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# Non-oscillation of half-linear differential equations with periodic coefficients

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**Abstract.** We consider half-linear Euler type differential equations with general periodic coefficients. It is well-known that these equations are conditionally oscillatory, i.e., there exists a border value given by their coefficients which separates oscillatory equations from non-oscillatory ones. In this paper, we study oscillatory properties in the border case. More precisely, we prove that the considered equations are non-oscillatory in this case. Our results cover the situation when the periodic coefficients do not have any common period.

**Keywords:** half-linear equations, oscillation theory, conditional oscillation, Prüfer angle, Riccati equation.

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## 1 Introduction

The so-called conditional oscillation of differential equations has been present in research papers for more than hundred years. In the last decades, many researchers have paid their attention to the half-linear (both differential and difference) equations and to the corresponding dynamic equations on time scales. Therefore, the conditional oscillation has become topical once again. It is worth to mention that a lot of results are not only generalizations of theorems from the linear case, but they give new results for linear equations as well.

Let us recall the conditional oscillation for half-linear differential equations in detail. We say that the equation of the form

$$\left[R(t)\Phi\left(x'\right)\right]' + \gamma S(t)\Phi(x) = 0, \qquad \Phi(x) = |x|^{p-1} \operatorname{sgn} x,\ p > 1, \tag{1.1}$$

where $R$ and $S$ are continuous functions, $R$ is positive, and $\gamma \in \mathbb{R}$, is conditionally oscillatory if there exists a (positive) constant $\Gamma$ such that (1.1) is oscillatory for $\gamma > \Gamma$ and non-oscillatory for $\gamma < \Gamma$. The constant $\Gamma$ is called the critical constant of (1.1). Of course, the critical constant is dependent on coefficients $R$ and $S$.

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The first result concerning the conditional oscillation was obtained by A. Kneser in [20], where the famous oscillation constant $\Gamma = 1/4$ was found for the linear equation

$$x'' + \frac{\gamma}{t^2} x = 0. \tag{1.2}$$

Next, we mention [15, 29], where the critical constant

$$\Gamma = \frac{\alpha^2}{4} \left( \int_0^\alpha \frac{\mathrm{d}\tau}{r(\tau)} \right)^{-1} \left( \int_0^\alpha s(\tau)\,\mathrm{d}\tau \right)^{-1}$$

was identified for the equation

$$[r(t)x']' + \frac{\gamma s(t)}{t^2} x = 0 \tag{1.3}$$

with positive $\alpha$-periodic coefficients $r, s$. We should also mention, at least as references, papers [21, 22, 23, 30] containing more general results (see also [13, 14]). Note that the critical case $\gamma = \Gamma$ of (1.3) was solved as non-oscillatory (see [30]).

We turn our attention to the half-linear equations. For the overview of the basic theory, we refer to books [1, 10]. It comes from [11] (see also [12]) that the equation

$$[\Phi(x')]' + \frac{\gamma}{t^p} \Phi(x) = 0$$

has the critical constant

$$\Gamma = \left( \frac{p-1}{p} \right)^p.$$

Further, we mention [16, 18, 36], where this result was extended up to the case of coefficients $r$ and $s$ having mean values in the equation

$$\left[ r^{1-p}(t) \Phi(x') \right]' + \frac{\gamma s(t)}{t^p} \Phi(x) = 0. \tag{1.4}$$

For the discrete counterpart concerning the conditional oscillation of the corresponding difference equations, we refer to [4, 17, 24, 35].

Nevertheless, there remains still an open problem. It is not known whether (1.4) with positive $\alpha$-periodic coefficient $r$ and $\beta$-periodic coefficient $s$ is oscillatory or not in the critical case ($r$ and $s$ do not need to have any common period, e.g., $\alpha = 1$, $\beta = \sqrt{2}$). In this paper, we prove that (1.4) is non-oscillatory in this case. We point out that coefficient $s$ can change its sign (in contrary to the situation common in the literature) and we remark that, according to our best knowledge, the result presented in this paper is new in the half-linear case as well as in the linear one (i.e., for $p = 2$). In addition, to prove our current result, we use another method than in previous works [16, 18, 36] which give the basic motivation for this paper.

The oscillation of half-linear equations is a subject of researches in the field of difference equations and dynamic equations on time scales as well. The discrete case is studied (and literature overviews are given), e.g., in [6, 19, 25, 38] and the dynamic equations on time scales are treated, e.g., in [26, 27, 28]. We add that the discrete counterpart of our current result is not known in the linear case.

Another direction of researches, which is related to the one presented here, is based on the oscillation of Euler type equations generalizing (1.2) in a different way. We point out at least papers [2, 3, 31, 32, 37], where the equations of the following form (and generalizations of this form)

$$x'' + f(t)g(x) = 0$$

are considered and oscillation theorems are proved.

The paper is organized as follows. In the next section, we shortly mention the half-linear Riccati equation and we recall the concept of the half-linear trigonometric functions. Then, based on [9], we introduce the modified half-linear Prüfer transformation which is the main tool in our paper. Section 3 is devoted to lemmas and remarks which are necessary to prove the announced result. All results together with concluding remarks and examples are collected in Section 4.

## 2 Preliminaries

In this section, we mention the used form of studied equations together with the corresponding Riccati equation, the notion of half-linear trigonometric functions, the concept of the modified Prüfer angle, and the definition of the mean value of functions. These tools will be applied in Sections 3 and 4.

It appears that it is useful to consider (1.1) in the Euler form, i.e., with $S(t) = s(t)/t^p$ for a continuous function $s$ (see also Introduction). Analogously, it is advantageous to consider coefficient $R$ in the form $R \equiv r^{-p/q}$, where $r$ is a continuous function and $q > 1$ is the number conjugated with $p$ (see the below given identity (2.6)). Altogether, we study the equation

$$\left[r^{-\frac{p}{q}}(t)\Phi\left(x'\right)\right]' + \frac{s(t)}{t^p}\Phi(x) = 0, \qquad \Phi(x) = |x|^{p-1}\operatorname{sgn} x,\ p > 1, \tag{2.1}$$

where $r, s\colon \mathbb{R}_a \to \mathbb{R}$, $\mathbb{R}_a := [a, \infty)$, $a \geq \mathrm{e}$ (e denotes the base of the natural logarithm log). Henceforth, let function $r$ be bounded and positive and $s$ be such that $\limsup_{t\to\infty} |s(t)| < \infty$. For further use, we denote

$$r^+ := \sup\{r(t);\, t \in \mathbb{R}_a\}, \qquad s^+ := \sup\{|s(t)|;\, t \in \mathbb{R}_a\}. \tag{2.2}$$

Let us recall the concept of the Riccati equation associated to (2.1). We define the function

$$w(t) = r^{-\frac{p}{q}}(t)\Phi\left(\frac{x'(t)}{x(t)}\right),$$

where $x$ is a non-trivial solution of (2.1). Note that, whenever $x(t) \neq 0$, function $w$ is well defined. By a direct computation, we can verify that $w$ solves the so-called Riccati equation

$$w' + \frac{s(t)}{t^p} + (p-1)r(t)|w|^q = 0 \tag{2.3}$$

associated to (2.1).

Now we mention the basic theory of the half-linear trigonometric functions. For more comprehensive description, we refer, e.g., to [10, Section 1.1.2]. The half-linear sine function, denoted by $\sin_p$, is defined as the odd $2\pi_p$-periodic extension of the solution of the initial problem

$$\left[\Phi\left(x'\right)\right]' + (p-1)\Phi(x) = 0, \qquad x(0) = 0, \qquad x'(0) = 1, \tag{2.4}$$

where

$$\pi_p := \frac{2}{p} B\left(\frac{1}{p}, \frac{1}{q}\right) = \frac{2\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{q}\right)}{p\Gamma\left(\frac{1}{p}+\frac{1}{q}\right)} = \frac{2\pi}{p \sin\frac{\pi}{p}}. \tag{2.5}$$

In the definition of $\pi_p$, we use the Euler beta and gamma functions

$$B(x,y) = \int_0^1 \tau^{x-1}(1-\tau)^{y-1}\,\mathrm{d}\tau, \quad x,y>0, \qquad \Gamma(x) = \int_0^\infty \tau^{x-1}\mathrm{e}^{-\tau}\,\mathrm{d}\tau, \quad x>0,$$

and the formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin[\pi x]}, \qquad x>0,$$

together with the identity (the conjugacy of the numbers $p$ and $q$)

$$\frac{1}{p}+\frac{1}{q} = 1, \quad \text{i.e.,} \quad p+q = pq. \tag{2.6}$$

The derivative of the half-linear sine function is called the half-linear cosine function and it is denoted by $\cos_p$. Note that the half-linear sine and cosine functions satisfy the half-linear Pythagorean identity

$$|\sin_p t|^p + |\cos_p t|^p = 1, \qquad t\in\mathbb{R}. \tag{2.7}$$

Especially, the half-linear trigonometric functions are bounded. Therefore, there exists $L>0$ such that

$$|\cos_p y|^p < L, \qquad \left|\Phi(\cos_p y)\sin_p y\right| < L, \qquad |\sin_p y|^p < L, \qquad y\in\mathbb{R}. \tag{2.8}$$

In fact, (2.8) is valid for any $L>1$.

Using the notion of the half-linear trigonometric functions, we can introduce the modified half-linear Prüfer transformation

$$x(t) = \rho(t)\sin_p\varphi(t), \qquad x'(t) = \frac{r(t)\rho(t)}{t}\cos_p\varphi(t). \tag{2.9}$$

Denote $v(t) = t^{p-1}w(t)$, where $w$ is a solution of (2.3). Considering the transformation given by (2.9), we get

$$v = \Phi\left(\frac{\cos_p\varphi}{\sin_p\varphi}\right). \tag{2.10}$$

From the fact that $\sin_p$ solves the equation in (2.4), we have

$$v' = (1-p)\left[1+\left|\frac{\cos_p\varphi}{\sin_p\varphi}\right|^p\right]\varphi'. \tag{2.11}$$

On the other hand, applying the Riccati equation (2.3), we obtain

$$v' = \left[t^{p-1}w\right]' = (p-1)t^{p-2}w + t^{p-1}w' = \frac{p-1}{t}\left[v - \frac{s(t)}{p-1} - r(t)\left|\frac{\cos_p\varphi}{\sin_p\varphi}\right|^p\right]. \tag{2.12}$$

Putting (2.11) and (2.12) together and using (2.10), we have

$$(1-p)\left[1+\left|\frac{\cos_p\varphi}{\sin_p\varphi}\right|^p\right]\varphi' = \frac{p-1}{t}\left[\Phi\left(\frac{\cos_p\varphi}{\sin_p\varphi}\right) - \frac{s(t)}{p-1} - r(t)\left|\frac{\cos_p\varphi}{\sin_p\varphi}\right|^p\right]. \tag{2.13}$$

Then, by a direct calculation starting with (2.13) and taking into account (2.7), we obtain the equation for the Prüfer angle $\varphi$ associated to (2.1) as

$$\varphi' = \frac{1}{t}\left[r(t)|\cos_p \varphi|^p - \Phi(\cos_p \varphi)\sin_p \varphi + s(t)\frac{|\sin_p \varphi|^p}{p-1}\right]. \tag{2.14}$$

For details, we can also refer to [9].

Finally, we recall that the mean value $M(f)$ of a continuous function $f\colon \mathbb{R}_a \to \mathbb{R}$ is defined as

$$M(f) := \lim_{t\to\infty} \frac{1}{t}\int_b^{b+t} f(\tau)\,\mathrm{d}\tau,$$

if the limit is finite and if it exists uniformly with respect to $b \ge a$.

## 3 Auxiliary results

To prove the announced result, we will use the following lemmas. The first four of them deal with (2.14).

**Lemma 3.1.** *For a solution $\varphi$ of (2.14) on $[a,\infty)$, it holds*

$$\limsup_{t\to\infty} \left|\frac{\varphi(t)}{\log t}\right| < \infty,$$

*i.e., there exists $N > 0$ with the property that*

$$|\varphi(t)| < N \log t, \qquad t \ge a.$$

*Proof.* Considering (2.2) and (2.8), one can directly calculate

$$\begin{aligned}
\limsup_{t\to\infty}\left|\frac{\varphi(t)-\varphi(t_0)}{\log t}\right| &\le \limsup_{t\to\infty}\left[\frac{1}{\log t}\int_{t_0}^{t}|\varphi'(\tau)|\,\mathrm{d}\tau\right] \\
&\le \limsup_{t\to\infty}\left[\frac{1}{\log t}\int_{t_0}^{t}\frac{1}{\tau}\Big(r(\tau)|\cos_p\varphi(\tau)|^p\right. \\
&\qquad\left. + \left|\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)\right| + |s(\tau)|\,\frac{|\sin_p\varphi(\tau)|^p}{p-1}\Big)\,\mathrm{d}\tau\right] \\
&\le \limsup_{t\to\infty}\left[\frac{1}{\log t}\int_{t_0}^{t}\frac{1}{\tau}\left(r^+L + L + \frac{s^+L}{p-1}\right)\mathrm{d}\tau\right] = K\limsup_{t\to\infty}\frac{\log t - \log t_0}{\log t} = K,
\end{aligned}$$

where $t_0 \in \mathbb{R}_a$ is arbitrarily given and

$$K := r^+L + L + \frac{s^+L}{p-1}. \tag{3.1}$$

□

**Lemma 3.2.** *If $\varphi$ is a solution of (2.14) on $[a, \infty)$, then the function $\psi\colon \mathbb{R}_a \to \mathbb{R}$ defined by*

$$\psi(t) := \int_t^{t+\sqrt{t}} \frac{\varphi(\tau)}{\sqrt{\tau}}\,\mathrm{d}\tau, \qquad t \geq a, \tag{3.2}$$

*satisfies*

$$|\varphi(t+s) - \psi(t)| \leq \frac{C \log t}{\sqrt{t}}, \qquad t \geq a,\ s \in \left[0, \sqrt{t}\right], \tag{3.3}$$

*for some $C > 0$.*

*Proof.* At first, we consider the function

$$\tilde{\psi}(t) := \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \varphi(\tau)\,\mathrm{d}\tau, \qquad t \geq a,$$

and we estimate its difference from $\psi$. For $t \geq a$, we have

$$\begin{aligned}
|\tilde{\psi}(t) - \psi(t)| &= \left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \varphi(\tau)\,\mathrm{d}\tau - \int_t^{t+\sqrt{t}} \frac{\varphi(\tau)}{\sqrt{\tau}}\,\mathrm{d}\tau \right| \\
&\leq \int_t^{t+\sqrt{t}} |\varphi(\tau)| \left( \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{\tau}} \right) \mathrm{d}\tau \\
&\leq \frac{\sqrt{t+\sqrt{t}} - \sqrt{t}}{t} \int_t^{t+\sqrt{t}} N \log \tau \,\mathrm{d}\tau \leq \frac{\sqrt{t+\sqrt{t}} - \sqrt{t}}{\sqrt{t}} N \log\left(t + \sqrt{t}\right),
\end{aligned}$$

where $N$ is taken from the statement of Lemma 3.1. Evidently, it holds

$$\lim_{t\to\infty} \sqrt{t+\sqrt{t}} - \sqrt{t} = \frac{1}{2}, \qquad \lim_{t\to\infty} \frac{\log\left(t+\sqrt{t}\right)}{\log t} = 1.$$

Thus, there exists $\widetilde{K} > 0$ for which

$$|\psi(t) - \tilde{\psi}(t)| \leq \frac{\widetilde{K} \log t}{\sqrt{t}}, \qquad t \geq a. \tag{3.4}$$

Since $\varphi$ is continuous, we have that, for any $t \geq a$, there exists $t_0 \in \left[t, t+\sqrt{t}\right]$ such that

$\tilde{\psi}(t) = \varphi(t_0)$. Hence, we have

$$
\begin{aligned}
|\varphi(t+s) - \tilde{\psi}(t)| = |\varphi(t+s) - \varphi(t_0)| &\leq \int_t^{t+\sqrt{t}} |\varphi'(\tau)| \, \mathrm{d}\tau \\
&\leq \frac{1}{t} \left[ \int_t^{t+\sqrt{t}} r(\tau) |\cos_p \varphi(\tau)|^p + \left|\Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau)\right| \, \mathrm{d}\tau \right. \\
&\qquad \left. + \int_t^{t+\sqrt{t}} \frac{|\sin_p \varphi(\tau)|^p}{p-1} |s(\tau)| \, \mathrm{d}\tau \right] \\
&\leq \frac{1}{t} \int_t^{t+\sqrt{t}} \left( Lr^+ + L + \frac{Ls^+}{p-1} \right) \mathrm{d}\tau \leq \frac{K}{\sqrt{t}}, \quad t \geq a, \, s \in \left[0, \sqrt{t}\right],
\end{aligned}
\tag{3.5}
$$

where $K$ is given in (3.1) ($r^+, s^+$ are defined in (2.2) and $L$ is from (2.8)). Combining (3.4) and (3.5), we obtain (3.3) for $C = \widetilde{K} + K$. □

**Remark 3.3.** From the above lemmas, it follows that there exists $U > 0$ for which

$$
|\psi(t)| < U \log t, \qquad t \geq a, \tag{3.6}
$$

where $\psi$ is defined in (3.2) for a solution $\varphi$ of (2.14) on $[a, \infty)$.

**Lemma 3.4.** *Let $\varphi$ be a solution of (2.14) on $[a, \infty)$. Then, there exist $P, \varrho > 0$ such that the function $\psi \colon \mathbb{R}_a \to \mathbb{R}$ defined in (3.2) satisfies the inequalities*

$$
\psi' \leq \frac{1}{t} \left[ \frac{|\cos_p \psi|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) \, \mathrm{d}\tau - \Phi(\cos_p \psi) \sin_p \psi + \frac{|\sin_p \psi|^p}{(p-1)\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau) \, \mathrm{d}\tau + \frac{P}{t^\varrho} \right] \tag{3.7}
$$

*and*

$$
\psi' \geq \frac{1}{t} \left[ \frac{|\cos_p \psi|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) \, \mathrm{d}\tau - \Phi(\cos_p \psi) \sin_p \psi + \frac{|\sin_p \psi|^p}{(p-1)\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau) \, \mathrm{d}\tau - \frac{P}{t^\varrho} \right]. \tag{3.8}
$$

*Proof.* For arbitrarily given $t > a$, we have

$$
\begin{aligned}
\psi'(t) &= \left(1 + \frac{1}{2\sqrt{t}}\right) \frac{\varphi\left(t + \sqrt{t}\right)}{\sqrt{t + \sqrt{t}}} - \frac{\varphi(t)}{\sqrt{t}} = \frac{1}{2\sqrt{t}} \cdot \frac{\varphi\left(t + \sqrt{t}\right)}{\sqrt{t + \sqrt{t}}} + \int_t^{t+\sqrt{t}} \left[\frac{\varphi(\tau)}{\sqrt{\tau}}\right]' \mathrm{d}\tau \\
&= \frac{1}{2\sqrt{t}} \cdot \frac{\varphi\left(t + \sqrt{t}\right)}{\sqrt{t + \sqrt{t}}} + \int_t^{t+\sqrt{t}} \frac{\varphi'(\tau)}{\tau^{\frac{1}{2}}} - \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}} \, \mathrm{d}\tau \\
&= \int_t^{t+\sqrt{t}} \frac{1}{\tau^{\frac{3}{2}}} \left[ r(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + \frac{|\sin_p \varphi(\tau)|^p}{p-1} s(\tau) \right] \mathrm{d}\tau \\
&\quad + \frac{1}{2\sqrt{t}} \cdot \frac{\varphi\left(t + \sqrt{t}\right)}{\sqrt{t + \sqrt{t}}} - \int_t^{t+\sqrt{t}} \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}} \, \mathrm{d}\tau.
\end{aligned}
\tag{3.9}
$$

Since

$$\lim_{t\to\infty} \frac{\left(t+\sqrt{t}\right)^{\frac{3}{2}} - t^{\frac{3}{2}}}{t} = \frac{3}{2},$$

there exists $V > 0$ for which

$$\frac{\left(t+\sqrt{t}\right)^{\frac{3}{2}} - t^{\frac{3}{2}}}{t^{\frac{5}{2}}} < \frac{V}{t^{\frac{3}{2}}}, \qquad t \geq a. \tag{3.10}$$

Thus, it holds (see again (2.2), (2.8), (3.1))

$$\begin{aligned}
&\left| \int_t^{t+\sqrt{t}} \frac{1}{\tau^{\frac{3}{2}}} \left[ r(\tau)|\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + \frac{|\sin_p \varphi(\tau)|^p}{p-1} s(\tau) \right] \mathrm{d}\tau \right. \\
&\qquad \left. - \frac{1}{t^{\frac{3}{2}}} \int_t^{t+\sqrt{t}} \left[ r(\tau)|\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + \frac{|\sin_p \varphi(\tau)|^p}{p-1} s(\tau) \right] \mathrm{d}\tau \right| \\
&\qquad \leq \int_t^{t+\sqrt{t}} \left[ r^+ L + L + \frac{s^+ L}{p-1} \right] \left[ \frac{1}{t^{\frac{3}{2}}} - \frac{1}{\tau^{\frac{3}{2}}} \right] \mathrm{d}\tau \\
&\qquad \leq K \frac{\left(t+\sqrt{t}\right)^{\frac{3}{2}} - t^{\frac{3}{2}}}{t^{\frac{5}{2}}} \leq \frac{KV}{t^{\frac{3}{2}}}, \qquad t \geq a.
\end{aligned} \tag{3.11}$$

We have (see (3.3) in Lemma 3.2 and (3.6) in Remark 3.3)

$$\begin{aligned}
&\left| \frac{1}{2\sqrt{t}} \cdot \frac{\varphi\left(t+\sqrt{t}\right)}{\sqrt{t+\sqrt{t}}} - \frac{\psi(t)}{2t} \right| \\
&\qquad = \frac{1}{2t} \left| \frac{\varphi\left(t+\sqrt{t}\right)}{\sqrt{1+\frac{1}{\sqrt{t}}}} - \psi(t) \right| \\
&\qquad \leq \frac{1}{2t} \left[ \left| \frac{\varphi\left(t+\sqrt{t}\right) - \psi(t)}{\sqrt{1+\frac{1}{\sqrt{t}}}} \right| + \left| \psi(t) \left( 1 - \frac{1}{\sqrt{1+\frac{1}{\sqrt{t}}}} \right) \right| \right] \\
&\qquad \leq \frac{1}{2t} \left[ \left| \varphi\left(t+\sqrt{t}\right) - \psi(t) \right| + |\psi(t)| \frac{\sqrt{1+\frac{1}{\sqrt{t}}} - 1}{\sqrt{1+\frac{1}{\sqrt{t}}}} \right] \\
&\qquad \leq \frac{1}{2t} \left[ \frac{C \log t}{\sqrt{t}} + \frac{U \log t}{\sqrt{t}} \cdot \frac{1}{\sqrt{1+\frac{1}{\sqrt{t}}} \left( \sqrt{1+\frac{1}{\sqrt{t}}} + 1 \right)} \right] \\
&\qquad \leq \frac{Q_1}{t^{\frac{4}{3}}}
\end{aligned} \tag{3.12}$$

for some $Q_1 > 0$ and for all $t \geq a$. We also have (see again (3.3), (3.6) with (3.10))

$$\begin{aligned}
\left| \frac{\psi(t)}{2t} - \int_t^{t+\sqrt{t}} \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}} \, \mathrm{d}\tau \right| &= \left| \int_t^{t+\sqrt{t}} \frac{\psi(t)}{2t^{\frac{3}{2}}} - \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}} \, \mathrm{d}\tau \right| \\
&\leq \left| \int_t^{t+\sqrt{t}} \frac{\psi(t)}{2t^{\frac{3}{2}}} - \frac{\psi(t)}{2\tau^{\frac{3}{2}}} \, \mathrm{d}\tau \right| + \left| \int_t^{t+\sqrt{t}} \frac{\psi(t)}{2\tau^{\frac{3}{2}}} - \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}} \, \mathrm{d}\tau \right| \\
&\leq \frac{U \log t}{2} \int_t^{t+\sqrt{t}} \left( \frac{1}{t^{\frac{3}{2}}} - \frac{1}{\tau^{\frac{3}{2}}} \right) \mathrm{d}\tau + \int_t^{t+\sqrt{t}} \frac{C \log t}{\sqrt{t}\, 2\tau^{\frac{3}{2}}} \, \mathrm{d}\tau \\
&\leq \frac{U \log t}{2} \cdot \frac{\left(t+\sqrt{t}\right)^{\frac{3}{2}} - t^{\frac{3}{2}}}{t^{\frac{5}{2}}} + \frac{C \log t}{2t^{\frac{3}{2}}} \leq \frac{(VU + C)\log t}{2t^{\frac{3}{2}}} \leq \frac{Q_2}{t^{\frac{4}{3}}}
\end{aligned} \tag{3.13}$$

for a number $Q_2 > 0$ and for all $t \geq a$. Considering (3.12) and (3.13), we get

$$\left| \frac{1}{2\sqrt{t}} \cdot \frac{\varphi\left(t+\sqrt{t}\right)}{\sqrt{t+\sqrt{t}}} - \int_t^{t+\sqrt{t}} \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}} \, \mathrm{d}\tau \right| \leq \frac{Q_1 + Q_2}{t^{\frac{4}{3}}}, \qquad t \geq a. \tag{3.14}$$

It means (see also (3.9) and (3.11)) that it suffices to consider the expression

$$\frac{1}{t} \left[ \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + \frac{|\sin_p \varphi(\tau)|^p}{p-1} s(\tau) \, \mathrm{d}\tau \right]$$

and that, to prove the statement of the lemma, it suffices to obtain the following inequalities

$$\left| \frac{|\cos_p \psi(t)|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) \, \mathrm{d}\tau - \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) |\cos_p \varphi(\tau)|^p \, \mathrm{d}\tau \right| \leq \frac{A_1 \log t}{\sqrt{t}}, \tag{3.15}$$

$$\left| \Phi(\cos_p \psi(t)) \sin_p \psi(t) - \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) \, \mathrm{d}\tau \right| \leq \frac{A_2}{t^{\varrho}}, \tag{3.16}$$

$$\left| \frac{|\sin_p \psi(t)|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau) \, \mathrm{d}\tau - \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau) |\sin_p \varphi(\tau)|^p \, \mathrm{d}\tau \right| \leq \frac{A_3 \log t}{\sqrt{t}} \tag{3.17}$$

for some constants $A_1, A_2, A_3 > 0$, for a number $\varrho > 0$, and for all $t \geq a$.

Since the half-linear trigonometric functions are continuously differentiable and periodic, there exists $B > 0$ with the property that

$$\left| |\cos_p y|^p - |\cos_p z|^p \right| \leq B|y - z|, \qquad y, z \in \mathbb{R}, \tag{3.18}$$

$$\left| \cos_p y - \cos_p z \right| \leq B|y - z|, \qquad y, z \in \mathbb{R}, \tag{3.19}$$

$$\left| |\sin_p y|^p - |\sin_p z|^p \right| \leq B|y - z|, \qquad y, z \in \mathbb{R}, \tag{3.20}$$

$$\left| \sin_p y - \sin_p z \right| \leq B|y - z|, \qquad y, z \in \mathbb{R}. \tag{3.21}$$

If $p \geq 2$, then function $\Phi$ has the Lipschitz property, i.e., there exists $\widetilde{B} \geq 2$ for which

$$|\Phi(y) - \Phi(z)| \leq \widetilde{B}|y - z|, \qquad y, z \in (-L, L). \tag{3.22}$$

If $p \in (1, 2)$, then

$$\left|y^{p-1} - z^{p-1}\right| \leq |y - z|^{p-1}, \qquad y, z \in [0, L), \tag{3.23}$$

and

$$|y|^{p-1} + |z|^{p-1} \leq 2|y + z|^{p-1}, \qquad y, z \in [0, L). \tag{3.24}$$

Considering (3.23) and (3.24), for $p \in (1, 2)$, we have

$$|\Phi(y) - \Phi(z)| \leq 2|y - z|^{p-1}, \qquad y, z \in (-L, L). \tag{3.25}$$

Thus, for all $p > 1$, (3.22) and (3.25) give

$$|\Phi(y) - \Phi(z)| \leq \widetilde{B}2L|y - z|^{\rho}, \qquad y, z \in (-L, L), \tag{3.26}$$

where $\rho := \min\{1, p - 1\}$ and where we use

$$|y - z| \leq 2L|y - z|^{\rho}, \qquad y, z \in (-L, L). \tag{3.27}$$

Altogether, it holds (see (2.8), (3.19), (3.21), (3.26), and (3.27))

$$\begin{aligned} &\left|\Phi(\cos_p y)\sin_p y - \Phi(\cos_p z)\sin_p z\,\right| \\ &\quad \leq \left|\Phi(\cos_p y)\sin_p y - \Phi(\cos_p z)\sin_p y\right| + \left|\Phi(\cos_p z)\sin_p y - \Phi(\cos_p z)\sin_p z\right| \\ &\quad \leq L\left|\Phi(\cos_p y) - \Phi(\cos_p z)\right| + L^{p-1}\left|\sin_p y - \sin_p z\right| \\ &\quad \leq 2L^2\widetilde{B}B^{\rho}\,|y - z|^{\rho} + 2L^p B|y - z|^{\rho} \end{aligned} \tag{3.28}$$

for all $y, z \in \mathbb{R}$ and $p > 1$. Of course, (3.28) guarantees the existence of $\overline{B} > 0$ such that

$$\left|\Phi(\cos_p y)\sin_p y - \Phi(\cos_p z)\sin_p z\right| \leq \overline{B}\,|y - z|^{\rho}, \qquad y, z \in \mathbb{R}. \tag{3.29}$$

Inequality (3.15) follows directly from (see (2.2), (3.3), and (3.18))

$$\begin{aligned} &\left|\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} r(\tau)\left(|\cos_p \psi(t)|^p - |\cos_p \varphi(\tau)|^p\right) \mathrm{d}\tau\right| \\ &\qquad \leq \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} r(\tau)B|\psi(t) - \varphi(\tau)|\,\mathrm{d}\tau \leq \frac{r^+BC\log t}{\sqrt{t}}, \qquad t \geq a. \end{aligned} \tag{3.30}$$

Applying (3.3) and (3.29), we have

$$\begin{aligned} &\left|\Phi(\cos_p \psi(t))\sin_p \psi(t) - \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} \Phi(\cos_p \varphi(\tau))\sin_p \varphi(\tau)\,\mathrm{d}\tau\right| \\ &\quad \leq \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} \left|\Phi(\cos_p \psi(t))\sin_p \psi(t) - \Phi(\cos_p \varphi(\tau))\sin_p \varphi(\tau)\right|\mathrm{d}\tau \\ &\quad \leq \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} \overline{B}\,|\psi(t) - \varphi(\tau)|^{\rho}\,\mathrm{d}\tau \leq \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} \frac{\overline{B}C^{\rho}\log^{\rho} t}{t^{\frac{\rho}{2}}}\,\mathrm{d}\tau = \frac{\overline{B}C^{\rho}\log^{\rho} t}{t^{\frac{\rho}{2}}}, \qquad t \geq a, \end{aligned}$$

i.e., (3.16) is true for some $A_2 > 0$ and $\varrho \in (0, \rho/2)$. Analogously as in (3.30) (consider (2.2), (3.3), and (3.20)), one can obtain (3.17) using

$$\begin{aligned}&\left| \frac{|\sin_p \psi(t)|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau)\,\mathrm{d}\tau - \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau)|\sin_p \varphi(\tau)|^p\,\mathrm{d}\tau \right| \\ &\qquad \le \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} |s(\tau)| B |\psi(t) - \varphi(\tau)|\,\mathrm{d}\tau \le \frac{s^+ BC \log t}{\sqrt{t}}, \qquad t \ge a.\end{aligned}$$

From the above calculations, we get (3.7) and (3.8) for a number $P > 0$ and any $\varrho$ such that $\varrho \in (0, \rho/2) = (0, \min\{p-1, 1\}/2)$ and $\varrho < 1/3$ (see (3.14)). □

**Lemma 3.5.** *Let function $r$ be $\alpha$-periodic and $s$ be $\beta$-periodic for arbitrary $\alpha, \beta > 0$. Let $\varphi$ be a solution of (2.14) on $[a, \infty)$. Then, there exist $\widetilde{P} > 0$ and $\tilde{\varrho} > 0$ such that the function $\psi\colon \mathbb{R}_a \to \mathbb{R}$ defined by (3.2) satisfies the inequality*

$$\psi' \le \frac{1}{t} \left[ |\cos_p \psi|^p M(r) - \Phi(\cos_p \psi) \sin_p \psi + M(s) \frac{|\sin_p \psi|^p}{p-1} + \frac{\widetilde{P}}{t^{\tilde{\varrho}}} \right]. \tag{3.31}$$

*Proof.* From Lemma 3.4 (see (3.7)), we know that $\psi$ satisfies the inequality

$$\psi' \le \frac{1}{t} \left[ \frac{|\cos_p \psi|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau)\,\mathrm{d}\tau - \Phi(\cos_p \psi) \sin_p \psi + \frac{|\sin_p \psi|^p}{(p-1)\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau)\,\mathrm{d}\tau + \frac{P}{t^{\varrho}} \right] \tag{3.32}$$

for some $P > 0$ and $\varrho \in (0, 1/3)$. Let $t \ge a$ be arbitrarily given. Let $n \in \mathbb{N} \cup \{0\}$ be such that $n\alpha \le \sqrt{t} < (n+1)\alpha$. Using the periodicity of function $r$ and (2.2), we obtain

$$\begin{aligned}\left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau)\,\mathrm{d}\tau - M(r) \right| &\le \left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau)\,\mathrm{d}\tau - \frac{1}{\sqrt{t}} \int_t^{t+n\alpha} r(\tau)\,\mathrm{d}\tau \right| \\ &\quad + \left| \frac{1}{\sqrt{t}} \int_t^{t+n\alpha} r(\tau)\,\mathrm{d}\tau - \frac{1}{n\alpha} \int_t^{t+n\alpha} r(\tau)\,\mathrm{d}\tau \right| \\ &\le \frac{r^+\alpha}{\sqrt{t}} + \left( \frac{1}{n\alpha} - \frac{1}{\sqrt{t}} \right) n\alpha M(r) \le \frac{[r^+ + M(r)]\,\alpha}{\sqrt{t}}.\end{aligned} \tag{3.33}$$

Analogously, we can obtain

$$\left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau)\,\mathrm{d}\tau - M(s) \right| \le \frac{[s^+ + M(s)]\,\beta}{\sqrt{t}}. \tag{3.34}$$

Obviously, inequalities (3.32), (3.33), and (3.34) give the statement of the lemma. □

Next, we deal with a perturbed equation and we state the equation for its Prüfer angle. We also mention a consequence of Lemma 3.4, as the below given Lemma 3.7, which will be essential in Section 4.

**Lemma 3.6.** *There exists* $\varepsilon > 0$ *such that the equation*

$$\left[\left(1+\frac{\varepsilon}{\log^2 t}\right)^{-\frac{p}{q}} \Phi\left(x'\right)\right]' + \frac{\Phi(x)}{t^p}\left(q^{-p}+\frac{\varepsilon}{\log^2 t}\right) = 0 \tag{3.35}$$

*is non-oscillatory.*

*Proof.* The lemma follows from [7, Theorem 4.1] (see also [8]). □

Considering (2.14), the equation for the Prüfer angle $\eta$ associated to (3.35) is

$$\eta' = \frac{1}{t}\left[\left(1+\frac{\varepsilon}{\log^2 t}\right)|\cos_p \eta|^p - \Phi(\cos_p \eta)\sin_p \eta + \left(q^{-p}+\frac{\varepsilon}{\log^2 t}\right)\frac{|\sin_p \eta|^p}{p-1}\right]. \tag{3.36}$$

**Lemma 3.7.** *Let* $\eta$ *be a solution of* (3.36) *on* $[a,\infty)$. *Then, there exist* $\widehat{P} > 0$ *and* $\hat{\varrho} > 0$ *such that the function* $\zeta\colon \mathbb{R}_a \to \mathbb{R}$ *defined as*

$$\zeta(t) := \int_t^{t+\sqrt{t}} \frac{\eta(\tau)}{\sqrt{\tau}}\,\mathrm{d}\tau, \qquad t \geq a,$$

*satisfies the inequality*

$$\begin{aligned}\zeta' \geq \frac{1}{t}\Bigg[&|\cos_p \zeta|^p\left(1+\frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \Phi(\cos_p \zeta)\sin_p \zeta \\ &+ \frac{|\sin_p \zeta|^p}{p-1}\left(q^{-p}+\frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \frac{\widehat{P}}{t^{\hat{\varrho}}}\Bigg].\end{aligned} \tag{3.37}$$

*Proof.* Since (3.35) is a special case of (2.1) for

$$r(t) = 1+\frac{\varepsilon}{\log^2 t}, \qquad s(t) = q^{-p}+\frac{\varepsilon}{\log^2 t},$$

we can use the above lemmas for $\zeta$ which corresponds to $\psi$.

Especially, from Lemma 3.4 (see (3.8)), we have

$$\begin{aligned}\zeta' &\geq \frac{1}{t}\Bigg[\frac{|\cos_p \zeta|^p}{\sqrt{t}}\int_t^{t+\sqrt{t}}\left(1+\frac{\varepsilon}{\log^2 \tau}\right)\mathrm{d}\tau - \Phi(\cos_p \zeta)\sin_p \zeta \\ &\qquad + \frac{|\sin_p \zeta|^p}{(p-1)\sqrt{t}}\int_t^{t+\sqrt{t}}\left(q^{-p}+\frac{\varepsilon}{\log^2 \tau}\right)\mathrm{d}\tau - \frac{P}{t^{\varrho}}\Bigg] \\ &\geq \frac{1}{t}\Bigg[|\cos_p \zeta|^p\left(1+\frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \Phi(\cos_p \zeta)\sin_p \zeta \\ &\qquad + \frac{|\sin_p \zeta|^p}{p-1}\left(q^{-p}+\frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \frac{P}{t^{\varrho}}\Bigg].\end{aligned}$$

It means that it suffices to put $\widehat{P} = P$ and $\hat{\varrho} = \varrho$ in (3.37). □

# 4 Results

Now we can prove the announced result.

**Theorem 4.1.** *If function $r$ is $\alpha$-periodic and has mean value $M(r) = 1$ and if function $s$ is $\beta$-periodic and has mean value $M(s) = q^{-p}$, then (2.1) is non-oscillatory.*

*Proof.* Taking into account the half-linear Pythagorean identity (see (2.7)), we observe

$$\max\left\{|\sin_p y|^p, |\cos_p y|^p\right\} \geq \frac{1}{2}, \qquad y \in \mathbb{R}.$$

Hence, for $\varepsilon > 0$ from the statement of Lemma 3.6, there exists $\delta > 0$ with the property that

$$\varepsilon \left|\cos_p y\right|^p + \frac{\varepsilon \left|\sin_p y\right|^p}{p-1} > \delta, \qquad y \in \mathbb{R};$$

i.e., the inequality

$$\frac{\varepsilon \left|\cos_p y\right|^p}{\log^2\left[t+\sqrt{t}\right]} + \frac{\varepsilon \left|\sin_p y\right|^p}{(p-1)\log^2\left[t+\sqrt{t}\right]} > \frac{D}{t^{\varrho}}, \qquad y \in \mathbb{R}, \tag{4.1}$$

holds for any constant $D > 0$ and $\varrho > 0$ and for all sufficiently large $t$.

Let $\varphi$ be a solution of (2.14) which is associated to (2.1). Lemma 3.5 says that the function $\psi$ defined by (3.2) satisfies inequality (3.31). Thus, considering (4.1), where $D = \widetilde{P} + \widehat{P}$ and $\varrho = \min\{\tilde{\varrho}, \hat{\varrho}\}$, we have

$$\begin{aligned}
\psi' &\leq \frac{1}{t}\left[|\cos_p \psi|^p - \Phi(\cos_p \psi)\sin_p \psi + q^{-p}\frac{|\sin_p \psi|^p}{p-1} + \frac{\widetilde{P}}{t^{\tilde{\varrho}}}\right] \\
&< \frac{1}{t}\left[|\cos_p \psi|^p\left(1 + \frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \Phi(\cos_p \psi)\sin_p \psi\right. \\
&\qquad \left. + \frac{|\sin_p \psi|^p}{p-1}\left(q^{-p} + \frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \frac{\widehat{P}}{t^{\hat{\varrho}}}\right]
\end{aligned} \tag{4.2}$$

for sufficiently large $t$. It is well-known that the non-oscillation of (2.1) is equivalent to the boundedness from above of the Prüfer angle $\varphi$ (given by (2.14)). We can refer, e.g., to [10, Section 1.1.3], [9], [30] (or consider directly (2.9) together with (2.14) when $\sin_p \varphi = 0$). We remark that the space of all values of $\varphi$ is unbounded if and only if $\lim_{t\to\infty} \varphi(t) = \infty$. It follows from the periodicity of the half-linear sine function and the right-hand side of (2.14) for values $\varphi$ satisfying $\sin_p \varphi = 0$ (when the derivative is positive).

Considering Lemma 3.6, we know that the Prüfer angle $\eta$ given by (3.36) is bounded. Lemma 3.2 says that $\varphi$ is bounded if and only if $\psi$ is bounded. In particular, $\zeta$ is bounded, because $\eta$, $\zeta$ are special cases of $\varphi$, $\psi$. Thus, Lemma 3.7 together with (4.2) guarantees that the considered solution $\varphi$ (given by (2.14)) is bounded, i.e., (2.1) is non-oscillatory. Indeed, it suffices to consider the solutions $\mu$, $\nu$ of the equations

$$\mu' = \frac{1}{t}\left[|\cos_p \mu|^p - \Phi(\cos_p \mu)\sin_p \mu + q^{-p}\frac{|\sin_p \mu|^p}{p-1} + \frac{\widetilde{P}}{t^{\tilde{\varrho}}}\right],$$

$$\nu' = \frac{1}{t}\left[|\cos_p \nu|^p\left(1 + \frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \Phi(\cos_p \nu)\sin_p \nu + \frac{|\sin_p \nu|^p}{p-1}\left(q^{-p} + \frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \frac{\widehat{P}}{t^{\hat{\varrho}}}\right]$$

determined by the same initial condition $\mu(T) = \nu(T) = 0$, where $T$ is sufficiently large. We have $\nu(t) \geq \mu(t)$, $t \geq T$. Therefore (see again (3.3)),

$$\limsup_{t\to\infty} \zeta(t) = \limsup_{t\to\infty} \eta(t) < \infty$$

gives

$$\limsup_{t\to\infty} \varphi(t) = \limsup_{t\to\infty} \psi(t) < \infty.$$

□

We recall the following known result.

**Theorem 4.2.** *Let $r$ be an $\alpha$-periodic function having mean value $M(r) = 1$ and let $s$ be a $\beta$-periodic function. Equation (2.1) is oscillatory if $M(s) > q^{-p}$; and (2.1) is non-oscillatory if $M(s) < q^{-p}$.*

*Proof.* See [36, Theorem 4] (and also [18]). □

**Remark 4.3.** In fact, the non-oscillatory part of Theorem 4.2 is also a consequence of our Theorem 4.1 and the half-linear Sturm comparison theorem (see, e.g., [10, Theorem 1.2.4]).

Using Theorem 4.2, we can improve Theorem 4.1 in the next form common in the literature.

**Theorem 4.4.** *Let function $f$ be $\alpha$-periodic, positive, and continuous and let function $h$ be $\beta$-periodic and continuous for arbitrary $\alpha, \beta > 0$. Consider the half-linear equation*

$$\left[f(t)\Phi\left(x'\right)\right]' + \frac{h(t)}{t^p}\Phi(x) = 0. \tag{4.3}$$

*Let*

$$\gamma := q^{-p}\left[M\left(f^{1-q}\right)\right]^{1-p} = q^{-p}\left[\frac{1}{\alpha}\int_0^\alpha f^{1-q}(\tau)\,\mathrm{d}\tau\right]^{1-p}. \tag{4.4}$$

*(i) If $M(h) > \gamma$, then (4.3) is oscillatory.*

*(ii) If $M(h) \leq \gamma$, then (4.3) is non-oscillatory.*

*Proof.* We rewrite (4.3) as

$$\left[\left[f^{1-q}(t)\right]^{-\frac{p}{q}}\Phi\left(x'\right)\right]' + \frac{h(t)}{t^p}\Phi(x) = 0,$$

i.e., it takes the form of (2.1) for

$$r(t) = \frac{f^{1-q}(t)}{M\left(f^{1-q}\right)}, \qquad s(t) = \left[M\left(f^{1-q}\right)\right]^{\frac{p}{q}} h(t).$$

Theorems 4.1 and 4.2 give that (4.3) is non-oscillatory if and only if

$$M(s) = \left[M\left(f^{1-q}\right)\right]^{\frac{p}{q}} M(h) = \left[M\left(f^{1-q}\right)\right]^{p-1} M(h) \leq q^{-p}.$$

Using $\gamma$ given in (4.4), we can reformulate this observation as follows. Equation (4.3) is non-oscillatory if and only if $M(h) \leq \gamma$. □

**Remark 4.5.** Let us consider the case when $M(h) = \gamma$. Note that it is not possible to generalize the result obtained above (see Theorem 4.4 or directly Theorem 4.1) for general function $h$ having mean value. It follows, e.g., from the main result of [9]. We conjecture that such a generalization is not true even for limit periodic and almost periodic functions in the place of $h$. Our conjecture is based on the constructions mentioned in [34] (or see [33, Theorem 3.5] together with [5, Theorem 1.27]).

Immediately, Theorem 4.4 guarantees the conditional oscillation of general periodic linear equations which is explicitly embodied in the corollary mentioned below.

**Corollary 4.6.** *Let $g_1$, $g_2$ be periodic and continuous functions and let $g_1$ be positive. The equation*

$$\left[\frac{x'}{g_1(t)}\right]' + \frac{g_2(t)}{t^2} x = 0$$

*is oscillatory if and only if $M\left(g_1\right) M\left(g_2\right) > 1/4$.*

*Proof.* It suffices to put $p = 2$ in Theorem 4.4. □

**Remark 4.7.** If $M\left(g_1\right) M\left(g_2\right) \neq 1/4$ and if $g_2$ is positive, then the statement of Corollary 4.6 follows from many known results (see Introduction).

To illustrate Theorem 4.4 and Corollary 4.6, we give the following two examples which are not generally solvable using known oscillatory criteria. We recall (see also Introduction) that the most general result concerning the conditional oscillation of (2.1) comes from [36]. In that paper, the conditional oscillation of equations with coefficients having mean values is analysed. The critical constant is found, but the critical case remains unsolved. Remark 4.5 is devoted to the description of this problem.

On the other hand, the critical case is studied in papers [8, 9], where the coefficients in the considered equations have the same period. The critical case with different periods of coefficients has not been analysed in the literature.

**Example 4.8.** Let $\alpha > 1/2$, $\beta_1, \beta_2 \neq 0$, $p = 3/2$. The coefficients of the half-linear equation

$$\left[\frac{\Phi\left(x'\right)}{\alpha + \cos\left[\beta_1 t\right] \sin\left[\beta_1 t\right]}\right]' + \frac{\left(\cos\left[\beta_2 t\right] \sin\left[\beta_2 t\right]\right)^2}{t^{\frac{3}{2}}} \Phi(x) = 0 \tag{4.5}$$

satisfy the conditions of Theorem 4.4. Since

$$M\left(\left(\cos\left[\beta_2 t\right] \sin\left[\beta_2 t\right]\right)^2\right) = \frac{1}{8}$$

and (see (4.4))

$$\gamma = 3^{-\frac{3}{2}} \left[ M \left( (\alpha + \cos[\beta_1 t] \sin[\beta_1 t])^2 \right) \right]^{-\frac{1}{2}} = \frac{1}{\sqrt{27\left(\alpha^2 + \frac{1}{8}\right)}},$$

(4.5) is non-oscillatory if and only if $1 + 8\alpha^2 \leq (8/3)^3$. We remark that this equivalence is new for all $\beta_1, \beta_2 \neq 0$ satisfying $\beta_1/\beta_2 \notin \mathbb{Q}$, because, in this case, the coefficient in the differential term and the coefficient in the potential of (4.5) do not have any common period.

**Example 4.9.** Let $\sigma(1), \sigma(2) > 1$ be arbitrary. The linear equations

$$\left[ \frac{x'}{2 + \sin_{\sigma(1)} t} \right]' + \frac{1 + \sin_{\sigma(2)} t}{8t^2} x = 0, \tag{4.6}$$

$$\left[ \frac{x'}{2 + \sin_{\sigma(1)} t} \right]' + \frac{1 + \cos_{\sigma(2)} t}{8t^2} x = 0, \tag{4.7}$$

$$\left[ \frac{x'}{2 + \cos_{\sigma(1)} t} \right]' + \frac{1 + \sin_{\sigma(2)} t}{8t^2} x = 0, \tag{4.8}$$

$$\left[ \frac{x'}{2 + \cos_{\sigma(1)} t} \right]' + \frac{1 + \cos_{\sigma(2)} t}{8t^2} x = 0 \tag{4.9}$$

are in the so-called border case $M(g_1)\, M(g_2) = 1/4$ (see Corollary 4.6), because

$$M(c + d \sin_\sigma t) = M(c + d \cos_\sigma t) = c, \qquad c, d \in \mathbb{R},\ \sigma > 1.$$

Nevertheless, we actually know that these equations are non-oscillatory. This fact does not follow from any previous result for, e.g., $\sigma(1) = 2$, $\sigma(2) = 3$. Indeed, in this case, the coefficients in the differential terms of (4.6), (4.7), (4.8), and (4.9) have the period $2\pi_2 = 2\pi$ and the coefficients in the potentials have the period $2\pi_3 = 8\pi\sqrt{3}/9$ (see (2.5)). Since $\pi_3/\pi_2 = 4\sqrt{3}/9 \notin \mathbb{Q}$, the coefficients do not have any common period for $\sigma(1) = 2$, $\sigma(2) = 3$.

Applying known comparison theorems, we can obtain several new results which follow from Theorem 4.4. We mention at least one known comparison theorem and a new result as Corollary 4.11 with the below given Example 4.12.

**Theorem 4.10.** *Let $r\colon \mathbb{R}_a \to \mathbb{R}$ be a continuous positive function satisfying*

$$\int_a^\infty r^{1-q}(\tau)\, d\tau = \infty \tag{4.10}$$

*and $s_1, s_2\colon \mathbb{R}_a \to \mathbb{R}$ be continuous functions satisfying*

$$\int_t^\infty s_2(\tau)\, d\tau \geq \left| \int_t^\infty s_1(\tau)\, d\tau \right|, \qquad t \geq T, \tag{4.11}$$

*for some $T \geq a$, where the integrals $\int_T^\infty s_1(\tau)\, d\tau$, $\int_T^\infty s_2(\tau)\, d\tau$ are convergent. Consider the equations*

$$\left[ r(t)\Phi(x') \right]' + s_1(t)\Phi(x) = 0, \tag{4.12}$$

$$\left[ r(t)\Phi(x') \right]' + s_2(t)\Phi(x) = 0. \tag{4.13}$$

*If (4.13) is non-oscillatory, then (4.12) is non-oscillatory as well.*

*Proof.* See [10, Theorem 2.3.1]. □

**Corollary 4.11.** *Let function $r$ be $\alpha$-periodic, positive, and continuous and let function $s$ be $\beta$-periodic and continuous for arbitrary $\alpha, \beta > 0$. Consider the equation*

$$\left[r^{-\frac{p}{q}}(t)\Phi\left(x'\right)\right]' + z(t)\Phi(x) = 0, \tag{4.14}$$

*where $z : \mathbb{R}_a \to \mathbb{R}$ is a continuous function satisfying*

$$\left|\int_a^\infty z(\tau)\,\mathrm{d}\tau\right| < \infty. \tag{4.15}$$

*If*

$$M(s) = \frac{1}{\beta}\int_0^\beta s(\tau)\,\mathrm{d}\tau \le q^{-p}\left[M\left(r\right)\right]^{1-p} = q^{-p}\left[\frac{1}{\alpha}\int_0^\alpha r(\tau)\,\mathrm{d}\tau\right]^{1-p} \tag{4.16}$$

*and if there exists $t_0 \ge a$ for which*

$$\int_t^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \ge \left|\int_t^\infty z(\tau)\,\mathrm{d}\tau\right|, \qquad t \ge t_0, \tag{4.17}$$

*then (4.14) is non-oscillatory.*

*Proof.* The corollary follows from Theorem 4.4, (ii) and Theorem 4.10. At first, we discuss the assumptions of Theorem 4.10. Putting $s_1(t) = z(t)$, $s_2(t) = s(t)/t^p$ for $t \ge a$, we consider (4.14) as (4.12) and the equation

$$\left[r^{-\frac{p}{q}}(t)\Phi\left(x'\right)\right]' + \frac{s(t)}{t^p}\Phi(x) = 0 \tag{4.18}$$

as (4.13), i.e., we replace function $r$ by $r^{-p/q}$. Since

$$\int_a^\infty \left[r^{-\frac{p}{q}}(\tau)\right]^{1-q}\mathrm{d}\tau = \int_a^\infty r(\tau)\,\mathrm{d}\tau = \lim_{n\to\infty} n\int_a^{a+\alpha} r(\tau)\,\mathrm{d}\tau = \infty,$$

condition (4.10) from Theorem 4.10 is fulfilled. The integral $\int_a^\infty s_1(\tau)\,\mathrm{d}\tau$ is convergent due to (4.15). The periodicity together with the continuity of function $s$ implies its boundedness. Therefore (consider that $p > 1$), we have

$$\left|\int_a^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau\right| \le \int_a^\infty \frac{|s(\tau)|}{\tau^p}\,\mathrm{d}\tau < \infty.$$

Hence, the integral $\int_a^\infty s_2(\tau)\,\mathrm{d}\tau$ is convergent as well. Moreover, (4.17) gives (4.11).

To finish the proof, it suffices to show that (4.18) is non-oscillatory which implies the non-oscillation of (4.14) (consider Theorem 4.10). Putting $f(t) = r^{-p/q}(t)$ and $h(t) = s(t)$ in Theorem 4.4, we can see that (4.16) ensures the validity of the inequality in Theorem 4.4, (ii). Indeed, it holds

$$\frac{1}{\alpha}\int_0^\alpha f^{1-q}(\tau)\,\mathrm{d}\tau = \frac{1}{\alpha}\int_0^\alpha \left[r^{-\frac{p}{q}}(\tau)\right]^{1-q}\mathrm{d}\tau = \frac{1}{\alpha}\int_0^\alpha r(\tau)\,\mathrm{d}\tau.$$

Thus, (4.18) is non-oscillatory and, consequently, (4.14) is non-oscillatory as well. □

**Example 4.12.** Let $a, b \neq 0$ be arbitrarily given. We define

$$\bar{z}(t) := \left(\frac{\pi}{4q}\right)^{p} (|\sin[bt]| + |\cos[bt]| + \tilde{z}(t)), \qquad t \in \mathbb{R}_3,$$

where

$$\tilde{z}(t) := \begin{cases} (t-2^n)\,\dfrac{n-1}{n}, & t \in \left[2^n, 2^n + \dfrac{1}{4}\right),\ n \in \mathbb{N} \setminus \{1\}; \\ \left(2^n + \dfrac{1}{2} - t\right)\dfrac{n-1}{n}, & t \in \left[2^n + \dfrac{1}{4}, 2^n + \dfrac{1}{2}\right),\ n \in \mathbb{N} \setminus \{1\}; \\ -2\left(t - 2^n - \dfrac{1}{2}\right)\dfrac{n-1}{n}, & t \in \left[2^n + \dfrac{1}{2}, 2^n + \dfrac{3}{4}\right),\ n \in \mathbb{N} \setminus \{1\}; \\ -2\,(2^n + 1 - t)\,\dfrac{n-1}{n}, & t \in \left[2^n + \dfrac{3}{4}, 2^n + 1\right],\ n \in \mathbb{N} \setminus \{1\}; \\ 0, & t \in \mathbb{R}_3 \setminus \bigcup\limits_{n \in \mathbb{N}\setminus\{1\}} [2^n, 2^n + 1]. \end{cases}$$

We consider the equation

$$\left[(|\sin[at]| + |\cos[at]|)^{-\frac{p}{q}}\, \Phi(x')\right]' + \frac{\bar{z}(t)}{t^p}\, \Phi(x) = 0 \tag{4.19}$$

which is in the form of (4.14) for $z(t) = \bar{z}(t)/t^p$. It is seen that

$$0 \leq \int_t^\infty z(\tau)\,\mathrm{d}\tau = \int_t^\infty |z(\tau)|\,\mathrm{d}\tau \leq \int_t^\infty \frac{H}{\tau^p}\,\mathrm{d}\tau < \infty, \qquad t \geq 3, \tag{4.20}$$

for some $H > 0$. We put

$$s(t) := \left(\frac{\pi}{4q}\right)^{p} (|\sin[bt]| + |\cos[bt]|), \qquad t \in \mathbb{R}_3.$$

Directly from $\lim_{t\to\infty} \left(\frac{t}{t+1}\right)^p = 1$, we get

$$\int_t^\infty \frac{\tilde{z}(\tau)}{\tau^p}\,\mathrm{d}\tau < 0, \quad \text{i.e.,} \quad \int_t^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau > \int_t^\infty z(\tau)\,\mathrm{d}\tau,$$

for all sufficiently large $t$. Hence (see also (4.20)), we have (4.17). Since

$$M(s) = \left(\frac{\pi}{4q}\right)^{p} \frac{4}{\pi} = q^{-p} \left[\frac{4}{\pi}\right]^{1-p} = q^{-p} \left[M\left(|\sin[at]| + |\cos[at]|\right)\right]^{1-p},$$

inequality (4.16) is satisfied as well. Finally, applying Corollary 4.11, we obtain the non-oscillation of (4.19) which does not follow from any known theorem.

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