f(t)h^{-q}(\tau(t))$. □

The following estimate is well known, see, e.g., [8].

Lemma 2.6. *Let* $c(t) \geq 0$ *and*

$$\int^{\infty} r^{1-q}(t)\,\mathrm{d}t = \infty. \tag{2.13}$$

Then eventually positive monotone solutions of (1.1) *satisfy*

$$\frac{x'(\tau(t))}{x'(t)} \geq \Phi^{-1}\left(\frac{r(t)}{r(\tau(t))}\right) \tag{2.14}$$

eventually.

Proof. The proof essentially follows the first part of the proof of [8, Theorem 1]. From the fact that x is an eventually positive solution it follows that $r(t)\Phi(x'(t))$ is nonincreasing eventually. Since $\Phi(\cdot)$ is increasing and $\tau(t) \leq t$, we have

$$\Phi^{-1}(r(\tau(t)))x'(\tau(t)) \geq \Phi^{-1}(r(t))x'(t).$$

Now (2.14) follows from the last inequality and from the fact that both r and x' are eventually positive (the positivity of r is obvious, the eventual positivity of x' is a well-known consequence of (2.13), see the first part of [8, Theorem 1] for more details). □

3 Oscillation criteria

Combining Lemma 2.2, the quadratic estimate (ii) from Lemma 2.3 and Lemma 2.6, we get the following comparison theorem. Note that taking $f(t) = \Phi^{-1}\left(\frac{r(t)}{r(\tau(t))}\right)$, the function $G(t)$ takes the form

$$G_1(t) = r(\tau(t))h(\tau(t))\Phi(h'(\tau(t))). \tag{3.1}$$

Theorem 3.1. *Let* $p \leq 2$, $c(t) \geq 0$, $\tau'(t) \geq 0$ *and* (2.13) *holds. Let h be a positive differentiable function and define*

$$\begin{aligned} R_1(t) &= r(\tau(t))\frac{1}{\tau'(t)}h^2(\tau(t))|h'(\tau(t))|^{p-2}, \\ C_1(t) &= h(\tau(t))\left[\left(r(\tau(t))\Phi(h'(\tau(t)))\right)' + c(t)\Phi(h(\tau(t)))\right]. \end{aligned} \tag{3.2}$$

If the ordinary linear differential equation

$$(R_1(t)y')' + \frac{p}{2}C_1(t)y = 0 \tag{3.3}$$

is oscillatory, then equation (1.1) *is oscillatory.*

Proof. Suppose, by contradiction, that (3.3) is oscillatory and for some $T > 0$ there exists a positive solution $x(t)$ of (1.1) on $[T, \infty)$. By Lemma 2.6, this solution satisfies the estimate (2.14). From Lemma 2.2 using $f(t) = \Phi^{-1}\big(\frac{r(t)}{r(\tau(t))}\big)$, $w(t)$ defined by (2.1) and with G_1 given by (3.1) we see that the function $v(t) = h^p(\tau(t))w(t) - G_1(t)$ satisfies the inequality

$$v'(t) + C_1(t) + (p-1)r^{1-q}(\tau(t))\tau'(t)h^{-q}(\tau(t))H(v(t), G_1(t)) \le 0. \tag{3.4}$$

Next, using estimate (ii) from Lemma 2.3 we get

$$v'(t) + C_1(t) + \frac{p}{2}r^{1-q}(\tau(t))\tau'(t)h^{-q}(\tau(t))|G_1(t)|^{q-2}v^2(t) \le 0,$$

i.e.,

$$v'(t) + C_1(t) + \frac{p}{2}\frac{v^2(t)}{R_1(t)} \le 0,$$

which is the Riccati inequality related to an equation which arises from (3.3) by multiplying with constant factor $2/p$. This means that (3.3) is nonoscillatory, thus the theorem is proved. $\square$

In the following theorem we use the estimate (iv) of Lemma 2.3 rather than (ii). The advantage of this method is that the estimate is sharper and thus its aplication allows to deduce sharper oscillation criteria with no restriction on p, but we have to ensure that $v(t)$ tends to 0.

Theorem 3.2. *Let $\tau'(t) \ge 0$, $c(t) \ge 0$ and* (2.13) *holds. Let h be a positive differentiable function and $C_1(t)$, $R_1(t)$ be given by* (3.2)*. Suppose that*

$$C_1(t) \ge 0, \qquad \int^{\infty} R_1^{-1}(t)\,\mathrm{d}t = \infty, \qquad \liminf_{t\to\infty} G_1(t) > 0$$

and

$$\limsup_{t\to\infty} G_1(t) < \infty \quad \text{or} \quad \lim_{t\to\infty} G_1(t) = \infty$$

hold. If the ordinary linear differential equation

$$(R_1(t)y')' + \frac{q-\varepsilon}{2}C_1(t)y = 0 \tag{3.5}$$

is oscillatory for some $\varepsilon > 0$, then equation (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution $x(t)$ which satisfies $x(t) > 0$ on $[T, \infty)$ for some $T > 0$. We prove that (3.5) is nonoscillatory for every $\varepsilon > 0$.

By Lemma 2.2, taking $f(t) = \Phi^{-1}\big(\frac{r(t)}{r(\tau(t))}\big)$ (see Lemma 2.6) there is a solution $v(t)$ of the modified Riccati inequality (3.4).

All conditions of Lemma 2.5 are satified. Really, conditions (2.9), (2.11) are assumed in the theorem (with G replaced by G_1 function, which is just a special case of G) and condition (2.12) is just another form of the assumption $\int^{\infty} R_1^{-1}(t)\,\mathrm{d}t = \infty$. Further, (2.10) takes the form $\int^\infty R_1^{-1}(t)|G_1(t)|^{2-q}\,\mathrm{d}t = \infty$ and thus it is a direct consequence of (2.12) and $0 < \liminf_{t\to\infty}|G_1(t)| \le \limsup_{t\to\infty}|G_1(t)| < \infty$. Thus $v(t) \ge 0$ eventually. Since (3.4) together with the nonnegativity of C_1 and $H(v, G_1)$ implies $v'(t) \le 0$, there exists a finite nonnegative limit $\lim_{t\to\infty} v(t)$.

We show that $\lim_{t\to\infty} v(t) = 0$. To achieve this, we integrate (3.4) from T_1 to t ($T_1 \ge T$) and since $v(t) \ge 0$ we get

$$v(T_1) \ge \int_{T_1}^t C_1(s)\,\mathrm{d}s + (p-1)\int_{T_1}^t r^{1-q}(\tau(s))\tau'(s)h^{-q}(\tau(s))H(v(s), G_1(s))\,\mathrm{d}s.$$

Both the integrals in the previous inequality are nonnegative and letting $t \to \infty$, we obtain

$$\int^\infty r^{1-q}(\tau(t))\tau'(t)h^{-q}(\tau(t))H(v(t), G_1(t))\,\mathrm{d}t < \infty.$$

Conditions $\liminf_{t\to\infty} G_1(t) > 0$ and $\lim_{t\to\infty} v(t) < \infty$ imply that there exists a positive constant M and $T_2 \ge T_1$ such that $\left|\frac{v(t)}{G_1(t)}\right| < M$ for $t \ge T_2$. Hence, by Lemma 2.3 (iii), there exists $K > 0$ such that

$$K|G_1|^{q-2}v^2(t) \le H(v(t), G_1(t)) \quad \text{for } t \ge T_2.$$

Using the relation $R_1(t)|G_1(t)|^{q-2} = \frac{r^{q-1}(\tau(t))h^q(\tau(t))}{\tau'(t)}$, the last inequality gives

$$K\frac{v^2(t)}{R_1(t)} \le r^{1-q}(\tau(t))h^{-q}(\tau(t))\tau'(t)H(v(t), G_1(t)) \quad \text{for } t \ge T_2.$$

Integrating this inequality from T_3 to t, where $T_3 \ge T_2$, and letting $t \to \infty$ we get

$$K\int_{T_3}^\infty \frac{v^2(t)}{R_1(t)}\,\mathrm{d}t \le \int_{T_3}^\infty r^{1-q}(\tau(t))h^{-q}(\tau(t))\tau'(t)H(v(t), G_1(t))\,\mathrm{d}t < \infty.$$

This inequality together with the assumption $\int^\infty R_1^{-1}(t)\,\mathrm{d}t = \infty$ shows that $\lim_{t\to\infty} v(t) = 0$.

Let $\varepsilon > 0$. By the local estimate (iv) from Lemma 2.3, there exists $T_4 \ge T_3$ such that

$$\frac{(q-\varepsilon)(q-1)}{2}|G_1(t)|^{q-2}v^2(t) \le H(v(t), G_1(t))$$

for $t \ge T_4$ and hence, using the relation between $G_1(t)$ and $R_1(t)$ and the obvious fact that $(p-1)(q-1) = 1$ we have

$$\frac{q-\varepsilon}{2}\frac{v^2(t)}{R_1(t)} \le (p-1)r^{1-q}(\tau(t))\tau'(t)h^{-q}(\tau(t))H(v(t), G_1(t))$$

for $t \ge T_4$. This inequality and (3.4) imply that $v(t)$ is a solution of the inequality

$$v'(t) + C_1(t) + \frac{q-\varepsilon}{2}\frac{v^2(t)}{R_1(t)} \le 0$$

on $[T_4, \infty)$ which is the Riccati inequality associated with (3.5). Hence (3.5) is nonoscillatory. □

The following theorem is a variant of Theorem 3.2. In contrast to Theorem 3.2, we do not suppose (2.13) and thus the estimate from Lemma 2.6 is not available.

Theorem 3.3. *Let h be a positive differentiable function such that $h'(t) \neq 0$. Let $f(t) > 0$ be a positive function and C, R and G be functions defined by (2.7),*

$$R(t) = \frac{r(t)}{\tau'(t)} h^2(\tau(t))|h'(\tau(t))|^{p-2} \frac{1}{\Phi(f(t))} \tag{3.6}$$

and (2.4), respectively. Suppose $\tau'(t) \geq 0$ and

$$\liminf_{t\to\infty} G(t) > 0, \tag{3.7}$$

$$C(t) \geq 0, \tag{3.8}$$

$$\int^{\infty} R^{-1}(t)\mathrm{d}t = \infty \tag{3.9}$$

and

$$\text{either} \quad \limsup_{t\to\infty} |G(t)| < \infty \quad \text{or} \quad \lim_{t\to\infty} |G(t)| = \infty.$$

Let

$$(R(t)y')' + \frac{q-\varepsilon}{2} C(t)y = 0 \tag{3.10}$$

be oscillatory for some $\varepsilon > 0$. Then every solution of (1.1) is either weakly oscillatory or in every neighborhood of ∞ there exists t^* such that

$$\frac{x'(\tau(t^*))}{x'(t^*)f(t^*)} < 1.$$

Proof. Suppose, by contradiction, that all the assumptions of the theorem are satisfied and that there exists t_0 such that (1.1) has a solution $x(t)$ satisfying

$$x(t) > 0, \quad x'(t) \neq 0, \quad \frac{x'(\tau(t))}{x'(t)f(t)} \geq 1$$

for $t \in [t_0, \infty)$. Let $\varepsilon > 0$. To prove the theorem is sufficient to show that the function $v(t)$ defined by (2.5) is a solution of

$$v'(t) + C(t) + \frac{q-\varepsilon}{2} \frac{v^2(t)}{R(t)} \leq 0, \tag{3.11}$$

since the existence of a solution of this inequality in some neighborhood of ∞ shows that (3.10) is nonoscillatory.

In notation of Lemma 2.2, the function v satisfies (2.6). It is easy to verify that

$$R^{-1}(t) = |G(t)|^{q-2} r^{1-q}(t)\tau'(t)h^{-q}(\tau(t))f(t)$$

and since (2.12) is a rewritten form of (3.9) and (2.12) with

$$0 < \liminf_{t\to\infty} |G(t)| \leq \limsup_{t\to\infty} |G(t)| < \infty$$

implies (2.10), the conditions of Lemma 2.5 are satisfied and there exists $t_1 > t_0$ such that $v(t) \geq 0$ on $[t_1, \infty)$. Since inequality (2.6) and nonnegativity of H implies $v'(t) \leq 0$, there exists a nonnegative finite limit $\lim_{t\to\infty} v(t)$.

We show that $v(t) \to 0$ as $t \to \infty$ and thus (iv) of Lemma 2.3 can be used. Integrating (2.6) on $[T, t]$ for $T \ge t_2$ we get

$$v(t) - v(T) + \int_T^t C(s)\,\mathrm{d}s + (p-1)\int_T^t r^{1-q}(s)\tau'(s)f(s)h^{-q}(\tau(s))H(v(s), G(s))\,\mathrm{d}s \le 0.$$

Since $v(t) \ge 0$, we have

$$v(T) \ge \int_T^t C(s)\,\mathrm{d}s + (p-1)\int_T^t r^{1-q}(s)\tau'(s)f(s)h^{-q}(\tau(s))H(v(s), G(s))\,\mathrm{d}s$$

and hence

$$\int_T^\infty r^{1-q}(s)\tau'(s)f(s)h^{-q}(\tau(s))H(v(s), G(s))\,\mathrm{d}s < \infty.$$

As in the proof of Theorem 3.2, there exists $t_3 > t_2$ and $K > 0$ such that

$$K|G(t)|^{q-2}v^2(t) \le H(v(t), G(t)) \quad \text{for } t \ge t_3$$

and hence

$$K\frac{v^2(t)}{R(t)} \le r^{1-q}(t)\tau'(t)f(t)h^{-q}(\tau(t))H(v(t), G(t)).$$

Integrating we get

$$K\int_{t_3}^\infty \frac{v^2(t)}{R(t)}\,\mathrm{d}t < \infty$$

and since (3.9) holds, we have $v(t) \to 0$ as $t \to \infty$.

As in the proof of Theorem 3.2, there exists t_4 such that

$$\frac{(q-\varepsilon)(q-1)}{2}|G(t)|^{q-2}v^2(t) \le H(v(t), G(t))$$

and hence

$$\frac{q-\varepsilon}{2}\frac{v^2(t)}{R(t)} \le (p-1)r^{1-q}(t)\tau'(t)h^{-q}(\tau(t))f(t)H(v(t), G(t))$$

holds for $t \ge t_4$. Using this estimate in (2.6) we see that $v(t)$ is on $[t_4, \infty)$ a solution of (3.11).

Thus, (3.10) is nonoscillatory and the theorem is proved. □

The following theorem allows to detect oscillation of the equation from oscillation of half-linear differential equation (rather than from linear equation). It is a direct delay variant of [5, Theorem 3] (note that [5, Theorem 3] is itself a generalization of a power type comparison result for half-linear equations [14, Theorem 1.1]).

Theorem 3.4. *Let h be a positive differentiable function such that $h'(t) > 0$, let $f(t) > 0$ be a positive continuous function and let C, R and G be defined by (2.7), (3.6) and (2.4), respectively. Suppose that $\tau'(t) \ge 0$, (3.8) holds and either (2.9), (2.10) or (2.11), (2.12) hold. Finally, suppose that there exists $\alpha > p$ such that the half-linear equation*

$$\Big(a(t)\Phi_\alpha(x')\Big)' + b(t)\Phi_\alpha(x) = 0, \tag{3.12}$$

where

$$\begin{aligned}
\Phi_\alpha(x) &= |x|^{\alpha-2}x,\\
a(t) &= \left(\frac{q}{\beta}\right)^{1-\alpha} r(t)\big(\tau'(t)\big)^{1-\alpha} f^{1-p}(t) h^{(q-\beta)(1-\alpha)}\big(\tau(t)\big)\big(h'(\tau(t))\big)^{p-\alpha},\\
b(t) &= \big(h(\tau(t))\big)^{q(1-\alpha)}\left[\left(\frac{q}{\beta}\tau'(t)-1\right)\frac{r(t)\big(h'(\tau(t))\big)^p}{\Phi(f(t))}+c(t)\big(h(\tau(t))\big)^p\right],\\
\beta &= \frac{\alpha}{\alpha-1},
\end{aligned}$$

is oscillatory. Then every solution of (1.1) *is either weakly oscillatory or in every neighborhood of* ∞ *there exists* t^* *such that*

$$\frac{x'(\tau(t^*))}{x'(t^*)f(t^*)} < 1.$$

Proof. Suppose, by contradiction, that there exists t_0 such that (1.1) has a solution $x(t)$ satisfying

$$x(t) > 0, \quad x'(t) \neq 0, \quad \frac{x'(\tau(t))}{x'(t)f(t)} \geq 1$$

for $t \in [t_0, \infty)$. We show that (3.12) is nonoscillatory for arbitrary $\alpha > p$.

By Lemma 2.2, the function $v(t)$ defined by (2.5) satisfies (2.6). Using the definition of $H(v, G)$ we get from (2.6) the inequality (the dependence on t is suppressed for brevity)

$$v' + C + (p-1)r^{1-q}\tau' f h^{-q}(\tau)|G|^q \left\{\left|\frac{v}{G}+1\right|^q - q\frac{v}{G} - 1\right\} \leq 0,$$

i.e.,

$$v' + C + (p-1)q r^{1-q}\tau' f h^{-q}(\tau)|G|^q \left\{\frac{1}{q}\left|\frac{v}{G}+1\right|^q - \left(\frac{v}{G}+1\right) + \frac{1}{p}\right\} \leq 0.$$

It follows from Lemma 2.5 that $v(t) > 0$ and since also $G(t) > 0$, we have by [5, Lemma 3] the inequality

$$\frac{1}{\beta}\left|\frac{v}{G}+1\right|^\beta - \left(\frac{v}{G}+1\right) + \frac{1}{\alpha} \leq \frac{\beta-1}{q-1}\left\{\frac{1}{q}\left|\frac{v}{G}+1\right|^q - \left(\frac{v}{G}+1\right) + \frac{1}{p}\right\}.$$

Hence

$$v' + C + \frac{(p-1)q(q-1)}{(\beta-1)\beta} r^{1-q}\tau' f h^{-q}(\tau)|G|^{q-\beta}\left\{|v+G|^\beta - \beta\Phi_\beta(G)v - |G|^\beta\right\} \leq 0.$$

Next for some differentiable positive function $F(t)$ which will be specified later denote $v(t) + G(t) = F(t)z(t)$. By a direct computation the previous inequality yields

$$\begin{aligned}
F'z + Fz' - G' + C &+ \frac{q(\alpha-1)}{\beta} r^{1-q}\tau' f h^{-q}(\tau)|G|^{q-\beta}F^\beta z^\beta\\
&- q(\alpha-1)r^{1-q}\tau' f h^{-q}(\tau)\Phi^{-1}(G)Fz + \frac{q}{\beta} r^{1-q}\tau' f h^{-q}(\tau)|G|^q \leq 0.
\end{aligned}$$

Using the definition of G (see (2.4)) we get

$$
\begin{aligned}
Fz' + \left(F' - q(\alpha-1)\tau' \frac{h'(\tau)}{h(\tau)} F\right) z \\
+ \frac{q(\alpha-1)}{\beta} r^{1-\beta} \tau' f^{1-(p-1)(q-\beta)} h^{-\beta}(\tau) (h'(\tau))^{(p-1)(q-\beta)} F^{\beta} z^{\beta} \\
+ \frac{q}{\beta} r\tau' f^{1-p} (h'(\tau))^p + C - \left(r\Phi\left(\frac{h'(\tau)}{f}\right)\right)' h(\tau) - r\frac{(h'(\tau))^p}{\Phi(f)} \le 0.
\end{aligned}
$$

The particular choice $F(t) = \big(h(\tau(t))\big)^{q(\alpha-1)}$ allows to eliminate the term linear in z and using definition of $C(t)$ (see (2.7)) we get

$$
\begin{aligned}
z' + \frac{q(\alpha-1)}{\beta} r^{1-\beta} \tau' f^{1-(p-1)(q-\beta)} h^{q-\beta}(\tau) \Big(h'(\tau)\Big)^{(p-1)(q-\beta)} z^{\beta} \\
+ h(\tau)^{q(1-\alpha)} \left[\frac{q}{\beta} r\tau' f^{1-p} (h'(t))^p + c(t) h^p(\tau) - r\frac{(h'(\tau))^p}{\Phi(f)}\right] \le 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
z' + (\alpha-1)\left[\left(\frac{q}{\beta}\right)^{1-\alpha} r(\tau')^{1-\alpha} f^{(1-\alpha)[1-(p-1)(q-\beta)]} h^{(q-\beta)(1-\alpha)}(\tau) (h'(\tau))^{(1-\alpha)(p-1)(q-\beta)}\right]^{1-\beta} z^{\beta} \\
+ (h(\tau))^{q(1-\alpha)} \left[\left(\frac{q}{\beta}\tau' - 1\right) \frac{r(h'(\tau))^p}{\Phi(f)} + c(t) h^p(\tau)\right] \le 0.
\end{aligned}
$$

Since $(1-\alpha)[1-(p-1)(q-\beta)] = 1-p$ and $(1-\alpha)(p-1)(q-\beta) = p-\alpha$, the above computation reveals that z solves the Riccati inequality associated to (3.12), hence (3.12) is nonoscillatory. □

Example 3.5. An important example of equation which is used to examine the sharpness of oscillation criteria is the Euler type equation which in the case of the second order delay differential equation reads as

$$
\big(\Phi(x'(t))\big)' + \frac{\beta}{t^p} \Phi\big(x(\lambda t)\big) = 0, \quad \lambda < 1. \tag{3.13}
$$

This equation is known to be oscillatory if

$$
\beta > \left(\frac{p}{p-1}\right)^p \frac{1}{\lambda^{p-1}} \tag{3.14}
$$

(see [10]). Note that in the case of ordinary differential equation (without delay) the oscillation constant (3.14) is known to be a boundary between oscillation and nonoscillation of the equation. In the case of delay equation we do not have such a sharp borderline between oscillation and nonoscillation, since oscillatory and nonoscillatory solutions may coexist. However, based on the results of [10] (compare also worse oscillation constant in [12] and the limit case $\lambda \to 1$) we suspect that the oscillation constant (3.14) is optimal and thus further refinements cannot be expected by decreasing β, but by studying perturbation of Euler type equation with critical constant. More precisely, we consider equation

$$
\big(\Phi(x'(t))\big)' + \left[\left(\frac{p-1}{p}\right)^p \frac{1}{\lambda^{p-1}} \frac{1}{t^p} + g(t)\right] \Phi\big(x(\lambda t)\big) = 0, \quad \lambda < 1. \tag{3.15}
$$

Denote $h(t) = t^{(p-1)/p}$. With this choice we have by direct computation (2.13), $G_1(t) = \left(\frac{p-1}{p}\right)^{p-1}$, $R_1(t) = \left(\frac{p-1}{p}\right)^{p-2} t$, $C_1 = \lambda^{p-1} t^{p-1} g(t)$ and by Theorem 3.2, (3.15) is oscillatory if

$$(ty')' + \left(\frac{p}{p-1}\right)^{p-2} \lambda^{p-1} \frac{q-\varepsilon}{2} t^{p-1} g(t) y = 0$$

is oscillatory for some $\varepsilon > 0$. By the classical Hille–Nehari oscillation criterion this linear equation is oscillatory if

$$\liminf_{t\to\infty} \ln t \int_t^\infty s^{p-1} g(s)\,\mathrm{d}s > \frac{1}{2q\lambda^{p-1}} \left(\frac{p-1}{p}\right)^{p-2} = \frac{1}{2\lambda^{p-1}} \left(\frac{p-1}{p}\right)^{p-1}. \tag{3.16}$$

An important particular case of the perturbed Euler equation in the theory of ODE's is the Riemann–Weber equation, see [9, 2, 6]. This equation corresponds to the case when the limes inferior in (3.16) becomes a nonzero finite number. A direct computation shows that taking $g(t) = \frac{\mu}{t^p \ln^2 t}$ in (3.15) we have

$$\liminf_{t\to\infty} \ln t \int_t^\infty s^{p-1} g(s)\,\mathrm{d}s = \mu$$

and thus the Riemann–Weber type equation

$$(\Phi(x'(t)))' + \left[\left(\frac{p-1}{p}\right)^p \frac{1}{\lambda^{p-1}} \frac{1}{t^p} + \frac{\mu}{t^p \ln^2 t}\right] \Phi(x(\lambda t)) = 0, \quad \lambda < 1. \tag{3.17}$$

is oscillatory if $\mu > \frac{1}{2\lambda^{p-1}} \left(\frac{p-1}{p}\right)^{p-1}$.

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