Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **51**, 1–13; http://www.math.u-szeged.hu/ejqtde/

Ulam stability for partial fractional differential inclusions via Picard operators theory

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Received 13 April 2014, appeared 21 October 2014
Communicated by Petru Jebelean

Abstract. In the present paper, we investigate, using the Picard operator technique, some existence and Ulam type stability results for the Darboux problem associated to some partial fractional order differential inclusions.

Keywords: fractional differential inclusion, left-sided mixed Riemann–Liouville integral, Caputo fractional order derivative, Darboux problem, multivalued weakly Picard operator, fixed point inclusion, Ulam–Hyers stability.

2010 Mathematics Subject Classification: 47H10, 34G20, 26A33.

1 Introduction

The fractional calculus deals with extensions of derivatives and integrals to noninteger orders. It represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [13, 24, 33]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [5], Kilbas *et al.* [19], Miller and Ross [20], the papers of Abbas *et al.* [1–4], Vityuk and Golushkov [35], and the references therein.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University (for more details see [34]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [14]. Thereafter, this type of stability is called the Ulam–Hyers stability. In 1978, Rassias [25] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations

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is how do the solutions of the inequality differ from those of the given functional equation? Considerable attention has been given to the study of the Ulam–Hyers and Ulam–Hyers–Rassias stability of all kinds of functional equations; one can see the monographs of [15, 16]. Bota-Boriceanu and Petrușel [7], Petru *et al.* [22, 23], and Rus [26, 27] discussed the Ulam–Hyers stability for operatorial equations and inclusions. Castro and Ramos [8], and Jung [18] considered the Hyers–Ulam–Rassias stability for a class of Volterra integral equations. Ulam stability for fractional differential equations with Caputo derivative are proposed by Wang *et al.* [36–38]. Some stability results for fractional integral equation are obtained by Wei *et al.* [39]. More details from historical point of view, and recent developments of such stabilities are reported in [17, 26, 39].

The theory of Picard operators was introduced by I. A. Rus (see [28–30] and their references) to study problems related to fixed point theory. This abstract approach was used later on by many mathematicians and it seemed to be a very useful and powerful method in the study of integral equations and inequalities, ordinary and partial differential equations (existence, uniqueness, differentiability of the solutions), etc. We recommend the monograph [30] and the references therein. The theory of Picard operators is a very powerful tool in the study of Ulam–Hyers stability of functional equations. We only have to define a fixed point equation from the functional equation we want to study, then if the defined operator is c-weakly Picard we also have immediately the Ulam–Hyers stability of the desired equation. Of course it is not always possible to transform a functional equation or a differential equation into a fixed point problem and actually this point shows a weakness of this theory. The uniform approach with Picard operators to the discussion of the stability problems of Ulam–Hyers type is due to Rus [27].

In this article, we discuss the Ulam–Hyers and the Ulam–Hyers–Rassias stability for the fractional partial differential inclusion

$$^{c}D^{r}_{\theta}u(x,y) \in F(x,y,u(x,y)); \text{ if } (x,y) \in J := [0,a] \times [0,b], \tag{1.1}$$

with the initial conditions

$$\begin{cases} u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{1.2}$$

where $a, b > 0$, $\theta = (0,0)$, $^{c}D^{r}_{\theta}$ is the fractional Caputo derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $F\colon J \times E \to \mathcal{P}(E)$ is a set-valued function with nonempty values in a (real or complex) separable Banach space E, and $\mathcal{P}(E)$ is the family of all nonempty subsets of E, and $\varphi\colon [0,a] \to E$, $\psi\colon [0,b] \to E$ are given absolutely continuous functions.

2 Preliminaries

Let $L^1(J)$ be the space of Bochner-integrable functions $u\colon J \to E$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x,y)\|_E \, dy \, dx,$$

where $\|\cdot\|_E$ denotes a complete norm on E. By $L^\infty(J)$ we denote the Banach space of measurable functions $u\colon J \to E$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : \|u(x,y)\|_E \le c, \text{ a.e. } (x,y) \in J\}.$$

As usual, by $\mathcal{C} := C(J)$ we denote the Banach space of all continuous functions from J into E with the norm $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(x,y)\in J} \|u(x,y)\|_E.$$

Let (X,d) be a metric space induced from the normed space $(X,\|\cdot\|)$. Denote $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(E) = \{Y \in \mathcal{P}(E) : Y \text{ compact}\}$ and $\mathcal{P}_{cp,cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ compact and convex}\}$.

Definition 2.1. A multivalued map $T \cdot X \to \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$, T has a fixed point if there is $x \in X$ such that $x \in T(x)$. The fixed point set of the multivalued operator T will be denoted by $Fix(T)$. The graph of T will be denoted by $Graph(F) := \{(u,v) \in X \times \mathcal{P}(X) : v \in T(u)\}$.

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to [0,\infty) \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{bd,cl}(X), H_d)$ is a Hausdorff metric space.

Notice that $A\colon X \to X$ is a selection for $T\colon X \to \mathcal{P}(X)$ if $A(u) \in T(u)$ for each $u \in X$. For each $u \in \mathcal{C}$, define the set of selections of the multivalued $F\colon J \times \mathcal{C} \to \mathcal{P}(\mathcal{C})$ by

$$S_{F,u} = \{v :\in L^1(J)\colon v(x,y) \in F(x,y,u(x,y));\ (x,y) \in J\}.$$

Definition 2.2. A multivalued map $G\colon J \to \mathcal{P}_{cl}(E)$, is said to be measurable if for every $v \in E$ the function $(x,y) \to d(v, G(x,y)) = \inf\{d(v,z) : z \in G(x,y)\}$ is measurable.

In what follows we will give some basic definitions and results on Picard operators [30]. Let (X,d) be a metric space and $A\colon X \to X$ be an operator. We denote by F_A the set of the fixed points of A. We also denote $A^0 := 1_X$, $A^1 := A, \ldots, A^{n+1} := A^n \circ A$; $n \in \mathbb{N}$ the iterate operators of the operator A.

Definition 2.3. The operator $A\colon X \to X$ is a Picard operator (briefly PO) if there exists $x^* \in X$ such that:

(i) $F_A = \{x^*\}$;

(ii) The sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2.4. The operator $A\colon X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.

Definition 2.5. If A is weakly Picard operator, then we consider the operator A^∞ defined by

$$A^\infty\colon X \to X;\ A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

Remark 2.6. It is clear that $A^\infty(X) = F_A$.

Definition 2.7. Let A be a weakly Picard operator and $c > 0$. The operator A is c-weakly Picard operator if

$$d(x, A^\infty(x)) \leq c\, d(x, A(x)); \quad x \in X.$$

In the multivalued case we have the following concepts (see [21,31]).

Definition 2.8. Let (X,d) be a metric space, and $F\colon X \to \mathcal{P}_{cl}(X)$ be a multivalued operator. By definition, F is a multivalued weakly Picard operator (briefly MWPO), if for each $u \in X$ and each $v \in F(x)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that

(i) $u_0 = u,\ u_1 = v$;

(ii) $u_{n+1} \in F(u_n)$ for each $n \in \mathbb{N}$;

(iii) the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Remark 2.9. A sequence $(u_n)_{n\in\mathbb{N}}$ satisfying condition (i) and (ii) in the above definition is called a sequence of successive approximations of F starting from $(x,y) \in Graph(F)$.

If $F\colon X \to \mathcal{P}_{cl}(X)$ is a MWPO, then we define $F^\infty : Graph(F) \to \mathcal{P}(Fix(F))$ by the formula $F^\infty(x,y) := \{x^* \in Fix(F)$: there exists a sequence of successive approximations of F starting from (x,y) that converges to $x^*\}$.

Definition 2.10. Let (X,d) be a metric space and let $\Psi\colon [0,\infty) \to [0,\infty)$ be an increasing function which is continuous at 0 and $\Psi(0) = 0$. Then $F\colon X \to \mathcal{P}_{cl}(X)$ is said to be a multivalued Ψ-weakly Picard operator (Ψ-MWPO) if it is a multivalued weakly Picard operator and there exists a selection $A^\infty : Graph(F) \to Fix(F)$ of F^∞ such that

$$d(u, A^\infty(u,v)) \leq \Psi(d(u,v)) \quad \text{for all } (u,v) \in Graph(F).$$

If there exists $c > 0$ such that $\Psi(t) = ct$ for each $t \in [0,\infty)$, then F is called a multivalued c-weakly Picard operator (c-MWPO).

Definition 2.11. A multivalued operator $N\colon X \to \mathcal{P}_{cl}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma \geq 0$ such that

$$H_d(N(u), N(v)) \leq \gamma d(u,v) \quad \text{for each } u, v \in X,$$

b) γ-contraction if and only if it is γ-Lipschitz with $\gamma \in [0,1)$.

Now, we introduce notations, definitions and preliminary lemmas concerning to partial fractional calculus theory.

Definition 2.12 ([35]). Let $\theta = (0,0)$, $r_1, r_2 \in (0,\infty)$ and $r = (r_1, r_2)$. For $f \in L^1(J)$, the expression

$$(I_\theta^r f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\, dt\, ds,$$

is called the left-sided mixed Riemann–Liouville integral of order r, where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}\, dt;\ \xi > 0$.

In particular,

$$(I_\theta^\theta f)(x,y) = f(x,y), \quad (I_\theta^\sigma f)(x,y) = \int_0^x \int_0^y f(s,t)\,dt\,ds \quad \text{for almost all } (x,y) \in J,$$

where $\sigma = (1,1)$.
For instance, $I_\theta^r f$ exists for all $r_1, r_2 \in (0,\infty)$, when $f \in L^1(J)$. Note also that when $u \in \mathcal{C}$, then $(I_\theta^r f) \in \mathcal{C}$, moreover

$$(I_\theta^r f)(x,0) = (I_\theta^r f)(0,y) = 0; \quad x \in [0,a],\ y \in [0,b].$$

Example 2.13. Let $\lambda, \omega \in (0,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2} \quad \text{for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$ the mixed second order partial derivative.

Definition 2.14 ([35])**.** Let $r \in (0,1] \times (0,1]$ and $f \in L^1(J)$. The Caputo fractional-order derivative of order r of f is defined by the expression

$${}^cD_\theta^r f(x,y) = (I_\theta^{1-r} D_{xy}^2 f)(x,y) = \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^x \int_0^y \frac{D_{st}^2 f(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}\,dt\,ds.$$

The case $\sigma = (1,1)$ is included and we have

$$({}^cD_\theta^\sigma f)(x,y) = (D_{xy}^2 f)(x,y) \quad \text{for almost all } (x,y) \in J.$$

Example 2.15. Let $\lambda, \omega \in (0,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$${}^cD_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2} \quad \text{for almost all } (x,y) \in J.$$

We need the following lemma.

Lemma 2.16 ([1])**.** *Let $h \in L^1(J)$, $0 < r_1, r_2 \le 1$, $\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0)$. A function $u \in \mathcal{C}$ is a solution of the fractional integral equation*

$$u(x,y) = \mu(x,y) + I_\theta^r h(x,y),$$

if and only if u is a solution of the problem

$$\begin{cases} {}^cD_\theta^r u(x,y) = h(x,y); & \text{if } (x,y) \in J := [0,a] \times [0,b], \\ u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0). \end{cases}$$

Remark 2.17. By Lemma 2.16, every solution of the problem (1.1)–(1.2) is a solution of the fixed point inclusion $u \in N(u)$, where $N\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is the multivalued operator defined by

$$N(u)(x,y) = \{\mu(x,y) + I_\theta^r f(x,y);\ f \in S_{F,u}\}; \quad (x,y) \in J,$$

and vice versa. So, the two problems are equivalent and we will focus on the fixed point problem $u \in N(u)$, where N is described above.

Let us give the definition of Ulam–Hyers stability of a fixed point inclusion due to Rus.

Definition 2.18 ([27])**.** Let (X,d) be a metric space and $A\colon X \to X$ be an operator. The fixed point equation $x = A(x)$ is said to be Ulam–Hyers stable if there exists a real number $c_A > 0$ such that: for each real number $\epsilon > 0$ and each solution y^* of the inequality $d(y, A(y)) \leq \epsilon$, there exists a solution x^* of the equation $x = A(x)$ such that

$$d(y^*, x^*) \leq \epsilon c_A; \quad x \in X.$$

In the multivalued case we have the following definition.

Definition 2.19 ([23])**.** Let (X,d) be a metric space and $A\colon X \to \mathcal{P}(X)$ be a multivalued operator. The fixed point inclusion $u \in A(u)$ is said to be generalized Ulam–Hyers stable if and only if there exists $\Psi\colon [0,\infty) \times [0,\infty)$ increasing, continuous at 0 and $\Psi(0) = 0$ such that for each $\epsilon > 0$ and for each solution v^* of the inequation $d(u, A(u)) \leq \epsilon$, there exists a solution u^* of the inclusion $u \in A(u)$ such that

$$d(u^*, v^*) \leq \Psi(\epsilon); \quad x \in X.$$

From the above definition, we shall give four types of Ulam stability of the fixed point inclusion $u \in N(u)$. Let ϵ be a positive real number and $\Phi\colon J \to [0,\infty)$ be a continuous function.

Definition 2.20. The fixed point inclusion $u \in N(u)$ is said to be Ulam–Hyers stable if there exists a real number $c_N > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality $H_d(u(x,y), N(u)(x,y)) \leq \epsilon$; $(x,y) \in J$, there exists a solution $v \in \mathcal{C}$ of the inclusion $u \in N(u)$ with

$$\|u(x,y) - v(x,y)\|_E \leq \epsilon c_N; \quad (x,y) \in J.$$

Definition 2.21. The fixed point inclusion $u \in N(u)$ is said to be generalized Ulam–Hyers stable if there exists an increasing function $\theta_N \in C([0,\infty),[0,\infty))$, $\theta_N(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality $H_d(u(x,y), N(u)(x,y)) \leq \epsilon$; $(x,y) \in J$, there exists a solution $v \in \mathcal{C}$ of the inclusion $u \in N(u)$ with

$$\|u(x,y) - v(x,y)\|_E \leq \theta_N(\epsilon); \quad (x,y) \in J.$$

Definition 2.22. The fixed point inclusion $u \in N(u)$ is said to be Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality $H_d(u(x,y), N(u)(x,y)) \leq \epsilon\Phi(x,y)$; $(x,y) \in J$, there exists a solution $v \in \mathcal{C}$ of the inclusion $u \in N(u)$ with

$$\|u(x,y) - v(x,y)\|_E \leq \epsilon c_{N,\Phi}\Phi(x,y); \quad (x,y) \in J.$$

Definition 2.23. The fixed point inclusion $u \in N(u)$ is said to be generalized Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each solution $u \in \mathcal{C}$ of the inequality $H_d(u(x,y), N(u)(x,y)) \leq \Phi(x,y)$; $(x,y) \in J$, there exists a solution $v \in \mathcal{C}$ of the inclusion $u \in N(u)$ with

$$\|u(x,y) - v(x,y)\|_E \leq c_{N,\Phi}\Phi(x,y); \quad (x,y) \in J.$$

Remark 2.24. It is clear that

(i) Definition 2.20 $\Rightarrow$ Definition 2.21,

(ii) Definition 2.22 $\Rightarrow$ Definition 2.23,

(iii) Definition 2.22 for $\Phi(x,y) = 1 \ \Rightarrow$ Definition 2.20.

Lemma 2.25 ([10]). *Let (X,d) be a complete metric space. If $A\colon X \to \mathcal{P}_{cl}(X)$ is a contraction, then A has fixed points.*

Now we present an important characterization lemma from the point of view of Ulam–Hyers stability.

Lemma 2.26 ([23]). *Let (X,d) be a metric space. If $A\colon X \to \mathcal{P}_{cp}(X)$ is a Ψ-MWPO, then the fixed point inclusion $u \in A(u)$ is generalized Ulam–Hyers stable. In particular, if A is a c-MWPO, then the fixed point inclusion $u \in A(u)$ is Ulam–Hyers stable.*

As a consequence we also have the following lemma.

Lemma 2.27 ([11,21]). *Let (X,d) be a Banach space. If $A : X \to \mathcal{P}_{cp}(X)$ is a q-contraction, then A is a c-MWPO, with $c = \frac{1}{1-q}$. Moreover, the fixed point inclusion $u \in A(u)$ is Ulam–Hyers stable.*

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 2.28 (Gronwall lemma [12]). *Let $v\colon J \to [0,\infty)$ be a real function and $\omega(\cdot,\cdot)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$v(x,y) \le \omega(x,y) + c\int_0^x\int_0^y \frac{v(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}\,dt\,ds,$$

then there exists a constant $\delta = \delta(r_1,r_2)$ such that

$$v(x,y) \le \omega(x,y) + \delta c\int_0^x\int_0^y \frac{\omega(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}\,dt\,ds,$$

for every $(x,y) \in J$.

3 Existence and Ulam stability results

Let us start in this section by giving conditions for the Ulam–Hyers stability of the problem (1.1)–(1.2).

Theorem 3.1. *Assume that the following hypotheses hold:*

(H_1) *the multifunction $F\colon J \times E \to \mathcal{P}_{cp}(E)$ has the property that $F(\cdot,\cdot,u)\colon J \to \mathcal{P}_{cp}(E)$ is jointly measurable for each $u \in E$;*

(H_2) *there exists $P \in L^\infty(J,[0,\infty))$ such that for each $u, v \in E$ and $(x,y) \in J$, we have*

$$H_d(F(x,y,u(x,y)), F(x,y,\overline{u}(x,y))) \le P(x,y)\|u-\overline{u}\|_E;$$

(H_3) *there exists an integrable function $q\colon [0,b] \to [0,\infty)$ such that for each $x \in [0,a]$ and $u \in E$, we have $F(x,y,u) \subset q(y)B(0,1)$, a.e. $y \in [0,b]$, where $B(0,1) = \{u \in E : \|u\|_E < 1\}$.*

If

$$M_F := \frac{p^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{3.1}$$

where $p^* = \|P\|_{L^\infty}$, *then the problem* (1.1)–(1.2) *has a solution on* J, *and* N *is a* (k_N*-MWPO) with* $k_N = \frac{1}{1-M_F}$. *Moreover the fixed point inclusion* $u \in N(u)$ *is Ulam–Hyers stable.*

Proof. Notice first that, for each $u \in \mathcal{C}$, the set $S_{F,u}$ is nonempty, since by (H_1), F has a measurable selection (see [9], Theorem III.6).

We shall show that N defined in Remark 2.17 satisfies the assumptions of Lemmas 2.25 and 2.27. The proof will be given in two steps.

Step 1: $N(u) \in P_{cp}(\mathcal{C})$ *for each* $u \in \mathcal{C}$.
From the continuity of μ and Theorem 2 in Rybiński [32] we have that for each $u \in \mathcal{C}$ there exists $f \in S_{F,u}$, for all $(x,y) \in J$, such that $f(x,y)$ is integrable with respect to y and continuous with respect to x. Then the function $v(x,y) = \mu(x,y) + I_\theta^r f(x,y)$ has the property $v \in N(u)$. Moreover, from (H_1) and (H_3), via Theorem 8.6.3 in Aubin and Frankowska [6], we get that $N(u)$ is a compact set, for each $u \in \mathcal{C}$.

Step 2: *There exists* $\gamma \in [0,1)$ *such that*

$$H_d(N(u), N(\overline{u})) \le \gamma \|u - \overline{u}\|_\infty \quad \text{for each } u, \overline{u} \in \mathcal{C}.$$

Let $u, \overline{u} \in \mathcal{C}$ and $h \in N(u)$. Then, there exists $f(x,y) \in F(x,y,u(x,y))$ such that for each $(x,y) \in J$, we have

$$h(x,y) = \mu(x,y) + I_\theta^r f(x,y).$$

From (H_2) it follows that

$$H_d(F(x,y,u(x,y)), F(x,y,\overline{u}(x,y))) \le P(x,y)\|u(x,y) - \overline{u}(x,y)\|_E.$$

Hence, there exists $w(x,y) \in F(x,y,\overline{u}(x,y)$ such that

$$\|f(x,y) - w(x,y)\|_E \le P(x,y)\|u(x,y) - \overline{u}(x,y)\|_E; \quad (x,y) \in J.$$

Consider $U \colon J \to \mathcal{P}(E)$ given by

$$U(x,y) = \{w \in E : \|f(x,y) - w(x,u)\|_E \le P(x,y)\|u(x,y) - \overline{u}(x,y)\|_E\}.$$

Since the multivalued operator $u(x,y) = U(x,y) \cap F(x,y,\overline{u}(x,y))$ is measurable (see Proposition III.4 in [9]), there exists a function $\overline{f}(x,y)$ which is a measurable selection for u. So, $\overline{f}(x,y) \in F(x,y,\overline{u}(x,y))$, and for each $(x,y) \in J$,

$$\|f(x,y) - \overline{f}(x,y)\|_E \le P(x,y)\|u(x,y) - \overline{u}(x,y)\|_E.$$

Let us define for each $(x,y) \in J$,

$$\overline{h}(x,y) = \mu(x,y) + I_\theta^r \overline{f}(x,y).$$

Then for each $(x,y) \in J$, we have

$$\begin{aligned}
\|h(x,y) - \overline{h}(x,y)\|_E &\le I_\theta^r \|f(x,y) - \overline{f}(x,y)\|_E \\
&\le I_\theta^r (P(x,y)\|u(x,y) - \overline{u}(x,y)\|_E) \\
&\le \|P\|_{L^\infty} \|u - \overline{u}\|_\infty \left(\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \, dt \, ds \right) \\
&\le \frac{p^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|u - \overline{u}\|_\infty.
\end{aligned}$$

Thus,

$$\|h - \overline{h}\|_\infty \le M_F \|u - \overline{u}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of u and $\overline{u}$, it follows that

$$H_d(N(u), N(\overline{u})) \le M_F \|u - \overline{u}\|_\infty.$$

Hence, by (3.1), N is a M_F-contraction. Consequently, by Lemma 2.25, N has a fixed point witch is a solution of the problem (1.1)–(1.2) on J.

Consequently, Lemma 2.27 implies that N is a (k_N-MWPO) with $k_N = \frac{1}{1-M_F}$ and the fixed point inclusion $u \in N(u)$ is Ulam–Hyers stable. □

Now, we present conditions for the generalized Ulam–Hyers–Rassias stability of the problem (1.1)–(1.2).

Theorem 3.2. *Assume that the assumptions* (H_1), (H_2) *and the following hypothesis hold*

(H_4) $\Phi \in L^1(J, [0, \infty))$ *and there exists* $\lambda_\Phi > 0$ *such that, for each* $(x, y) \in J$ *we have*

$$(I_\theta^r \Phi)(x, y) \le \lambda_\Phi \Phi(x, y).$$

If the condition (3.1) *holds, then the fixed point inclusion* $u \in N(u)$ *is generalized Ulam–Hyers–Rassias stable.*

Proof. Let $u \in \mathcal{C}$ be a solution of the inequality $H_d(u, N(u)) \le \Phi(x, y)$; $(x, y) \in J$. By Lemma 2.25 there is v a solution of the fixed point inclusion $u \in N(u)$. Then we have

$$v(x, y) = \mu(x, y) + I_\theta^r f_v(x, y); \quad f_v \in S_{F,v}, (x, y) \in J.$$

Then, for each $(x, y) \in J$, it follows that

$$\begin{aligned} \|u(x, y) - v(x, y)\|_E &\le H_d(u, N(v)) \\ &\le H_d(u, N(u)) + H_d(N(u), N(v)) \\ &\le \Phi(x, y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(s, t) - f_v(s, t)\|_E \, dt\, ds. \end{aligned}$$

where $f \in S_{F,u}$. Thus, for each $(x, y) \in J$, we have

$$\|u(x, y) - v(x, y)\|_E \le \Phi(x, y) + \int_0^x \int_0^y \frac{p^*(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|u(s, t) - v(s, t)\|_E \, dt\, ds.$$

From Lemma 2.28, there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\begin{aligned} \|u(x, y) - v(x, y)\|_E &\le \Phi(x, y) + \frac{\delta p^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \Phi(s, t)\, dt\, ds \\ &= \Phi(x, y) + \delta p^* (I_\theta^r \Phi)(x, y). \end{aligned}$$

Hence, by (H_4) for each $(x, y) \in J$, we get

$$\begin{aligned} \|u(x, y) - v(x, y)\|_E &\le (1 + \delta p^* \lambda_\Phi)\Phi(x, y) \\ &=: c_{f,\Phi} \Phi(x, y). \end{aligned}$$

Finally, the fixed point inclusion $u \in N(u)$ is generalized Ulam–Hyers–Rassias stable. □

4 An Example

Let $E = l^1 = \{w = (w_1, w_2, \dots, w_n, \dots) : \sum_{n=1}^{\infty} |w_n| < \infty\}$, be the Banach space with norm $\|w\|_E = \sum_{n=1}^{\infty} |w_n|$, and consider the following partial functional fractional order differential inclusion of the form

$$^cD_\theta^r u(x,y) \in F(x,y,u(x,y)); \quad \text{a.e. } (x,y) \in J = [0,1] \times [0,1], \tag{4.1}$$

with the initial conditions

$$\begin{cases} u(x,0) = x; & x \in [0,1], \\ u(0,y) = y^2; & y \in [0,1], \end{cases} \tag{4.2}$$

where $(r_1, r_2) \in (0,1] \times (0,1]$,

$$u = (u_1, u_2, \dots, u_n, \dots), \quad {}^cD_\theta^r u = ({}^cD_\theta^r u_1, {}^cD_\theta^r u_2, \dots, {}^cD_\theta^r u_n, \dots),$$

and

$$F(x,y,u(x,y)) \\ = \{v \in C([0,1]\times[0,1], \mathbb{R}) : \|f_1(x,y,u(x,y))\|_E \le \|v\|_E \le \|f_2(x,y,u(x,y))\|_E\};$$

$(x,y) \in [0,1] \times [0,1]$, where $f_1, f_2 \colon [0,1] \times [0,1] \times E \to E$,

$$f_k = (f_{k,1}, f_{k,2}, \dots, f_{k,n}, \dots); \quad k \in \{1,2\},\ n \in \mathbb{N},$$

$$f_{1,n}(x,y,u_n(x,y)) = \frac{xy^2 u_n}{(1 + \|u_n\|_E)e^{10+x+y}}; \quad n \in \mathbb{N},$$

and

$$f_{2,n}(x,y,u_n(x,y)) = \frac{xy^2 u_n}{e^{10+x+y}}; \quad n \in \mathbb{N}.$$

We assume that F is compact valued. We can see that the solutions of the problem (4.1)–(4.2) are solutions of the fixed point inclusion $u \in A(u)$ where $A \colon C([0,1]\times[0,1], \mathbb{R}) \to \mathcal{P}(C([0,1]\times[0,1], \mathbb{R}))$ is the multifunction operator defined by

$$(Au)(x,y) = \{x + y^2 + I_\theta^r f(x,y);\ f \in S_{F,u}\}, \quad (x,y) \in [0,1] \times [0,1].$$

For each $(x,y) \in [0,1] \times [0,1]$ and all $z_1, z_2 \in E$, we have

$$\|f_2(x,y,z_2) - f_1(x,y,z_1)\|_E \le xy^2 e^{-10-x-y} \|z_2 - z_1\|_E.$$

Thus, the hypotheses (H_1)–(H_3) are satisfied with $P(x,y) = xy^2 e^{-10-x-y}$ and $q(y) = y^2 e^{-10-y}$. We shall show that condition (3.1) holds with $a = b = 1$. Indeed, $p^* = e^{-10}$, $\Gamma(1 + r_i) > \frac{1}{2}$; $i = 1,2$. A simple computation shows that

$$M_F := \frac{p^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 4e^{-10} < 1.$$

Consequently, by Theorem 3.1, A is a (k_N-MWPO) with $k_N = \frac{1}{1-M_F}$ and the fixed point inclusion $u \in A(u)$ is Ulam–Hyers stable.

Next, we can see that the hypothesis (H_4) is satisfied with $\Phi(x,y) = xy^2$ and $\lambda_\Phi \le \frac{2}{\Gamma(2+r_1)\Gamma(3+r_2)}$. Indeed, for each $(x,y) \in [0,1] \times [0,1]$, we get

$$(I_\theta^r \Phi)(x,y) = \frac{2}{\Gamma(2+r_1)\Gamma(3+r_2)} x^{1+r_1} y^{2+r_2} \le \lambda_\Phi \Phi(x,y).$$

Consequently, Theorem 3.2 implies that the fixed point inclusion $u \in A(u)$ is generalized Ulam–Hyers–Rassias stable.

Acknowledgements

For the third author, this work benefits of the financial support of a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0094.

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