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Oscillation of trinomial differential equations with positive and negative terms

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Abstract. In the paper, we offer a new technique for investigation of properties of trinomial differential equations with positive and negative terms

$$\Big(b(t)\,\big(a(t)x'(t)\big)'\Big)' + p(t)f(x(\tau(t))) - q(t)h(x(\sigma(t))) = 0.$$

We offer criteria for every solution to be oscillatory. We support our results with illustrative examples.

Keywords: third order differential equations, delay argument, oscillation.

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1 Introduction

We consider the third order trinomial differential equation with positive and negative terms

$$\Big(b(t)\,\big(a(t)x'(t)\big)'\Big)' + p(t)f(x(\tau(t))) - q(t)h(x(\sigma(t))) = 0, \tag{E}$$

where

(H_1) $a(t), b(t), p(t), q(t), \tau(t), \sigma(t) \in C([t_0, \infty))$ are positive;

(H_2) $f(u), h(u) \in C(\mathbb{R})$, $uf(u) > 0$, $uh(u) > 0$ for $u \neq 0$, h is bounded, f is nondecreasing;

(H_3) $f(uv) \geq f(u)f(v)$ for $uv > 0$;

(H_4) $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \sigma(t) = \infty$.

We consider the canonical case of (E), that is

(H_5) $\displaystyle\int_{t_0}^{\infty} \frac{1}{b(s)}\,\mathrm{d}s = \int_{t_0}^{\infty} \frac{1}{a(s)}\,\mathrm{d}s = \infty,$

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and throughout the paper we assume that

(H_6) $\int_{t_0}^{\infty} \frac{1}{a(t)} \int_t^{\infty} \frac{1}{b(s)} \int_s^{\infty} q(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t < \infty.$

By a solution of (E) we understand a function $x(t)$ with both quasi-derivatives $a(t)x'(t)$, $b(t)\left(a(t)x'(t)\right)'$ continuous on $[T_x, \infty))$, $T_x \geq t_0$, which satisfies Eq. (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (E) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Equation (E) includes the couple of binomial differential equations

$$\left(b(t)\left(a(t)x'(t)\right)'\right)' + p(t)f(x(\tau(t))) = 0 \qquad (E_1)$$

and

$$\left(b(t)\left(a(t)x'(t)\right)'\right)' - q(t)h(x(\sigma(t))) = 0. \qquad (E_2)$$

Properties of both equations have been studied by many authors. See papers of Baculíková et al. [1, 2], Candan and Dahiya [3], Grace et al. [4], Thandapani and Li [9], Tiryaki and Atkas [10].

We reveal that the solutions' spaces of (E_1) and (E_2) are absolutely different. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of considered equations, then for (E_1) the set $\mathcal{N}$ has the following decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2,$$

where positive solution

$$x(t) \in \mathcal{N}_0 \iff a(t)x'(t) < 0, \quad b(t)\left(a(t)x'(t)\right)' > 0, \quad \left(b(t)\left(a(t)x'(t)\right)'\right)' < 0,$$
$$x(t) \in \mathcal{N}_2 \iff a(t)x'(t) > 0, \quad b(t)\left(a(t)x'(t)\right)' > 0, \quad \left(b(t)\left(a(t)x'(t)\right)'\right)' < 0.$$

On the other hand, for (E_2) the set $\mathcal{N}$ has the following reduction

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3,$$

with positive solution

$$x(t) \in \mathcal{N}_1 \iff a(t)x'(t) > 0, \quad b(t)\left(a(t)x'(t)\right)' < 0, \quad \left(b(t)\left(a(t)x'(t)\right)'\right)' > 0,$$
$$x(t) \in \mathcal{N}_3 \iff a(t)x'(t) > 0, \quad b(t)\left(a(t)x'(t)\right)' > 0, \quad \left(b(t)\left(a(t)x'(t)\right)'\right)' > 0.$$

Consequently, the nonoscillatory solutions' space of (E) with positive and negative part is unclear.

Another method frequently used in the oscillation theory of trinomial differential equations is to omit one term. And so, if we omit the negative part of (E), we are led to the differential inequality

$$\left\{\left(b(t)\left(a(t)x'(t)\right)'\right)' + p(t)f(x(\tau(t)))\right\}\operatorname{sgn} x(t) \geq 0. \qquad (E_3)$$

But it is well known that properties of the corresponding differential equation (E_1) are connected with the opposite differential inequality. Similarly omitting the positive term of (E) yields the differential inequality

$$\left\{\left(b(t)\left(a(t)x'(t)\right)'\right)' - q(t)h(x(\sigma(t)))\right\} \operatorname{sgn} x(t) \le 0, \qquad (E_4)$$

which is again opposite to those that we need. So there is only a limited number of papers dealing (E) with positive and negative parts. In this paper we use a method that overcomes those difficulties appearing due to negative and positive terms of (E).

2 Main results

In this paper we reduce the investigation of trinomial equations to oscillation of a suitable first order differential equation. We establish a new comparison method for investigating properties of trinomial differential equations with positive and negative terms.

We denote

$$J(t) = \int_{t_1}^{\tau(t)} \frac{1}{a(s)} \int_{t_1}^{s} \frac{1}{b(u)}\, \mathrm{d}u\, \mathrm{d}s,$$

with t_1 large enough.

Theorem 2.1. *Assume that*

$$\int_{t_1}^{\infty} \frac{1}{a(v)} \int_{v}^{\infty} \frac{1}{b(s)} \int_{s}^{\infty} p(u)\, \mathrm{d}u\, \mathrm{d}s\, \mathrm{d}v = \infty. \qquad (2.1)$$

Let the first order delay differential equation

$$y'(t) + p(t) f(J(t)) f(y(\tau(t))) = 0 \qquad (E_0)$$

be oscillatory. Then every solution of (E) either oscillates or converges to zero as $t \to \infty$.

Proof. Assume that (E) possesses a nonoscillatory solution $x(t)$. Without loss of generality we may assume that $x(t)$ is eventually positive. We introduce the auxiliary function

$$w(t) = x(t) + \int_{t}^{\infty} \frac{1}{a(v)} \int_{v}^{\infty} \frac{1}{b(s)} \int_{s}^{\infty} q(u)h(x(\sigma(u)))\, \mathrm{d}u\, \mathrm{d}s\, \mathrm{d}v. \qquad (2.2)$$

Note that condition (H_6) and the fact that $h(u)$ is bounded implies that $w(t)$ exists for all t and so the definition of $w(t)$ is correct. Moreover, $w(t) > x(t) > 0$, $w'(t) < x'(t)$ and

$$\left(b(t)\left(a(t)w'(t)\right)'\right)' = -p(t)f(x(\tau(t))) < 0. \qquad (2.3)$$

Therefore, condition (H_5) together with a modification of Kiguradze's lemma [5, 6] imply that either

$$w(t) \in \mathcal{N}_0 \iff a(t)w'(t) < 0, \quad b(t)\left(a(t)w'(t)\right)' > 0,$$

or

$$w(t) \in \mathcal{N}_2 \iff a(t)w'(t) > 0, \quad b(t)\left(a(t)w'(t)\right)' > 0,$$

eventually, let us say for $t \ge t_1$. First assume that $w(t) \in \mathcal{N}_2$. Using the fact that $b(t)(a(t)w'(t))'$ is decreasing, we have

$$a(t)w'(t) \ge \int_{t_1}^{t} b(s)\left(a(s)w'(s)\right)' \frac{1}{b(s)}\, \mathrm{d}s \ge b(t)\left(a(t)w'(t)\right)' \int_{t_1}^{t} \frac{1}{b(s)}\, \mathrm{d}s.$$

Using the last estimate and properties of $w(t)$ and $x(t)$ one can see that

$$\begin{aligned} x(t) &\geq \int_{t_1}^{t} x'(s)\,\mathrm{d}s \geq \int_{t_1}^{t} w'(s)\,\mathrm{d}s \geq \int_{t_1}^{t} \frac{b(s)\,\big(a(s)w'(s)\big)'}{a(s)} \int_{t_1}^{s} \frac{1}{b(u)}\,\mathrm{d}u\,\mathrm{d}s \\ &\geq b(t)\,\big(a(t)w'(t)\big)' \int_{t_1}^{t} \frac{1}{a(s)} \int_{t_1}^{s} \frac{1}{b(u)}\,\mathrm{d}u\,\mathrm{d}s, \end{aligned}$$

which in view of (2.3) and (H_3) ensures that $z(t) = b(t)(a(t)w'(t))'$ is a positive solution of the differential inequality

$$z'(t) + p(t)f(J(t))f\big(z(\tau(t))\big) \leq 0.$$

It follows from Theorem 1 in [8] that the corresponding differential equation (E_0) also has a positive solution. A contradiction and the case $w(t) \in \mathcal{N}_2$ is impossible.

Now we assume that $w(t) \in \mathcal{N}_0$. Since $w(t)$ is positive and decreasing, there exists $\lim_{t\to\infty} w(t) = 2\ell \geq 0$. It follows from (2.2) that $\lim_{t\to\infty} x(t) = 2\ell$. If we assume that $\ell > 0$, then $x(\tau(t)) \geq \ell > 0$, eventually. An integration of (2.3) yields

$$b(t)\,\big(a(t)w'(t)\big)' \geq \int_{t}^{\infty} p(s)f\big(x(\tau(s))\big)\,\mathrm{d}s \geq f(\ell) \int_{t}^{\infty} p(s)\,\mathrm{d}s.$$

Integrating from t to ∞ and then from t_1 to ∞ one gets

$$w(t_1) \geq f(\ell) \int_{t_1}^{\infty} \frac{1}{a(v)} \int_{v}^{\infty} \frac{1}{b(s)} \int_{s}^{\infty} p(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}v,$$

which contradicts to (2.1) and the proof is complete. □

For a special case of (E) we have the following easily verifiable criterion.

Corollary 2.2. *Assume that* (2.1) *holds and*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)J(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}. \tag{P_1}$$

Then every solution of the trinomial differential equation

$$\Big(b(t)\,\big(a(t)x'(t)\big)'\Big)' + p(t)x(\tau(t)) - q(t)h(x(\sigma(t))) = 0 \tag{E_L}$$

either oscillates or converges to zero.

Proof. Theorem 2.1.1 in [7] guarantees oscillation (E_0) with $f(u) = u$. The assertion of the corollary now follows from Theorem 2.1. □

As a matter of fact we are able to provide a general criterion for the studied property of (E).

Corollary 2.3. *Assume that* (2.1) *holds,* $\tau(t)$ *is nondecreasing and*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)f\,(J(s))\,\mathrm{d}s > \limsup_{u\to 0} \frac{u}{f(u)}. \tag{P_2}$$

Then every solution of (E) *either oscillates or converges to zero.*

Proof. By Theorem 2.1 it is sufficient to show that (E_0) is oscillatory. Assume on the contrary that (E_0) possesses a nonoscillatory, let us say positive solution $y(t)$. It follows from (E_0) that $y'(t) < 0$. Thus, there exists $\lim_{t\to\infty} y(t) = c \geq 0$. An integration of ($E_0$) from $\tau(t)$ to t provides

$$\begin{aligned} y(\tau(t)) &= y(t) + \int_{\tau(t)}^{t} p(s) f(J(s)) f\big(y(\tau(s))\big)\,\mathrm{d}s \\ &\geq y(t) + f\big(y(\tau(t))\big) \int_{\tau(t)}^{t} p(s) f(J(s))\,\mathrm{d}s. \end{aligned}$$

The last inequality together with (P_2) implies that $c = 0$ and what is more,

$$\frac{y(\tau(t))}{f\big(y(\tau(t))\big)} \geq \int_{\tau(t)}^{t} p(s) f(J(s))\,\mathrm{d}s.$$

Taking limit superior on both sides, we get a contradiction with (P_2). □

For the function $f(u) = u^{\beta}$ we immediately get the following corollary.

Corollary 2.4. *Let $\beta \in (0,1)$. Assume that (2.1) holds, $\tau(t)$ is nondecreasing and*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s) J^{\beta}(s)\,\mathrm{d}s > 0. \tag{P_3}$$

Then every solution of

$$\Big(b(t)\,\big(a(t)x'(t)\big)'\Big)' + p(t)x^{\beta}(\tau(t)) - q(t)h(x(\sigma(t))) = 0 \tag{E_S}$$

either oscillates or converges to zero.

Example 2.5. Consider the third order trinomial differential equation

$$\left(t^{1/3}\left(t^{1/2}x'(t)\right)'\right)' + \frac{p}{t^{13/6}} x(\lambda\, t) - \frac{q}{t^3} \arctan\big(x(\sigma(t))\big) = 0, \tag{E_x}$$

with $p > 0$, $q > 0$, $\lambda \in (0,1)$. Now $h(u) = \arctan(u)$ is bounded, condition (2.1) holds true and (P_1) takes the form

$$p\lambda^{7/6} \ln\left(\frac{1}{\lambda}\right) > \frac{7}{9\,\mathrm{e}}, \tag{2.4}$$

which implies that every nonoscillatory solution of (E_x) tends to zero as $t \to \infty$. For $\lambda = 1/2$ condition (2.4) reduces to $p > 0.9267$.

Employing the additional condition, we achieve the oscillation of (E). We use the auxiliary function $\xi(t) \in C^1([t_0,\infty))$ satisfying

$$\xi'(t) > 0, \quad \xi(t) > t, \quad \eta(t) = \xi(\xi(\tau(t))) < t \tag{2.5}$$

and we use the notation

$$I(t) = \int_{\tau(t)}^{\xi(\tau(t))} \frac{1}{a(s)} \int_{s}^{\xi(s)} \frac{1}{b(u)}\,\mathrm{d}u\,\mathrm{d}s$$

and in the rest of this paper, we assume that

(H_7) $\displaystyle\int_{t_0}^{\infty} \frac{1}{a(t)} \int_{t_0}^{t} \frac{1}{b(s)} \int_{t_0}^{s} q(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t < \infty.$

Theorem 2.6. *Let (2.5) hold and (E_0) be oscillatory. Let the first order delay differential equation*

$$y'(t) + p(t)f\left(I(t)\right)f\big(y(\eta(t))\big) = 0 \tag{E_5}$$

be oscillatory. Then (E) is oscillatory.

Proof. Assume that (E) has a positive solution $x(t)$. Let $w(t)$ be defined by (2.2). Proceeding exactly as in the proof of Theorem 2.1, we verify that $w(t) \in \mathcal{N}_0$ and there exists a finite $\lim_{t\to\infty} w(t) = c \geq 0$. By ($H_5$), this implies $\lim_{t\to\infty} b(t)\left(a(t)w'(t)\right)' = \lim_{t\to\infty} a(t)w'(t) = 0$. Taking (2.2) into account, we see that

$$\lim_{t\to\infty} x(t) = c, \quad \lim_{t\to\infty} a(t)x'(t) = \lim_{t\to\infty} b(t)\left(a(t)x'(t)\right)' = 0. \tag{2.6}$$

We introduce another auxiliary function

$$z(t) = x(t) + \int_t^\infty \frac{1}{a(v)} \int_{t_1}^v \frac{1}{b(s)} \int_{t_1}^s q(u)h(x(\sigma(u)))\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}v. \tag{2.7}$$

Note that condition (H_7) and the fact that $h(u)$ is bounded implies that $z(t)$ exists for all t. It is easy to verify that

$$z(t) > x(t) > 0, \quad a(t)z'(t) < a(t)x'(t), \quad b(t)\left(a(t)z'(t)\right)' < b(t)\left(a(t)x'(t)\right)'$$

and

$$\Big(b(t)\left(a(t)z'(t)\right)'\Big)' = -p(t)f(x(\tau(t))) < 0. \tag{2.8}$$

Therefore, condition (H_5) together with a modification of Kiguradze's lemma implies that either

$$z(t) \in \mathcal{N}_0 \iff a(t)z'(t) < 0, \quad b(t)\left(a(t)z'(t)\right)' > 0,$$

or

$$z(t) \in \mathcal{N}_2 \iff a(t)z'(t) > 0, \quad b(t)\left(a(t)z'(t)\right)' > 0,$$

eventually. But, if we let $z(t) \in \mathcal{N}_2$, then condition $\lim_{t\to\infty} a(t)x'(t) = 0$ together with $a(t)z'(t) < a(t)x'(t)$ implies $\lim_{t\to\infty} a(t)z'(t) = 0$. A contradiction and we conclude that $z(t) \in \mathcal{N}_0$.

On the other hand, an integration of $(a(t)z'(t))' < (a(t)x'(t))'$ from t to ∞ in view of (2.6) yields

$$-a(t)x'(t) \geq \int_t^\infty b(s)\left(a(s)z'(s)\right)' \frac{1}{b(s)}\,\mathrm{d}s \geq \int_t^{\xi(t)} b(s)\left(a(s)z'(s)\right)' \frac{1}{b(s)}\,\mathrm{d}s.$$

Using the monotonicity of $y(t) = b(t)(a(t)z'(t))'$, the last inequality implies

$$-a(t)x'(t) \geq y(\xi(t)) \int_t^{\xi(t)} \frac{1}{b(s)}\,\mathrm{d}s.$$

Dividing by $a(t)$ and integrating form $\tau(t)$ to $\xi(\tau(t))$, we have

$$x(\tau(t)) \geq \int_{\tau(t)}^{\xi(\tau(t))} \frac{y(\xi(u))}{a(u)} \int_u^{\xi(u)} \frac{1}{b(s)}\,\mathrm{d}s\,\mathrm{d}u \geq y(\eta(t))I(t).$$

Setting into (2.8), one can see that $y(t) = b(t)(a(t)z'(t))'$ is a positive solution of differential inequality

$$y'(t) + p(t)f\left(I(t)\right)f\big(y(\eta(t))\big) \leq 0.$$

It follows from Theorem 1 in [8] that the corresponding differential equation (E_5) also has a positive solution. A contradiction and thus the case $z(t) \in \mathcal{N}_2$ is also impossible and we conclude that (E) is oscillatory. □

Remark 2.7. Employing sufficient conditions for oscillation of (E_5) together with those for (E_0), we obtain oscillatory criteria for (E).

The following results are obvious.

Corollary 2.8. *Assume that* (2.5) *and* (P_1) *hold. If*

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} p(s)I(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{P_4}$$

then (E_L) *is oscillatory.*

Corollary 2.9. *Assume that* (2.5), (P_2) *holds and* $\tau(t)$ *is nondecreasing. If*

$$\limsup_{t\to\infty} \int_{\eta(t)}^{t} p(s)f\left(I(s)\right)\,\mathrm{d}s > \limsup_{u\to 0} \frac{u}{f(u)}, \tag{P_5}$$

then (E) *is oscillatory.*

Corollary 2.10. *Let* $\beta \in (0,1)$*. Assume that* (2.5), (P_3) *hold and* $\tau(t)$ *is nondecreasing. If*

$$\limsup_{t\to\infty} \int_{\eta(t)}^{t} p(s)I^{\beta}(s)\,\mathrm{d}s > 0, \tag{P_6}$$

then (E_S) *oscillates.*

Example 2.11. Consider once more the differential equation

$$\left(t^{1/3}\left(t^{1/2}x'(t)\right)'\right)' + \frac{p}{t^{13/6}}x(\lambda\, t) - \frac{q}{t^3}\arctan\left(x\left(\sigma(t)\right)\right) = 0. \tag{E_x}$$

We set $\xi(t) = \alpha\, t$, where $\alpha = \sqrt{\dfrac{\lambda+1}{2\lambda}}$. Then $\eta(t) = \dfrac{\lambda+1}{2}$ and

$$I(t) = \frac{9}{7}\left(\alpha^{2/3} - 1\right)\left(\alpha^{7/6} - 1\right)\lambda^{7/6}\, t^{7/6}.$$

Simple computation reveals that (P_4) takes the form

$$\frac{9p}{7}\left(\alpha^{2/3} - 1\right)\left(\alpha^{7/6} - 1\right)\lambda^{7/6}\ln\frac{2}{1+\lambda} > \frac{1}{\mathrm{e}}. \tag{2.9}$$

By Corollary 2.8, (E_x) is oscillatory if both conditions (2.4) and (2.9) are satisfied. For $\lambda = 1/2$ it happens provided that $p > 57.8225$.

3 Comparison with existing results

The results obtained provide a new technique for studying oscillation and asymptotic properties of trinomial third order differential equations with positive and negative terms via oscillation of a suitable first order equations.

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