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Boundedness character of a max-type system of difference equations of second order

Stevo Stević[✉ 1, 2], **Mohammed A. Alghamdi**[2], **Abdullah Alotaibi**[2] and **Naseer Shahzad**[2]

[1]Mathematical Institute of the Serbian Academy of Sciences,
Knez Mihailova 36/III, 11000 Beograd, Serbia
[2]King Abdulaziz University, Department of Mathematics,
P.O. Box 80203, Jeddah 21589, Saudi Arabia

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Abstract. The boundedness character of positive solutions of the next max-type system of difference equations

$$x_{n+1} = \max\left\{A, \frac{y_n^p}{x_{n-1}^q}\right\}, \quad y_{n+1} = \max\left\{A, \frac{x_n^p}{y_{n-1}^q}\right\}, \quad n \in \mathbb{N}_0,$$

with $\min\{A, p, q\} > 0$, is characterized.

Keywords: max-type system of difference equations, positive solutions, bounded solutions, unbounded solutions.

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1 Introduction

Difference equations and systems which do not stem from the differential ones have attracted some attention in last few decades (see, e.g., [1–47]). Some of the systems that are of interest are symmetric or those obtained from symmetric by modifications of their parameters (see, for example, [5,9,13–19,22,23,36,39–44] and the related references therein). Another subarea, of interest deals with *max-type* difference equations and systems (see, for example, [1,7,10–12,17,19,21,28–35,38,40,42,43,45–47] and the related references therein). However, there are only a few papers which belong to both areas (see [17,19,21,40,42,43]). Although majority of the papers in the area treat equations or systems with integer powers of their variables, there are some papers on equations or systems with non-integer powers of their variables (see, for example, [3, 4, 8, 20, 27–33, 35, 47]). Paper [29] is one of the first such papers on max-type difference equations. It studies positive solutions of the difference equation

$$x_{n+1} = \max\left\{a, \frac{x_n^p}{x_{n-1}^p}\right\}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

✉ Corresponding author. Email: sstevic@ptt.rs

with $\min\{a, p\} > 0$.

Motivated by [29], in [43], S. Stević studied the boundedness character and global attractivity of positive solutions of the following symmetric system of max-type difference equations

$$x_{n+1} = \max\left\{a, \frac{y_n^p}{x_{n-1}^p}\right\}, \quad y_{n+1} = \max\left\{a, \frac{x_n^p}{y_{n-1}^p}\right\}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

with $\min\{a, p\} > 0$.

For related max-type difference equations see also [28,30,33,35].

Here we continue the line of investigations by studying the boundedness character of positive solutions of the next system of max-type difference equations

$$x_{n+1} = \max\left\{A, \frac{y_n^p}{x_{n-1}^q}\right\}, \quad y_{n+1} = \max\left\{A, \frac{x_n^p}{y_{n-1}^q}\right\}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $\min\{A, p, q\} > 0$.

Two of our results (Theorem 2.2 and Theorem 2.4) are natural extensions of the results on the boundedness character of positive solutions of system (1.2) appearing in [43]. For the other two results (Theorem 2.1 and Theorem 2.3) we need some other methods, different from the ones used in studying system (1.2). Generally speaking, the paper is also a continuation of studying special cases of the next systems of difference equations

$$x_{n+1} = \max\left\{A_n, \frac{y_{n-k}^p}{x_{n-l}^q}\right\}, \quad y_{n+1} = \max\left\{A_n, \frac{x_{n-k}^p}{y_{n-l}^q}\right\}, \quad n \in \mathbb{N}_0,$$

$$x_{n+1} = A_n + \frac{y_{n-k}^p}{x_{n-l}^q}, \quad y_{n+1} = A_n + \frac{x_{n-k}^p}{y_{n-l}^q}, \quad n \in \mathbb{N}_0,$$

where $k, l \in \mathbb{N}$, $\min\{p, q\} > 0$ and $(A_n)_{n\in\mathbb{N}_0}$ is a sequence of positive numbers, as well as special cases of their scalar counterparts

$$x_{n+1} = \max\left\{A_n, \frac{x_{n-k}^p}{x_{n-l}^q}\right\}$$

$$x_{n+1} = A_n + \frac{x_{n-k}^p}{x_{n-l}^q}, \quad n \in \mathbb{N}_0,$$

where $k, l \in \mathbb{N}$, $\min\{p, q\} > 0$ and $(A_n)_{n\in\mathbb{N}_0}$ is a sequence of positive numbers.

For some results in the area see, for example, [2,4,6,8,14,20,24,25,28–30,33] and the related references therein.

Solution $(x_n, y_n)_{n\geq-1}$ of system (1.3) is bounded if there is an $M \geq 0$ such that

$$\|(x_n, y_n)\|_2 = \sqrt{x_n^2 + y_n^2} \leq M, \quad n \geq -1. \tag{1.4}$$

If

$$\sup_{n\geq-1} \sqrt{x_n^2 + y_n^2} = +\infty$$

we say that the solution is unbounded.

2 Boundedness character of positive solutions of system (1.3)

In this section we prove the main results of this paper, which give a complete picture for the boundedness character of positive solutions of system (1.3).

Theorem 2.1. *Assume that $A > 0$, $2\sqrt{q} \le p < 1+q$ and $q \in (0,1)$. Then all positive solutions of system (1.3) are bounded.*

Proof. First note that from (1.3) we have

$$\min\{x_n, y_n\} \ge A, \quad n \in \mathbb{N}. \tag{2.1}$$

It is not difficult to see that the conditions $2\sqrt{q} \le p < 1+q$ and $q \in (0,1)$ imply that the polynomial $P(t) = t^2 - pt + q$ has zeroes t_1 and t_2 such that $0 < t_2 < t_1 < 1$.

We have

$$x_{n+1} = \max\left\{A, \frac{y_n^{t_1+t_2}}{x_{n-1}^{t_1 t_2}}\right\},$$
$$y_{n+1} = \max\left\{A, \frac{x_n^{t_1+t_2}}{y_{n-1}^{t_1 t_2}}\right\}, \quad n \in \mathbb{N}_0,$$

which along with (2.1) implies that

$$\frac{x_{n+1}}{y_n^{t_1}} = \max\left\{\frac{A}{y_n^{t_1}}, \frac{y_n^{t_2}}{x_{n-1}^{t_1 t_2}}\right\} \le \max\left\{A^{1-t_1}, \left(\frac{y_n}{x_{n-1}^{t_1}}\right)^{t_2}\right\} \tag{2.2}$$

$$\frac{y_{n+1}}{x_n^{t_1}} = \max\left\{\frac{A}{x_n^{t_1}}, \frac{x_n^{t_2}}{y_{n-1}^{t_1 t_2}}\right\} \le \max\left\{A^{1-t_1}, \left(\frac{x_n}{y_{n-1}^{t_1}}\right)^{t_2}\right\} \tag{2.3}$$

for every $n \in \mathbb{N}$, and consequently

$$\max\left\{\frac{x_{n+1}}{y_n^{t_1}}, \frac{y_{n+1}}{x_n^{t_1}}\right\} \le \max\left\{A^{1-t_1}, \max\left\{\frac{x_n}{y_{n-1}^{t_1}}, \frac{y_n}{x_{n-1}^{t_1}}\right\}^{t_2}\right\}, \quad n \in \mathbb{N}. \tag{2.4}$$

Let

$$u_n = \max\left\{\frac{x_n}{y_{n-1}^{t_1}}, \frac{y_n}{x_{n-1}^{t_1}}\right\}, \quad n \in \mathbb{N},$$

and

$$v_{n+1} = \max\left\{A^{1-t_1}, v_n^{t_2}\right\}, \quad n \in \mathbb{N}, \tag{2.5}$$

with

$$v_1 = u_1.$$

By induction, we have

$$u_n \le v_n, \quad n \in \mathbb{N}. \tag{2.6}$$

The fact $t_2 \in (0,1)$ implies that the equation $g(x) = x$, where

$$g(x) = \max\left\{A^{1-t_1}, x^{t_2}\right\}, \quad x \in (0,\infty), \tag{2.7}$$

has a unique fixed point $\bar{x} \geq 1$ and

$$(g(x) - x)(x - \bar{x}) < 0, \quad x \in \mathbb{R}_+ \setminus \{\bar{x}\}. \tag{2.8}$$

Hence, for $v_1 \in (0, \bar{x}]$, we have

$$v_n \leq v_{n+1} \leq \bar{x}, \quad n \in \mathbb{N},$$

and for $v_1 \geq \bar{x}$, we have

$$v_n \geq v_{n+1} \geq \bar{x}, \quad n \in \mathbb{N}.$$

Hence, $(v_n)_{n\in\mathbb{N}}$ is bounded, which along with (2.6) implies that

$$u_n \leq L_1, \quad n \in \mathbb{N}_0,$$

for some $L_1 \geq \bar{x} \geq 1$.

Therefore

$$x_{n+1} \leq L_1 y_n^{t_1}, \quad y_{n+1} \leq L_1 x_n^{t_1}, \quad n \in \mathbb{N}_0. \tag{2.9}$$

From (2.9) we easily get

$$x_n + y_n \leq 2L_1 \big(x_{n-1} + y_{n-1}\big)^{t_1}, \quad n \in \mathbb{N}, \tag{2.10}$$

from which it easily follows that

$$x_n + y_n \leq (2L_1)^{\frac{1-{t_1}^n}{1-t_1}} \big(x_0 + y_0\big)^{{t_1}^n} \leq (2L_1)^{\frac{1}{1-t_1}} \max\big\{1, x_0 + y_0\big\}. \tag{2.11}$$

From (2.1) and (2.11) the boundedness of sequences $(x_n)_{n\geq -1}$ and $(y_n)_{n\geq -1}$, and consequently the theorem follows. □

Theorem 2.2. *Assume that $A > 0$, $p > 0$ and $p^2 < 4q$. Then all positive solutions of system (1.3) are bounded.*

Proof. Let sequence $(p_n)_{n\in\mathbb{N}_0}$ be defined as follows

$$p_{k+1} = \frac{q}{p - p_k}, \qquad p_0 = 0. \tag{2.12}$$

Using (1.3) and (2.12) we have

$$\begin{aligned} x_{n+1} &= \max\left\{A, \frac{y_n^p}{x_{n-1}^q}\right\} = \max\left\{A, \left(\frac{y_n}{x_{n-1}^{q/p}}\right)^p\right\} \\ &= \max\left\{A, \max\left\{\frac{A}{x_{n-1}^{q/p}}, \frac{x_{n-1}^{p-\frac{q}{p}}}{y_{n-2}^q}\right\}^p\right\} \\ &= \max\left\{A, \max\left\{\frac{A}{x_{n-1}^{q/p}}, \left(\frac{x_{n-1}}{y_{n-2}^{q/(p-\frac{q}{p})}}\right)^{p-\frac{q}{p}}\right\}^p\right\} \\ &= \max\left\{A, \max\left\{\frac{A}{x_{n-1}^{q/p}}, \max\left\{\frac{A}{y_{n-2}^{q/(p-\frac{q}{p})}}, \frac{y_{n-2}^{p-\frac{q}{p-\frac{q}{p}}}}{x_{n-3}^q}\right\}^{p-\frac{q}{p}}\right\}^p\right\} \end{aligned} \tag{2.13}$$

$$= \max\left\{A, \max\left\{\frac{A}{x_{n-1}^{q/p}}, \max\left\{\frac{A}{y_{n-2}^{q/(p-\frac{q}{p})}}, \left(\frac{y_{n-2}}{x_{n-3}^{q/(p-\frac{q}{p-\frac{q}{p}})}}\right)^{p-\frac{q}{p-\frac{q}{p}}}\right\}^{p-\frac{q}{p}}\right\}^{p}\right\}$$

$$= \cdots$$

$$= \max\left\{A, \max\left\{\frac{A}{x_{n-1}^{q/p}}, \max\left\{\frac{A}{y_{n-2}^{q/(p-\frac{q}{p})}}, \cdots \max\left\{\frac{A}{y_{n-2k}^{p_{2k}}}, \frac{y_{n-2k}^{p-p_{2k}}}{x_{n-(2k+1)}^{q}}\right\}^{p-p_{2k-1}} \cdots\right\}^{p-\frac{q}{p}}\right\}^{p}\right\}, \tag{2.14}$$

$$= \max\left\{A, \max\left\{\frac{A}{x_{n-1}^{q/p}}, \max\left\{\cdots \max\left\{\frac{A}{y_{n-2k}^{p_{2k}}}, \left(\frac{y_{n-2k}}{x_{n-(2k+1)}^{q/(p-p_{2k})}}\right)^{p-p_{2k}}\right\}^{p-p_{2k-1}} \cdots\right\}^{p-\frac{q}{p}}\right\}^{p}\right\},$$

$$= \max\left\{A, \max\left\{\frac{A}{x_{n-1}^{q/p}}, \max\left\{\cdots \max\left\{\frac{A}{y_{n-2k}^{p_{2k}}}, \max\left\{\frac{A}{x_{n-(2k+1)}^{q/(p-p_{2k})}}, \frac{x_{n-(2k+1)}^{p-\frac{q}{p-p_{2k}}}}{y_{n-(2k+2)}^{q}}\right\}^{p-p_{2k}}\right\}^{p-p_{2k-1}} \cdots\right\}^{p-\frac{q}{p}}\right\}^{p}\right\}, \tag{2.15}$$

$$= \max\left\{A, \max\left\{\frac{A}{x_{n-1}^{q/p}}, \max\left\{\cdots \max\left\{\frac{A}{x_{n-(2k+1)}^{p_{2k+1}}}, \left(\frac{x_{n-(2k+1)}}{y_{n-(2k+2)}^{q/(p-p_{2k+1})}}\right)^{p-p_{2k+1}}\right\}^{p-p_{2k}} \cdots\right\}^{p-\frac{q}{p}}\right\}^{p}\right\}.$$

If $p^2 \le q$, then by using (2.1) in (2.13), for $n \ge 3$, we get

$$x_{n+1} = \max\left\{A, \max\left\{\frac{A}{x_{n-1}^{q/p}}, \frac{x_{n-1}^{p-\frac{q}{p}}}{y_{n-2}^{q}}\right\}^{p}\right\} \le \max\left\{A, A^{p-q}, \frac{1}{A^{pq+q-p^2}}\right\},$$

so $(x_n)_{n\ge -1}$ is bounded, in this case.

The monotonicity of $g(x) = q/(p-x)$ on the interval $(0, p)$ along with the fact $0 = p_0 < p_1 = q/p$ implies that p_k is increasing as far as $p_k < p$. If $p_k < p$ for every $k \in \mathbb{N}_0$, then there would exist $\lim_{k\to\infty} p_k := \hat{p}$ and $(\hat{p})^2 - p\hat{p} + q = 0$, but the equation does not have real roots because of the condition $p^2 < 4q$.

Therefore, there is an $l_0 \in \mathbb{N}$ such that

$$p_{l_0-1} < p \quad \text{and} \quad p_{l_0} \ge p.$$

If $l_0 = 2k$, then by using (2.1) in (2.14), we get

$$x_{n+1} = \max\left\{A, \max\left\{\frac{A}{x_{n-1}^{q/p}}, \max\left\{\frac{A}{y_{n-2}^{q/(p-\frac{q}{p})}}, \cdots \max\left\{\frac{A}{y_{n-2k}^{p_{2k}}}, \frac{y_{n-2k}^{p-p_{2k}}}{x_{n-(2k+1)}^{q}}\right\}^{p-p_{2k-1}} \cdots\right\}^{p-\frac{q}{p}}\right\}^{p}\right\}$$

$$\le \max\left\{A, \max\left\{\frac{A}{A^{q/p}}, \max\left\{\frac{A}{A^{q/(p-\frac{q}{p})}}, \cdots \max\left\{\frac{A}{A^{p_{2k}}}, \frac{1}{A^{q-p+p_{2k}}}\right\}^{p-p_{2k-1}} \cdots\right\}^{p-\frac{q}{p}}\right\}^{p}\right\},$$

for $n \ge 2k+2$, from which the boundedness of $(x_n)_{n\ge -1}$ follows in this case.

If $l_0 = 2k+1$, then by using (2.1) in (2.15), we get

$$x_{n+1}=\max\left\{A,\max\left\{\frac{A}{x_{n-1}^{q/p}},\max\left\{\cdots\max\left\{\frac{A}{y_{n-2k}^{p_{2k}}},\max\left\{\frac{A}{x_{n-(2k+1)}^{p_{2k+1}}},\frac{x_{n-(2k+1)}^{p-p_{2k+1}}}{y_{n-(2k+2)}^{q}}\right\}^{p-p_{2k}}\right\}^{p-p_{2k-1}}\cdots\right\}^{p-\frac{q}{p}}\right\}^{p}\right\}$$

$$\leq\max\left\{A,\max\left\{\frac{A}{A^{q/p}},\max\left\{\cdots\max\left\{\frac{A}{A^{p_{2k}}},\max\left\{\frac{A}{A^{p_{2k+1}}},\frac{1}{A^{q-p+p_{2k+1}}}\right\}^{p-p_{2k}}\right\}^{p-p_{2k-1}}\cdots\right\}^{p-\frac{q}{p}}\right\}^{p}\right\},$$

for $n\geq 2k+3$, from which the boundedness of $(x_n)_{n\geq-1}$ follows in this case.

Since the system (1.3) is symmetric, the boundedness of $(x_n)_{n\geq-1}$ imply the boundedness of $(y_n)_{n\geq-1}$, finishing the proof of the theorem. □

Theorem 2.3. *Assume that $A>0$, $p=1+q$, and $q\in(0,1)$. Then all positive solutions of system (1.3) are bounded.*

Proof. First note that by using the change of variables

$$x_n=A\hat{x}_n,\qquad y_n=A\hat{y}_n,\quad n\in\mathbb{N}_0,$$

system (1.3), in this case, is reduced to the same system with $A=1$. Hence we may assume that $A=1$.

Assume that the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are defined by

$$\begin{aligned} a_0&=q, & b_0&=q+1,\\ a_{2n+1}&=(q+1)b_{2n}-a_{2n}, & b_{2n+1}&=qb_{2n},\quad n\in\mathbb{N}_0,\\ b_{2n+2}&=(q+1)a_{2n+1}-b_{2n+1}, & a_{2n+2}&=qa_{2n+1},\quad n\in\mathbb{N}_0. \end{aligned}\tag{2.16}$$

From this, by using (1.3) and a simple inductive argument, we have

$$\begin{aligned} x_{n+1}&=\max\left\{1,\frac{y_n^{q+1}}{x_{n-1}^{q}}\right\}=\max\left\{1,\frac{y_n^{b_0}}{x_{n-1}^{a_0}}\right\}\\ &=\max\left\{1,\frac{1}{x_{n-1}^{a_0}},\frac{x_{n-1}^{(q+1)b_0-a_0}}{y_{n-2}^{qb_0}}\right\}=\max\left\{1,\frac{1}{x_{n-1}^{a_0}},\frac{x_{n-1}^{a_1}}{y_{n-2}^{b_1}}\right\}\\ &=\max\left\{1,\frac{1}{x_{n-1}^{a_0}},\frac{1}{y_{n-2}^{b_1}},\frac{y_{n-2}^{(q+1)a_1-b_1}}{x_{n-3}^{qa_1}}\right\}=\max\left\{1,\frac{1}{x_{n-1}^{a_0}},\frac{1}{y_{n-2}^{b_1}},\frac{y_{n-2}^{b_2}}{x_{n-3}^{a_2}}\right\}\\ &=\max\left\{1,\frac{1}{x_{n-1}^{a_0}},\frac{1}{y_{n-2}^{b_1}},\frac{1}{x_{n-3}^{a_2}},\frac{x_{n-3}^{(q+1)b_2-a_2}}{y_{n-4}^{qb_2}}\right\}\\ &=\max\left\{1,\frac{1}{x_{n-1}^{a_0}},\frac{1}{y_{n-2}^{b_1}},\frac{1}{x_{n-3}^{a_2}},\frac{x_{n-3}^{a_3}}{y_{n-4}^{b_3}}\right\}\\ &=\max\left\{1,\frac{1}{x_{n-1}^{a_0}},\frac{1}{y_{n-2}^{b_1}},\frac{1}{x_{n-3}^{a_2}},\frac{1}{y_{n-4}^{b_3}},\frac{y_{n-4}^{(q+1)a_3-b_3}}{x_{n-5}^{qa_3}}\right\}\\ &=\max\left\{1,\frac{1}{x_{n-1}^{a_0}},\frac{1}{y_{n-2}^{b_1}},\frac{1}{x_{n-3}^{a_2}},\frac{1}{y_{n-4}^{b_3}},\frac{y_{n-4}^{b_4}}{x_{n-5}^{a_4}}\right\}\\ &=\cdots\\ &=\max\left\{1,\frac{1}{x_{n-1}^{a_0}},\frac{1}{y_{n-2}^{b_1}},\ldots,\frac{y_{n-2k}^{b_{2k}}}{x_{n-2k-1}^{a_{2k}}}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \max\left\{1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \dots, \frac{x_{n-2k-1}^{(q+1)b_{2k}-a_{2k}}}{y_{n-2k-2}^{qb_{2k}}}\right\} \\
&= \max\left\{1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \dots, \frac{x_{n-2k-1}^{a_{2k+1}}}{y_{n-2k-2}^{b_{2k+1}}}\right\} \qquad (2.17)\\
&= \max\left\{1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \dots, \frac{1}{y_{n-2k-2}^{b_{2k+1}}}, \frac{y_{n-2k-2}^{(q+1)a_{2k+1}-b_{2k+1}}}{x_{n-2k-3}^{qa_{2k+1}}}\right\} \\
&= \max\left\{1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \dots, \frac{1}{y_{n-2k-2}^{b_{2k+1}}}, \frac{y_{n-2k-2}^{b_{2k+2}}}{x_{n-2k-3}^{a_{2k+2}}}\right\}, \qquad (2.18)
\end{aligned}
$$

for every $k \in \mathbb{N}$.

From (2.16) we have that

$$b_{2n} = \frac{a_{2n+1} + a_{2n}}{q+1}, \quad n \in \mathbb{N}_0.$$

Applying this to the following relation

$$b_{2n+2} = (q+1)a_{2n+1} - qb_{2n}, \quad n \in \mathbb{N}_0,$$

we get

$$a_{2n+3} + a_{2n+2} - (q^2+q+1)a_{2n+1} + qa_{2n} = 0, \quad n \in \mathbb{N}_0.$$

From this and the relation $a_{2n+2} = qa_{2n+1}$, we get

$$a_{2n+3} - (q^2+1)a_{2n+1} + q^2 a_{2n-1} = 0, \quad n \in \mathbb{N}. \qquad (2.19)$$

It is easy to see that the general solution of difference equation (2.19) is

$$a_{2n+1} = c_1 + c_2 q^{2n}, \quad n \in \mathbb{N}_0.$$

From this and since

$$
\begin{gathered}
a_1 = (q+1)b_0 - a_0 = q^2+q+1, \qquad b_1 = qb_0 = q^2+q, \\
a_2 = qa_1 = q^3+q^2+q, \qquad b_2 = (q+1)a_1 - b_1 = (q+1)(q^2+1), \\
a_3 = (q+1)b_2 - a_2 = q^4+q^3+q^2+q+1,
\end{gathered}
$$

we have that

$$c_1 + c_2 = q^2+q+1, \qquad c_1 + c_2 q^2 = q^4+q^3+q^2+q+1$$

and consequently

$$c_1 = \frac{1}{1-q}, \qquad c_2 = \frac{q^4+q^3}{q^2-1} = \frac{q^3}{q-1}.$$

Hence

$$a_{2n+1} = \frac{1-q^{2n+3}}{1-q}, \quad n \in \mathbb{N}_0. \qquad (2.20)$$

Letting $n \to +\infty$ we get

$$\lim_{n\to+\infty} a_{2n+1} = \frac{1}{1-q}. \qquad (2.21)$$

From this and (2.16) we also have that

$$\lim_{n\to+\infty} a_{2n} = \lim_{n\to+\infty} b_{2n+1} = \frac{q}{1-q} = q \lim_{n\to+\infty} b_{2n}. \tag{2.22}$$

Now note that from (2.17) and (2.18) we have that

$$x_{2n+1} = \max\left\{1, \frac{1}{x_{2n-1}^{a_0}}, \frac{1}{y_{2n-2}^{b_1}}, \dots, \frac{1}{y_0^{b_{2n-1}}}, \frac{y_0^{b_{2n}}}{x_{-1}^{a_{2n}}}\right\}$$
$$x_{2n} = \max\left\{1, \frac{1}{x_{2n-2}^{a_0}}, \frac{1}{y_{2n-3}^{b_1}}, \dots, \frac{1}{x_0^{a_{2n-2}}}, \frac{x_0^{a_{2n-1}}}{y_{-1}^{b_{2n-1}}}\right\},$$

for $n \in \mathbb{N}$.

From this, since $\min\{x_n, y_n\} \ge 1$ for $n \in \mathbb{N}$, and by using (2.21) and (2.22) the boundedness of the sequence $(x_n)_{n\ge -1}$ easily follows.

Since system (1.3) is symmetric, the boundedness of $(x_n)_{n\ge-1}$ imply the boundedness of $(y_n)_{n\ge-1}$, finishing the proof of the theorem. □

The following theorem shows that positive solutions of system (1.3) are unbounded in the other cases.

Theorem 2.4. *Assume that $A > 0$. If $p^2 \ge 4q \ge 4$, or $p > 1 + q$ and $q \in (0,1)$, then system (1.3) has positive unbounded solutions.*

Proof. Assume that $p^2 \ge 4q \ge 4$ and $p \ne 2$. From (1.3) we have

$$x_{n+1} \ge \frac{y_n^p}{x_{n-1}^q}, \quad y_{n+1} \ge \frac{x_n^p}{y_{n-1}^q}, \quad n \in \mathbb{N}_0. \tag{2.23}$$

Let $a_n = \ln(x_n y_n)$, $n \ge -1$. Then from (2.23), it follows that

$$a_{n+1} - pa_n + qa_{n-1} \ge 0, \quad n \in \mathbb{N}_0. \tag{2.24}$$

The polynomial $P(t) = t^2 - pt + q$ has the zeroes $t_{1,2} = (p \pm \sqrt{p^2 - 4q})/2$, and $t_1 > 1$, and $t_2 > 0$.

From (2.24) we get

$$a_{n+1} - t_1 a_n - t_2(a_n - t_1 a_{n-1}) \ge 0, \quad n \in \mathbb{N}_0, \tag{2.25}$$

that is,

$$\frac{x_{n+1}y_{n+1}}{(x_n y_n)^{t_1}} \ge \left(\frac{x_n y_n}{(x_{n-1}y_{n-1})^{t_1}}\right)^{t_2}, \quad n \in \mathbb{N}_0, \tag{2.26}$$

which implies that

$$\frac{x_{n+1}y_{n+1}}{(x_n y_n)^{t_1}} \ge \left(\frac{x_0 y_0}{(x_{-1}y_{-1})^{t_1}}\right)^{t_2^{n+1}}, \quad n \in \mathbb{N}_0. \tag{2.27}$$

Let x_i, y_i, $i \in \{-1, 0\}$ be chosen such that

$$x_0 y_0 > 1 \quad \text{and} \quad x_0 y_0 = (x_{-1}y_{-1})^{t_1}. \tag{2.28}$$

This, along with (2.27), yields

$$x_n y_n \geq \left(\frac{x_0 y_0}{(x_{-1}y_{-1})^{t_1}}\right)^{t_2^n} (x_{n-1}y_{n-1})^{t_1} = (x_{n-1}y_{n-1})^{t_1}, \quad n \in \mathbb{N}_0, \tag{2.29}$$

from which we get

$$x_n y_n \geq (x_0 y_0)^{t_1^n}, \quad n \in \mathbb{N}_0. \tag{2.30}$$

Letting $n \to \infty$ in (2.30), using the first assumption in (2.28) and $t_1 > 1$, it follows that

$$x_n y_n \to +\infty \quad \text{as} \quad n \to \infty, \tag{2.31}$$

which along with the inequality between arithmetic and geometric means implies

$$\sqrt{x_n^2 + y_n^2} \to +\infty \quad \text{as} \quad n \to \infty, \tag{2.32}$$

from which it follows that $(x_n, y_n)_{n \geq -1}$ is unbounded.

The proof in the case $p > 1 + q$ and $q \in (0,1)$ is similar, since then

$$t_1 = (p + \sqrt{p^2 - 4q})/2 > 1.$$

If $p = q + 1 = 2$, then $t_1 = t_2 = 1$. If we choose x_i, y_i, $i \in \{-1, 0\}$ such that

$$x_0 y_0 > x_{-1} y_{-1} > 0, \tag{2.33}$$

then from (2.27) we get

$$x_n y_n \geq \frac{x_0 y_0}{x_{-1} y_{-1}} x_{n-1} y_{n-1}, \quad n \in \mathbb{N}_0,$$

and consequently

$$x_n y_n \geq \left(\frac{x_0 y_0}{x_{-1} y_{-1}}\right)^n x_0 y_0, \quad n \in \mathbb{N}_0. \tag{2.34}$$

Letting $n \to \infty$ in (2.34) we get (2.31) and consequently (2.32), which implies that $(x_n, y_n)_{n \geq -1}$ is unbounded, finishing the proof of the theorem. □

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