Electronic Journal of Qualitative Theory of Differential Equations
2008, No. **19**, 1-10; **http://www.math.u-szeged.hu/ejqtde/**

OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF A NEUTRAL DIFFERENTIAL EQUATION WITH OSCILLATING COEFFICIENTS

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ABSTRACT. In this paper, we obtain sufficient conditions so that every solution of

$$\big(y(t) - \sum_{i=1}^{n} p_i(t)y(\delta_i(t))\big)' + \sum_{i=1}^{m} q_i(t)y(\sigma_i(t)) = f(t)$$

oscillates or tends to zero as $t \to \infty$. Here the coefficients $p_i(t), q_i(t)$ and the forcing term $f(t)$ are allowed to oscillate; such oscillation condition in all coefficients is very rare in the literature. Furthermore, this paper provides an answer to the open problem 2.8.3 in [7, p. 57]. Suitable examples are included to illustrate our results.

1. INTRODUCTION

During the previous two decades, oscillation of solutions to neutral delay differential equations has been studied extensively. In this article, we extend some results from equation with fixed-sign coefficients to equations with oscillating coefficients. In particular, we obtain sufficient conditions for every solution of the first-order non-homogeneous nonlinear neutral delay differential equation

$$\big(y(t) - \sum_{i=1}^{n} p_i(t)y(\delta_i(t))\big)' + \sum_{i=1}^{m} q_i(t)g_i\big(y(\sigma_i(t))\big) = f(t), \tag{1.1}$$

to oscillate or to tend to zero as t tends to infinity. Here $f, g, p_i, q_i, \delta_i, \sigma_i$ are continuous, p_i, δ_i are differentiable, and f, g, p_i, q_i can assume positive and negative values.

The main motivation of this work is the open problem [7, Problem 2.8.3, p.57]:

> Extend the following result to equations with oscillating coefficients.
> Theorem 2.3.1 in [7]: Under the assumptions that $q(t) \geq 0$ and
>
> $$\liminf_{t\to\infty} \int_{t-\tau}^{t} q(s)ds > e^{-1} \tag{1.2}$$
>
> every solution of
>
> $$y'(t) + q(t)y(t-\tau) = 0, \quad t \geq t_0 \tag{1.3}$$
>
> oscillates.

2000 *Mathematics Subject Classification.* 34c10, 34c15, 34k40.
Key words and phrases. Oscillatory solution; neutral differential equation; asymptotic behaviour.

In most of the references it is assumed that the coefficients p_i, q_i are positive [4, 8, 9, 10, 11, 12]. However, in [1, 6], p_i oscillates, but the sign of q_i remains constant. In [3, Theorem 6(ii)], p oscillates, but the proof is wrong because the conditions needed to apply [7, Lemma 2.2] are not met. In [9, Theorem 2.4], p is periodic, oscillates and is restricted by inequalities similar to (H2).

It seems, that (1.1) is least studied when the functions q_i oscillate. This is so because the techniques used in the other cases fail. Ladde [5, Theorem 2.2.2] shows that (1.3) has only oscillatory solutions when $q(t) > 0$ on a sequence of intervals of length 2τ, whose end points approach $+\infty$, and $\int_{t-\tau}^{t} q(s)\,ds > 1/e$ on the right half of those intervals.

Our approach here is to separate the positive part and the negative part of the function q_i. Our assumptions are stated as follows:

(H1) $\delta_i(t) \le t$, $\lim_{t\to\infty} \delta_i(t) = \infty$, $\sigma_i(t) \le t$, $\lim_{t\to\infty} \sigma_i(t) = \infty$ for all i.
(H2) There exist constants $t_0 \ge 0$, $r_i \le 0$ and $R_i \ge 0$ such that $\sum_{i=1}^{n}(R_i - r_i) < 1$ and $r_i \le p_i(t) \le R_i$ for $t \ge t_0$.
(H3) The functions g_i are bounded.
(H4) $yg_i(y) > 0$ for $y \ne 0$ and $i = 1, \dots m$.
(H5) $\int_0^\infty \sum_{i=1}^{m} q_i^+(s)\,ds = \infty$, where $q^+(t) = \max\{q(t), 0\}$.
(H6) $\int_0^\infty \sum_{i=1}^{m} q_i^-(s)\,ds < \infty$, where $q^-(t) = \max\{-q(t), 0\}$.
(H7) $\int_0^\infty |f(s)|\,ds < \infty$.

A prototype of a function satisfying (H3)-(H4) is $g(u) = ue^{-u^2}$ which decreases for some values of u; therefore the results in [1, 4, 8, 9, 10, 11, 12] can not be applied here.

From the definitions of the functions $q^+(t)$ and $q^-(t)$, it follows that $q^+(t) \ge 0$, $q^-(t) \ge 0$, and $q(t) = q^+(t) - q^-(t)$. Then using this decomposition, (1.1) can be rewritten as

$$\big(y(t) - \sum_{i=1}^{n} p_i(t)y(\delta_i(t))\big)' + \sum_{i=1}^{m} q_i^+(t)g_i\big(y(\sigma_i(t))\big) - \sum_{i=1}^{m} q_i^-(t)g_i\big(y(\sigma_i(t))\big) = f(t)\,. \tag{1.4}$$

By a solution y of (1.1), we mean a real-valued function which is continuous and differentiable on some interval $[t_y, \infty)$, such that (1.1) is satisfied. As far as existence and uniqueness of solutions we refer the reader to [7]. In this work we assume the existence of solutions and study only their qualitative behaviour.

A solution of (1.1), is said to be oscillatory if it has arbitrarily large zeros. Otherwise it is said to be non oscillatory. In the sequel, unless otherwise specified, when we write a functional inequality, it will be assumed to hold for all sufficiently large values of t.

2. Main results

Theorem 2.1. *Under assumptions* (H1)–(H7)*, every solution of* (1.1) *oscillates or tends to zero as* $t \to \infty$.

Proof. We shall show that for every solutions which does not oscillate, approaches zero as $t \to \infty$.

Case 1: There exits t_0 such that $y(t) > 0$ for $t \geq t_0$. If necessary, increment the t_0 here to exceed the one in (H2), and by (H1), to have

$$y(\delta_i(t)) > 0, \quad y(\sigma_i(t)) > 0, \quad \text{for } t \geq t_0\,. \tag{2.1}$$

For simplicity of notation, define

$$z(t) = y(t) - \sum_{i=1}^{n} p_i(t) y(\delta_i(t))\,.$$

Using that $\int_0^\infty q_i^- < \infty$ and that g_i is bounded, we define

$$w(t) = z(t) + \int_t^\infty \sum_{i=1}^{m} q_i^-(s) g_i(y(\sigma_i(s)))\,ds - \int_{t_0}^t f(s)\,ds\,.$$

Then using (1.4),

$$w'(t) = -\sum_{i=1}^{m} q_i^+ g_i(y(\sigma_i(t)))\,. \tag{2.2}$$

Since $y > 0$, by (H4), $w'(t) \leq 0$; so that $w(t)$ is non-increasing. Then

$$w(t_0) \geq w(t) \geq y(t) - \sum_{i=1}^{n} p_i(t) y(\delta_i(t)) - \int_{t_0}^t f(s)\,ds\,.$$

By (H7) the function $\int_0^t f(s)\,ds$ is bounded, and by (2.1),

$$w(t_0) + \sup_{t\geq t_0} \int_{t_0}^t f(s)\,ds \geq y(t) - \sum_{i=1}^{n} p_i(t) y(\delta_i(t)) \geq y(t) - \sum_{i=1}^{n} R_i y(\delta_i(t))\,. \tag{2.3}$$

Using a contradiction argument, we prove that $y(t)$ is bounded above (y is continuous on $[0, t_0]$ and is bounded below by zero on $[t_0, \infty)$). Assuming that y is unbounded, we define a sequence $\{t_k\}_{k=1}^\infty$ such that $t_k \to \infty$ and $y(t_k) = \max\{y(t) : t_0 \leq t \leq t_k\}$. Then $y(t_k) \to \infty$ and by (H1), for each i, $y(\delta_i(t_k)) \to \infty$ as $k \to \infty$. Since $\delta_i(t) \leq t$, from (2.3), it follows that for each t_k,

$$w(t_0) + \sup_{t\geq t_0} \int_{t_0}^t f(s)\,ds \geq (1 - \sum_{i=1}^{n} R_i) y(t_k)\,.$$

By (H2), $(1 - \sum_{i=1}^n R_i) > 0$; so that the right-hand side approaches $+\infty$, as $k \to \infty$. This is a contradiction that proves y being bounded.

Since y and p_i are bounded functions, so is z. Then by (H6), (H7), it follows that w is bounded. Since w is bounded and non-increasing, it must converge as $t \to \infty$. Also by the definition of $w(t)$, the function $z(t)$ converges. Let

$$l := \lim_{t\to\infty} z(t) = \lim_{t\to\infty} \Big(y(t) - \sum_{i=1}^{n} p_i(t) y(\delta_i(t))\Big). \tag{2.4}$$

Now, using a contradiction argument, we show that $\liminf_{t\to\infty} y(t) = 0$. Suppose $\liminf_{t\to\infty} y(t) > 0$. Then by (H1), $\liminf_{t\to\infty} y(\sigma_i(t)) > 0$. From the definition of $\liminf$, there exist constants y_1 and t_1 such that $y(\sigma_i(t)) \geq y_1 > 0$ for all $t \geq t_1$, and all $1 \leq i \leq m$. Let y_2 be an upper bound for y. Because g_i's are

continuous, by (H4), there exists a positive lower bound m_1 for all g_i's on $[y_1, y_2]$; i.e., $0 < m_1 \le g_i(y(\sigma_i(s)))$ for all $s \ge t_1$, $i = 1, \dots, m$. Then integrating (2.2),

$$w(t_1) - w(t) = \int_{t_1}^{t} \sum_{i=1}^{m} q_i^+(s) g_i\big(y(\sigma_i(s))\big)\, ds \ge m_1 \int_{t_1}^{t} \sum_{i=1}^{m} q_i^+(s)\, ds\,.$$

Since the left-hand side is a bounded function while, by (H5), the right-hand side approaches $+\infty$, we have a contradiction. Therefore, $\liminf_{t\to\infty} y(t) = 0$.

Now we prove that $\limsup_{t\to\infty} y(t) = 0$. Since $y \ge 0$, from assumption (H2), it follows that $y(t) - \sum_{i=1}^{n} p_i(t) y(\delta_i(t)) \ge y(t) - \sum_{i=1}^{n} R_i y(\delta_i(t))$. Recall that for bounded functions, $\limsup\{f(t)+g(t)\} \ge \limsup\{f(t)\} + \liminf\{g(t)\}$. Taking the lim sup in (2.4), we have

$$\begin{aligned} l &\ge \limsup_{t\to\infty}\{y(t) + \sum_{i=1}^{n} -R_i y(\delta_i(t))\} \\ &\ge \limsup_{t\to\infty}\{y(t)\} + \sum_{i=1}^{n} \liminf_{t\to\infty}\{-R_i y(t)\} \\ &= \limsup_{t\to\infty}\{y(t)\} - \sum_{i=1}^{n} R_i \limsup_{t\to\infty}\{y(t)\} \\ &= (1 - \sum_{i=1}^{n} R_i) \limsup_{t\to\infty}\{y(t)\}\,. \end{aligned} \tag{2.5}$$

In the equality above, we use that $-R_i \le 0$.

Since $y \ge 0$, from assumption (H2), it follows that $y(t) - \sum_{i=1}^{n} p_i(t) y(\delta_i(t)) \le y(t) - \sum_{i=1}^{n} r_i y(\delta_i(t))$. Recall that for bounded functions, $\liminf\{f(t) + g(t)\} \le \liminf\{f(t)\} + \limsup\{g(t)\}$. Taking lim inf in (2.4), we have

$$\begin{aligned} l &\le \liminf_{t\to\infty}\{y(t) + \sum_{i=1}^{n} -r_i y(\delta_i(t))\} \\ &\le \liminf_{t\to\infty}\{y(t)\} + \sum_{i=1}^{n} \limsup_{t\to\infty}\{-r_i y(t)\} \\ &= 0 - \sum_{i=1}^{n} r_i \limsup_{t\to\infty}\{y(t)\}\,. \end{aligned}$$

In the the equality above, we use that $-r_i \ge 0$. From (2.5) and the above inequality,

$$\big(1 - \sum_{i=1}^{n} (R_i - r_i)\big) \limsup_{t\to\infty}\{y(t)\} \le 0\,.$$

Since $y \ge 0$, by (H2), it follows that $\limsup_{t\to\infty}\{y(t)\} = 0$. The proof of case 1 is complete.

Case 2: There exists t_0 such that $y(t) < 0$ for all $t \ge t_0$. If necessary increment the t_0 here to exceed the one in (H2), and by (H1) to have

$$y(\delta_i(t)) < 0, \quad y(\sigma_i(t)) < 0, \quad \text{for } t \ge t_0\,. \tag{2.6}$$

We define $z(t)$ and $w(t)$ as in case 1. Then $w'(t) \geq 0$ and $w(t)$ is non-decreasing; so that

$$w(t_0) \leq w(t) \leq y(t) - \sum_{i=1}^{n} p_i(t) y(\delta_i(t)) - \int_{t_0}^{t} f(s)\, ds\,.$$

By (H2),

$$w(t_0) + \inf_{t \geq t_0} \int_{t_0}^{t} f(s)\, ds \leq y(t) - \sum_{i=1}^{n} p_i(t) y(\delta_i(t)) \leq y(t) - \sum_{i=1}^{n} R_i y(\delta_i(t))\,. \quad (2.7)$$

We claim that $y(t)$ is bounded (y is continuous on $[0, t_0]$ and is bounded above by zero on $[t_0, \infty)$). On the contrary suppose that y is unbounded, we define a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to \infty$ and $y(t_k) = \min\{y(t) : t_0 \leq t \leq t_k\}$. Then $y(t_k) \to -\infty$ and by (H1), for each i, $y(\delta_i(t_k)) \to -\infty$ as $k \to \infty$. From (2.7), it follows that for each t_k,

$$w(t_0) + \sup_{t \geq t_0} F(t) \leq (1 - \sum_{i=1}^{n} R_i) y(t_k)\,.$$

By (H2), $(1 - \sum_{i=1}^{n} R_i) > 0$, so that the right-hand side approaches $-\infty$. This contradiction implies y being bounded.

Since y and p_i are bounded functions, so is z. Then by (H6), (H7), it follows that w is bounded. Since w is bounded and non-decreasing, it must converge as $t \to \infty$. Also by the definition of $w(t)$, it follows that $z(t)$ converges. Let

$$l := \lim_{t \to \infty} z(t) = \lim_{t \to \infty} \Big(y(t) - \sum_{i=1}^{n} -p_i(t) y(\delta_i(t)) \Big)\,. \quad (2.8)$$

Now, using a contradiction argument, we prove $\limsup_{t\to\infty} y(t) = 0$. Suppose $\limsup_{t\to\infty} y(t) < 0$. Then by (H1), $\limsup_{t\to\infty} y(\sigma_i(t)) < 0$. From the definition of lim sup, there exist constants y_2 and t_1 such that $y(\sigma_i(t)) \leq y_2 < 0$ for all $t \geq t_1$, and all $1 \leq i \leq n$. Let y_1 be a lower bound for y. Then there exists a negative upper bound m_2 for all g_i's on $[y_2, y_1]$; i.e., $g_i(y(\sigma_i(s))) \leq m_2 < 0$ for all $s \geq t_1$, $i = 1, \ldots m$. Then integrating on (2.2),

$$w(t) - w(t_1) = -\int_{t_1}^{t} \sum_{i=1}^{m} q_i^+(s) g_i\big(y(\sigma_i(s))\big)\, ds \geq -m_2 \int_{t_1}^{t} \sum_{i=1}^{m} q_i^+(s)\, ds\,.$$

The left-hand side is a bounded function and, by (H5), the right-hand side approaches $+\infty$ as $t \to \infty$. This contradiction implies $\limsup_{t\to\infty} y(t) = 0$.

Now we prove that $\liminf_{t\to\infty} y(t) = 0$. Since $y \leq 0$, from assumption (H2), it follows that $y(t) - \sum_{i=1}^{n} p_i(t) y(\delta_i(t)) \geq y(t) - \sum_{i=1}^{n} r_i y(\delta_i(t))$. Recall that for bounded functions, $\limsup\{f(t) + g(t)\} \geq \limsup\{f(t)\} + \liminf\{g(t)\}$. Taking the

lim sup in (2.8), we have

$$\begin{aligned} l &\geq \limsup_{t\to\infty}\{y(t) + \sum_{i=1}^{n} -r_i y(\delta_i(t))\} \\ &\geq \limsup_{t\to\infty}\{y(t)\} + \sum_{i=1}^{n} \liminf_{t\to\infty}\{-r_i y(t)\} \\ &= 0 - \sum_{i=1}^{n} r_i \liminf_{t\to\infty}\{y(t)\}\,. \end{aligned} \tag{2.9}$$

In the equality above, we use that $-r_i \geq 0$.

Since $y \leq 0$, from assumption (H2), it follows that $y(t) - \sum_{i=1}^{n} p_i(t)y(\delta_i(t)) \leq y(t) - \sum_{i=1}^{n} R_i y(\delta_i(t))$. Recall that for bounded functions, $\liminf\{f(t)+g(t)\} \leq \liminf\{f(t)\} + \limsup\{g(t)\}$. Taking lim inf in (2.4), we have

$$\begin{aligned} l &\leq \liminf_{t\to\infty}\{y(t) + \sum_{i=1}^{n} -R_i y(\delta_i(t))\} \\ &\leq \liminf_{t\to\infty}\{y(t)\} + \sum_{i=1}^{n} \limsup_{t\to\infty}\{-R_i y(t)\} \\ &= \liminf_{t\to\infty}\{y(t)\} + \sum_{i=1}^{n} -R_i \liminf_{t\to\infty}\{y(t)\} \\ &= (1 - \sum_{i=1}^{n} R_i) \liminf_{t\to\infty}\{y(t)\}\,. \end{aligned}$$

For the equality above, we use that $-R_i \leq 0$. From the (2.9) and the above inequality,

$$0 \leq \big(1 - \sum_{i=1}^{n}(R_i - r_i)\big) \liminf_{t\to\infty}\{y(t).$$

Since $y \leq 0$, by (H2), it follows that $\liminf_{t\to\infty}\{y(t)\} = 0$. The proof of case 2 is complete.

In summary, every solution does not oscillate approaches zero. □

Note that in the above theorem, (H3) requires g_i being bounded. However, the open problem in [7] does not satisfy this condition. To address this shortcoming, we introduce the following hypotheses, and state another theorem.

(H8) There exists a positive constant τ such that $\delta_i(t) \leq t - \tau$ and $\sigma_i(t) \leq t - \tau$ for all $t \geq 0$ and all i's.

(H9) There exist non-negative constants a, b such that

$$|g_i(u)| \leq a|u| + b \quad \text{for all } u,\ 1 \leq i \leq m\,.$$

Theorem 2.2. *Assume* (H1)-(H2), (H4)-(H9) *hold. Then every solution of* (1.1) *oscillates or tends to zero as* $t \to \infty$.

Proof. As in Theorem 2.1, we prove that every solution which does not oscillate, approaches zero as $t \to \infty$.

Case 1 There exists a t_0 such that $y(t) > 0$ for $t \geq t_0$. If necessary increment t_0 so that (2.1) is satisfied, and, by (H6),

$$\alpha := (a+b) \int_{t_0}^{\infty} \sum_{i=1}^{m} q_i^-(s)\, ds < 1 - \sum_{i=1}^{n} R_i \,. \tag{2.10}$$

Using that $\delta_i(t)$ and $\sigma_i(t)$ are continuous and both tend to ∞ as $t \to \infty$, we define the values $\delta_0 = \inf\{\delta_i(t) : t \geq t_0,\ 1 \leq i \leq n\}$ and $\sigma_0 = \inf\{\sigma_i(t) : t \geq t_0,\ 1 \leq i \leq m\}$. Select a constant M large enough such that

$$\begin{gathered} 1 \leq M \,, \\ |y(t)| \leq M \quad \text{for } \min\{\delta_0, \sigma_0\} \leq t \leq t_0 \,, \\ \alpha + \sum_{i=1}^{n} R_i \leq \frac{M}{M + |y(t_0)| + |\sum_{i=1}^{n} p_i(t_0) y(\delta_i(t_0))| + \int_{t_0}^{\infty} |f|} \,. \end{gathered} \tag{2.11}$$

Such M exists that

$$0 \leq y(t) \leq M + |y(t_0)| + |\sum_{i=1}^{n} p_i(t_0) y(\delta_i(t_0))| + \int_{t_0}^{\infty} |f| := M_2 \quad \text{for } t \leq t_0 \tag{2.12}$$

by (H2), (2.10) and $0 \leq \alpha + \sum_{i=1}^{n} R_i < 1$. Then Now for $t_0 \leq t \leq t_0 + \tau$, we integrate (1.4), to obtain

$$\begin{aligned} y(t) &= \sum_{i=1}^{n} p_i(t) y(\delta_i(t)) + y(t_0) - \sum_{i=1}^{n} p_i(t_0) y(\delta_i(t_0)) \\ &\quad - \int_{t_0}^{t} \sum_{i=1}^{m} q_i^+(s) g_i(y(\sigma_i(s)))\, ds + \int_{t_0}^{t} \sum_{i=1}^{m} q_i^-(s) g_i(y(\sigma_i(s)))\, ds + \int_{t_0}^{t} f(s)\, ds \,. \end{aligned} \tag{2.13}$$

Because of $\delta_i(t) \leq t - \tau$ and $\sigma_i(t) \leq t - \tau$, we can use (2.12) to estimate each term in the above expression. Using $p_i(t) \leq R_i$, we obtain

$$|\sum_{i=1}^{n} p_i(t) y(\delta_i(t))| \leq \sum_{i=1}^{n} R_i M_2 \,.$$

Since $y g_i(y) > 0$, the fourth term on the right-hand side of (2.13) can be estimated by zero. Using that $y(\sigma_i(s)) \leq M_2$ and $M_2 \geq 1$, by (H9), we have $|g(y(\sigma_i(s)))| \leq a|y(\sigma_i(s))| + b \leq aM_2 + b \leq (a+b)M_2$. Then by (2.10),

$$\Big| \int_{t_0}^{t} \sum_{i=1}^{m} q_i^-(s) g_i(y(\sigma_i(s)))\, ds \Big| \leq \alpha M_2 \,.$$

From the two inequalities above, (2.13), and (2.11) we obtain

$$0 \leq y(t) \leq \big(\alpha + \sum_{i=1}^{n} R_i\big) M_2 + |y(t_0)| + |\sum_{i=1}^{n} p_i(t_0) y(\delta_i(t_0))| + \int_{t_0}^{\infty} |f| \leq M_2$$

for $t_0 \leq t \leq t_0 + \tau$. Recursively, we show that $0 \leq y(t) \leq M_2$ on the intervals $[t_0 + \tau, t_0 + 2\tau]$, $[t_0 + 2\tau, t_0 + 3\tau]$, ...

Next we define $z(t)$ and $w(t)$ as in Theorem 2.1. Then prove that $\lim_{t\to\infty} y(t) = 0$ by the same methods as in the proof of Theorem 2.1.

Case2 There exists t_0 such that $y(t) < 0$ for $t \geq t_0$. The proof is similar to the proof of case 1; so we just sketch it. If necessary increment t_0 so that (2.6) and (2.10) are satisfied. Let M be defined as in (2.11). Then

$$0 \leq -y(t) \leq M + |y(t_0)| + |\sum_{i=1}^{n} p_i(t_0) y(\delta_i(t_0))| + \int_{t_0}^{\infty} |f| := M_2 \quad \text{for } t \leq t_0\,. \tag{2.14}$$

Now for $t_0 \leq t \leq t_0 + \tau$, from (1.4), we have

$$\begin{aligned} -y(t) &= \sum_{i=1}^{n} p_i(t)(-y(\delta_i(t))) - y(t_0) + \sum_{i=1}^{n} p_i(t_0) y(\delta_i(t_0)) \\ &\quad + \int_{t_0}^{t} \sum_{i=1}^{m} q_i^+(s) g_i(y(\sigma_i(s)))\, ds - \int_{t_0}^{t} \sum_{i=1}^{m} q_i^-(s) g_i(y(\sigma_i(s)))\, ds - \int_{t_0}^{t} f(s)\, ds\,. \end{aligned}$$

Because $y g_i(y) > 0$, the fourth term on the right-hand side can be estimated by zero. Because $\delta_i(t) \leq t - \tau$ and $\sigma_i(t) \leq t - \tau$, we can use (2.14) to obtain

$$0 \leq -y(t) \leq (\alpha + \sum_{i=1}^{n} R_i) M_2 + |y(t_0)| + |\sum_{i=1}^{n} p_i(t_0) y(\delta_i(t_0))| + \int_{t_0}^{\infty} |f| \leq M_2,$$

for $t_0 \leq t \leq t_0 + \tau$. Recursively, we show that $0 \leq -y(t) \leq M_2$ on the intervals $[t_0 + \tau, t_0 + 2\tau]$, $[t_0 + 2\tau, t_0 + 3\tau]$, ... For the rest of the proof, we proceed as in Theorem 2.1. □

The results in Theorems 2.1 and 2.2 hold for bounded solutions, as follows.

Theorem 2.3. *Under assumptions* (H1)-(H2), (H4)–(H7)*, every bounded solution of* (1.1) *oscillates or tends to zero as* $t \to \infty$*.*

Regarding the open problem in [7], we have the following result.

Corollary 2.4. *Assume that*

$$\int_0^{\infty} q^+(s)\, ds = \infty, \quad \text{and} \quad \int_0^{\infty} q^-(s)\, ds < \infty\,.$$

Then every solution of (1.3) *oscillates or tends to zero as* $t \to \infty$*.*

Proof. The delay equation (1.3) is a particular case of (1.1) where $n = m = 1$, $p_1(t) = 0$, $q_1(t) = q(t)$, $g_1(u) = u$, $\sigma_1(t) = t - \tau$, and $f(t) = 0$. Condition (H3) is not satisfied, but (H9) is satisfied with $a = 1$, $b = 0$. Since (H1)-(H2), (H4)-(H9) are satisfied, we apply Theorem 2.2 and obtain the desired result. □

Remark 2.5. Condition (1.2) implies (H5). In fact, from the definition of lim inf, there exists t_0 such that for $t \geq t_0$,

$$\int_{t-\tau}^{t} q^+(s)\, ds \geq \int_{t-\tau}^{t} q(s)\, ds \geq \frac{1}{2e}\,.$$

Partitioning the interval of integration $[t_0, \infty)$ in intervals of length τ, we have

$$\int_{t_0}^{\infty} q^+(s)\, ds = \int_{t_0}^{t_0+\tau} q^+(s)\, ds + \int_{t_0+\tau}^{t_0+2\tau} q^+(s)\, ds + \cdots \geq \sum_{i=1}^{\infty} \frac{1}{2e} = \infty$$

which implies (H5). Also note that (1.2) does not imply (H6).

Example. To present an equation where Theorem 2.2 applies, we define

$$q(t) = \big(\sin(t)\big)^+ - \frac{1}{t^2+1}\big(\sin(t)\big)^-.$$

Then $q^+(t) = \big(\sin(t)\big)^+$ and $\int_0^\infty q^+ = \infty$; so (H5) is satisfied. Also $0 \le q^-(t) = \frac{1}{t^2+1}\big(\sin(t)\big)^- \le \frac{1}{t^2+1}$ and $\int_0^\infty q^- \le \pi/2$; so (H6) is satisfied. Consider the delay equation

$$y'(t) + q(t)y(t-1) = e^{-t}\big(q(t)e - 1\big)\,.$$

Since q is bounded, the right-hand side is integrable, in absolute value, and (H7) is satisfied. In fact, (H1)-(H2), (H4)-(H9) are satisfied and the solution to the above equation is $y(t) = e^{-t}$ which approaches zero as $t \to \infty$.

Example. To emphasize the need for (H3), or for (H6), we present the delay equation

$$y'(t) - \cos(t)y(t-2\pi) = 0\,,$$

where $q(t) = -\cos(t)$ which does not satisfy (H6), and $g(y) = y$ which does not satisfy (H3). Note that (H1), (H2), (H4), (H5), (H7)–(H9) hold, but we can not apply Theorem 2.1 or Theorem 2.2. Also note that the solution is $y(t) = \exp\sin(t)$ which does not oscillate and does not tend to zero as $t \to \infty$.

Acknowledgments. The authors are thankful to the anonymous referee for his/her helpful comments to improve the presentation of this paper.

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(Received February 11, 2008)

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