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On a Parabolic Strongly Nonlinear Problem on Manifolds

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Dedicated to Professor Marcondes Clark on the occasion of his 54th birthday

Abstract: In this work we will prove the existence uniqueness and asymptotic behavior of weak solutions for the system (*) involving the pseudo Laplacian operator and the condition $\frac{\partial u}{\partial t} + \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\nu_i + |u|^\rho u = f$ on Σ_1, where Σ_1 is part of the lateral boundary of the cylinder $Q = \Omega \times (0, T)$ and f is a given function defined on Σ_1.

Key Words: Weak solutions, manifolds, monotonic operator, compactness.

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^n (n \geq 1)$ with smooth boundary Γ. We consider $\{\Gamma_0, \Gamma_1\}$ a partition of Γ, that is, $\Gamma = \Gamma_0 \cup \Gamma_1$, with Γ_0 and Γ_1 having positive Lebesgue measure and with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

Let ν be the outward normal to Γ and $T > 0$ is a real number. We denote by $Q = \Omega \times (0, T)$ the cylinder of the $\mathbb{R}^{n+1}$.

The goal of this work is to solve following strongly nonlinear boundary problem:

$$(*) \left| \begin{array}{lr} \mathcal{A}u = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ u = 0 & \text{on} \quad \Sigma_0 = \Gamma_0 \times (0, T) \\ \dfrac{\partial u}{\partial t} + \displaystyle\sum_{i=1}^{n} \Big|\frac{\partial u}{\partial x_i}\Big|^{p-2} \frac{\partial u}{\partial x_i} \nu_i + |u|^\rho u = f & \text{on} \quad \Sigma_1 = \Gamma_1 \times (0, T) \\ u(x, 0) = u_0(x) & \text{on } \Gamma, \end{array} \right.$$

where $\mathcal{A}$ is the pseudo Laplacian operator defined by

$$\begin{array}{rcccl} \mathcal{A} & : & W_0^{1,p}(\Omega) & \rightarrow & W^{-1,p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1 \\ & & w & \mapsto & \mathcal{A}w \end{array}$$

with $\mathcal{A}w = -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Big(\Big|\frac{\partial w}{\partial x_i}\Big|^{p-2} \frac{\partial w}{\partial x_i} \Big)$, $2 < p < \infty$; ρ is a positive real constant satisfying the conditions (1) and f is a known real value function.

As the solution of system depend of x and t and the equation $(*)_1$ does not have temporal derivative of the function u, this system is not Cauchy-Kovalevsky type.

This problem associated with evolution equation on lateral boundary, with $p = 2$, was study in Araruna-Antunes-Medeiros [1] and Domingos-Cavalcante [4], both motivated by the idea applied in Lions ([6], pp. 134), which consists to reduce the problem in a model of mathematical physics on the manifolds Σ_1. Also, Araruna-Araujo in [2] studied the system $(*)$ in your form more simple, that is, $p = 2$. Recently, O.A.Lima, at al has been researching in

Partial Differential Equations involving the pseudo Laplacian operator [10]. In this work we use a technique due to Lions [6], which transforms the system $(*)$ in a Cauchy-Kovalevsky type one by means of a suitable perturbations in the equation $(*)_1$. The solution of $(*)$ is obtained as limit of solutions of the perturbed problem.

For $p > 2$, the operator $\mathcal{A}$ brings great difficulties, because it is non-linear, mainly to establish concepts of solutions, in passage to the limit, to work with the trace application and immersion in spaces $W^{s,p}(\Omega)$, $s \in \mathbb{R}$ (for this we consult Něcas [5]) and to obtain a estimative for derived of the approximate function(here we use strongly the proprieties of the trace application). Finally all the difficulties will be overcome through careful handling of the proprieties of the operator $\mathcal{A}$.

This paper is organized as follows: In Section 1, we will give some notations, hypothesis and results. In Section 2, we will introduce the perturbed problem. In Section 3, we will prove the existence of the solutions for the perturbed problem. In Section 4, we will treat of the uniqueness for the solution of the perturbed problem. Finally in Section 5, we will prove the main result of this work.

2 Hypotheses and Notations

We denote by $W^{\frac{1}{p'},p}(\Gamma)$ the vectorial space of functions $v|_\Gamma$ when $v \in W^{1,p}(\Omega)$, for $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. By $W^{-\frac{1}{p'},p'}(\Gamma)$ denotes the dual of the space $W^{\frac{1}{p'},p}(\Gamma)$.
Let $p > 2$ be and V_0 the Banach space given by $V_0 = \left\{ v \in W^{1,p}(\Omega);\ v|_{\Gamma_0} = 0 \right\}$ equipped with the norm $\|v\|_{V_0} = \left(\displaystyle\sum_{i=1}^{n} \int_\Omega |\frac{\partial v}{\partial x_i}|^p \, dx \right)^{\frac{1}{p}}$. Note that the application $\gamma : V_0 \longrightarrow W^{\frac{1}{p'},p}(\Gamma_1)$ is linear, continuous and surjective.

In $(*)$ assume that ρ is a real number such that

$$\rho > 0, \quad \text{if } \ n = 1 \qquad \text{or} \qquad 0 < \rho \leq \frac{(n+2)p - 2(n+1)}{n - p + 1} \quad \text{se } \ n \geq 2. \qquad (1)$$

With the choice we have $W^{\frac{1}{p'},p}(\Gamma_1) \subset L^{\rho+2}(\Gamma_1)$ with continuous and dense immersion, for $\dfrac{1}{p'} + \dfrac{1}{p} = 1$. Therefore $L^{\frac{\rho+2}{\rho+1}}(\Gamma_1) = \big(L^{\rho+2}(\Gamma_1)\big)' \subset W^{-\frac{1}{p'},p'}(\Gamma_1)$ with immersions are dense and continuous.

Let $a : V_0 \times V_0 \to \mathbb{R}$ defined by

$$a(u,v) = \sum_{i=1}^{n} \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx. \tag{2}$$

which is linear with respect the second variable. Note that, the application $v \to a(u,v)$ is continuous on V_0 for $u \in V_0$ fix.

Let $V = L^p(0,T;V_0)$ and the operator $\mathcal{B}$ from V in V' defined by

$$(\mathcal{B}(u), v)_{V'\times V} = \int_0^T a(u(t), v(t))\ dt, \qquad \forall\ \ u, v \in V. \tag{3}$$

Thus $\mathcal{B}$ is a hemicontinuous, monotonic operator and $\|\mathcal{B}u\|_{V'} \leq C\|u\|_V^{p-1}, \quad \forall\, u \in V$.

To facilitate the understand of this work, introduce the followings notations:

$$(f,g)_\Omega = \int_\Omega fg\ dx, \qquad (\varphi,\psi)_\Gamma = \int_\Gamma \varphi\psi\ d\Gamma.$$

3 Perturbed Problem

The Problem $(*)$ is not of the Cauchy-Kowaleska's type. Thus, consider the following perturbed problem:

For all $\varepsilon > 0$, the family of functions $u_\varepsilon(x,t)$ is defined by:

$$(**)\quad \left|\begin{array}{ll}
\varepsilon\dfrac{\partial u_\varepsilon}{\partial t} + \mathcal{A}u_\varepsilon = 0 & \text{in}\quad Q = \Omega\times(0,T),\\
u_\varepsilon = 0 & \text{on}\quad \Sigma_0 = \Gamma_0 \times (0,T),\\
\dfrac{\partial u_\varepsilon}{\partial t} + \displaystyle\sum_{i=1}^{n} \Big|\frac{\partial u_\varepsilon}{\partial x_i}\Big|^{p-2}\frac{\partial u_\varepsilon}{\partial x_i}\nu_i + |u_\varepsilon|^\rho u_\varepsilon = f & \text{on}\quad \Sigma_1 = \Gamma_1\times(0,T),\\
u_\varepsilon(x,0) = w_0(x) & \qquad\qquad x \in \Omega,
\end{array}\right.$$

where $w_0 = \gamma^{-1}u_0 \in V_0$.

The solution concept for $(**)$ is established by Gauss's Theorem as follows:

For $v \in V_0 \cap C^2(\overline{\Omega})$ we have

$$\int_\Omega \frac{\partial}{\partial x_i}\Big\{(|\frac{\partial u_\varepsilon}{\partial x_i}|^{p-2}\frac{\partial u_\varepsilon}{\partial x_i}).v\Big\}dx = \int_{\Gamma_1} |\frac{\partial u_\varepsilon}{\partial x_i}|^{p-2}\frac{\partial u_\varepsilon}{\partial x_i}.\nu_i.v\ d\Gamma,$$

hence

$$\int_\Omega \frac{\partial}{\partial x_i}(|\frac{\partial u_\varepsilon}{\partial x_i}|^{p-2}\frac{\partial u_\varepsilon}{\partial x_i}).v\ dx + \int_\Omega |\frac{\partial u_\varepsilon}{\partial x_i}|^{p-2}\frac{\partial u_\varepsilon}{\partial x_i}.\frac{\partial v}{\partial x_i}\ dx = \int_{\Gamma_1} |\frac{\partial u_\varepsilon}{\partial x_i}|^{p-2}\frac{\partial u_\varepsilon}{\partial x_i}.\nu_i.v\ d\Gamma.$$

Summing up from $i = 1$ to n on both sides of the above equation yields:

$$(\mathcal{A}(u_\varepsilon), v)_\Omega = a(u_\varepsilon, v) - \sum_{i=1}^{n}\int_{\Gamma_1} |\frac{\partial u_\varepsilon}{\partial x_i}|^{p-2}\frac{\partial u_\varepsilon}{\partial x_i}.\nu_i.v\ d\Gamma.$$

From this and observing that (*see* $(**)_3$)

$$-\sum_{i=1}^{n}\int_{\Gamma_1} |\frac{\partial u_\varepsilon}{\partial x_i}|^{p-2}\frac{\partial u_\varepsilon}{\partial x_i}.\nu_i.v\ d\Gamma = \int_{\Gamma_1}\frac{\partial u_\varepsilon}{\partial t}v d\Gamma_1 + \int_{\Gamma_1}|u_\varepsilon|^\rho u_\varepsilon v d\Gamma_1 - \int_{\Gamma_1} f v d\Gamma_1,$$

we obtain $(\mathcal{A}(u_\varepsilon), v)_\Omega = a(u_\varepsilon, v) + (\gamma u'_\varepsilon, \gamma v)_{\Gamma_1} + (|\gamma u_\varepsilon|^\rho \gamma u_\varepsilon, \gamma v)_{\Gamma_1} - (f, \gamma v)_{\Gamma_1}$. Substituting this identity in $(**)_1$ we get:

$$\varepsilon(u'_\varepsilon, v)_\Omega + (\gamma u'_\varepsilon, \gamma v)_{\Gamma_1} + a(u_\varepsilon, v) + (|\gamma u_\varepsilon|^\rho \gamma u_\varepsilon, \gamma v)_{\Gamma_1} = (f, \gamma v)_{\Gamma_1},$$

where u'_ε means $\dfrac{\partial u_\varepsilon}{\partial t}$. Therefore, a solution of the problem $(**)$ is understood in the following sense.

Definition 3.1. *A real value function $u_\varepsilon(x,t)$ is a solution of the problem $(**)$ if, only if,*

$$u_\varepsilon \in L^p(0,T;V_0) \cap L^\infty(0,T;L^2(\Omega)), \tag{4}$$

$$\gamma u_\varepsilon \in L^{\rho+2}(0,T;L^{\rho+2}(\Gamma_1)) \cap L^\infty(0,T;L^2(\Gamma_1)), \tag{5}$$

$$u'_\varepsilon \in L^{p'}(0,T;W^{-1,p'}(\Omega)), \tag{6}$$

$$\gamma u'_\varepsilon \in L^{p'}(0,T;W^{\frac{-1}{p'},p'}(\Gamma_1)), \tag{7}$$

$$\varepsilon(u'_\varepsilon, v)_\Omega + (\gamma u'_\varepsilon, \gamma v)_{\Gamma_1} + a(u_\varepsilon, v) + (|\gamma u_\varepsilon|^\rho \gamma u_\varepsilon, \gamma v)_{\Gamma_1} = (f, \gamma v)_{\Gamma_1}, \quad \forall\, v \in V_0. \tag{8}$$

and satisfying the initial conditions

$$u_\varepsilon(0) = w_0 \quad in \quad \Omega, \tag{9}$$

$$\gamma u_\varepsilon(0) = u_0 \quad in \quad \Gamma_1 \tag{10}$$

with u_0 belongs to $W^{\frac{1}{p'},p}(\Gamma_1)$.

4 Existence Theorem

In this section we will establish a theorem of existence of solutions.

Theorem 1. *Suppose* $f \in L^{p'}(0,T,W^{-\frac{1}{p'},p'}(\Gamma_1))$ *and* $w_0 \in V_0$. *Then, for each* $\varepsilon > 0$ *the problem* $(**)$ *has a unique solution* u_ε *in the sense of Definition* (3.1).

Remark 1. *Note that, the date* w_0 *is taken such that* $\gamma w_0 = u_0$, *since, given* $u_0 \in W^{\frac{1}{p'},p}(\Gamma_1)$ *there exists* $w_0 \in V_0$ *such that* $\gamma w_0 = u_0$ *because the application* $\gamma : V_0 \longrightarrow W^{\frac{1}{p'},p}(\Gamma_1)$ *is surjective.*

Proof: We will employ the Faedo-Galerkin's method. In fact, for V_0 we construct a special Hilbertian basis $(w_\mu)_{\mu\in\mathbb{N}}$ of V_0. By $V_{0m} = \big[w_1, \ldots.., w_m\big]$ we will denote the subspace spanned by the m first vectors of V_0. The approximated problem consist to find a function $u_{\varepsilon m}(t) \in V_{0m}$ of the type $u_{\varepsilon m}(x,t) = \displaystyle\sum_{j=1}^{m} g_{j\varepsilon m}(t) w_j(x)$ solution of the initial value problem for the system of ordinary differential equations:

$$\left|\begin{array}{l} \varepsilon(u'_{\varepsilon m}(t), v)_\Omega + (\gamma u'_{\varepsilon m}(t), \gamma v)_{\Gamma_1} + a(u_{\varepsilon m}(t), v) + \\ + (|\gamma u_{\varepsilon m}(t)|^\rho \gamma u_{\varepsilon m}(t), \gamma v)_{\Gamma_1} = (f(t), \gamma v)_{\Gamma_1} \quad \forall\ v \in V_0 \\ u_{\varepsilon m}(0) = u_{\varepsilon 0m} \longrightarrow w_0 \quad \text{in} \quad V_0. \end{array}\right. \tag{11}$$

The system (11) has a local solution on the interval $[0, t_m[$, with $t_m < T$. This solution can be extended to the whole interval $[0, T]$ as consequence of the a priori estimates that shall be proved in the next step.

Estimates

Taking $v = u_{\varepsilon m}(t)$ in (11) and integrating from 0 to $t < t_m$ we obtain

$$\frac{\varepsilon}{2}|u_{\varepsilon m}(t)|^2_{L^2(\Omega)} + \frac{1}{2}\|\gamma u_{\varepsilon m}(t)\|^2_{L^2(\Gamma_1)} + \int_0^t \|u_{\varepsilon m}(s)\|^p_{V_0} ds + \int_0^t \|\gamma u_{\varepsilon m}(s)\|^{\rho+2}_{L^{\rho+2}(\Gamma_1)} ds \leq$$
$$\int_0^t \|f(s)\|_{W^{-\frac{1}{p'},p'}(\Gamma_1)} \|\gamma u_{\varepsilon m}(s)\|_{W^{\frac{1}{p'},p}(\Gamma_1)} ds \leq C \int_0^t \|f(s)\|_{W^{-\frac{1}{p'},p'}(\Gamma_1)} \|u_{\varepsilon m}(s)\|_{V_0} ds +$$
$$\frac{\varepsilon}{2}|u_{\varepsilon 0m}|^2_{L^2(\Omega)} + \frac{1}{2}\|\gamma u_{\varepsilon 0m}\|^2_{L^2(\Gamma_1)} \leq \frac{C}{p'} \int_0^T \|f(s)\|^{p'}_{W^{-\frac{1}{p'},p'}(\Gamma_1)} + \frac{1}{p}\int_0^t \|u_{\varepsilon m}(s)\|^p_{V_0} ds +$$
$$\frac{\varepsilon}{2}|u_{\varepsilon 0m}|^2_{L^2(\Omega)} + \frac{1}{2}\|\gamma u_{\varepsilon 0m}\|^2_{L^2(\Gamma_1)}.$$

From hypotheses about the initial conditions and the continuity of the application γ, we obtain:

$$\begin{aligned}&\frac{\varepsilon}{2}|u_{\varepsilon m}(t)|^2_{L^2(\Omega)} + \frac{1}{2}\|\gamma u_{\varepsilon m}(t)\|^2_{L^2(\Gamma_1)} + \frac{1}{p'}\int_0^t \|u_{\varepsilon m}(s)\|^p_{V_0} ds + \\ &\int_0^t \|\gamma u_{\varepsilon m}(s)\|^{\rho+2}_{L^{\rho+2}(\Gamma_1)} ds \leq C\end{aligned} \tag{12}$$

where C is constant which is independent of t and m. This estimate implies that we can prolong the approximate solution $u_{\varepsilon m}(t)$ to interval $[0, T]$ and too we obtain:

$$\left|\begin{array}{l} (u_{\varepsilon m}) \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)); \\ (\gamma u_{\varepsilon m}) \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Gamma_1)); \\ (u_{\varepsilon m}) \quad \text{is bounded in} \quad L^p(0, T; V_0); \\ (\gamma u_{\varepsilon m}) \quad \text{is bounded in} \quad L^p(0, T; W^{\frac{1}{p'},p}(\Gamma_1)); \\ (\gamma u_{\varepsilon m}) \quad \text{is bounded in} \quad L^{\rho+2}(0, T; L^{\rho+2}(\Gamma_1)); \\ (u_{\varepsilon m}(T)) \quad \text{is bounded in} \quad L^2(\Omega); \\ (\gamma u_{\varepsilon m}(T)) \quad \text{is bounded in} \quad L^2(\Gamma_1). \end{array}\right. \tag{13}$$

Note that $\displaystyle\int_0^T \int_{\Gamma_1} \left||\gamma u_{\varepsilon m}|^\rho \gamma u_{\varepsilon m}\right|^{\frac{\rho+2}{\rho+1}} dt = \int_0^T \int_{\Gamma_1} |\gamma u_{\varepsilon m}|^{\rho+2} dt \leq C$. Thus

$$\left(|\gamma u_{\varepsilon m}|^\rho \gamma u_{\varepsilon m}\right) \quad \text{is bounded in} \quad L^{\frac{\rho+2}{\rho+1}}(0, T; L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)). \tag{14}$$

From $(11)_1$ we get

$$\left\langle \left\{\varepsilon u'_{\varepsilon m}, \gamma u'_{\varepsilon m}\right\}, \left\{v, \gamma v\right\} \right\rangle = (f(t) - |\gamma u_{\varepsilon m}(t)|^\rho \gamma u_{\varepsilon m}(t), \gamma v)_{\Gamma_1} - a(u_{\varepsilon m}(t), v),$$

where $< ., . >$ represent the duality paring between $V'_0 \times W^{-\frac{1}{p'},p'}(\Gamma_1)$ and $\left\{V_0 \times W^{\frac{1}{p'},p}(\Gamma_1)\right\}$.

Hence

$$\left|\left\langle \{\varepsilon u'_{\varepsilon m}, \gamma u'_{\varepsilon m}\}, \{v, \gamma v\} \right\rangle\right| \leq |a(u_{\varepsilon m}(t), v)| +$$
$$\left(\|f(t)\|_{W^{-\frac{1}{p'},p'}(\Gamma_1)} + \||\gamma u_{\varepsilon m}|^{\rho}\gamma u_{\varepsilon m}\|_{W^{-\frac{1}{p'},p'}(\Gamma_1)}\right)\|\gamma v\|_{W^{\frac{1}{p'},p}(\Gamma_1)} \leq$$
$$\left(\|f(t)\|_{W^{-\frac{1}{p'},p'}(\Gamma_1)} + C\||\gamma u_{\varepsilon m}|^{\rho}\gamma u_{\varepsilon m}\|_{L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)}\right)\|\gamma v\|_{W^{\frac{1}{p'},p}(\Gamma_1)} +$$
$$\|u_{\varepsilon m}(t)\|_{V_0}^{p-1}\|v\|_{V_0} \leq \left(\|\gamma v\|_{W^{\frac{1}{p'},p}(\Gamma_1)} + \|v\|_{V_0}\right) \times$$
$$\left[\|f(t)\|_{W^{-\frac{1}{p'},p'}(\Gamma_1)} + C\||\gamma u_{\varepsilon m}|^{\rho}\gamma u_{\varepsilon m}\|_{L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)} + \|u_{\varepsilon m}(t)\|_{V_0}^{p-1}\right].$$

From estimates above we have

$$\left|\left\langle \{\varepsilon u'_{\varepsilon m}, \gamma u'_{\varepsilon m}\}, \{v, \gamma v\} \right\rangle\right| \leq$$
$$\left[\|f(t)\|_{W^{-\frac{1}{p'},p'}(\Gamma_1)} + C\||\gamma u_{\varepsilon m}|^{\rho}\gamma u_{\varepsilon m}\|_{L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)} + \|u_{\varepsilon m}(t)\|_{V_0}^{p-1}\right] \times$$
$$\left\|\{v, \gamma v\}\right\|_{V_0 \times W^{\frac{1}{p'},p}(\Gamma_1)}.$$

Therefore, from bounded (14) and $(13)_3$, we get

$$\left(\{\varepsilon u'_{\varepsilon m}, \gamma u'_{\varepsilon m}\}\right) \quad \text{is bounded in} \quad L^{p'}(0,T; V'_0 \times W^{-\frac{1}{p'},p'}(\Gamma_1)). \tag{15}$$

Passage to the Limit

From estimates (13), (14) and (15) we obtain

$$\left|\begin{array}{l}
u_{\varepsilon m} \overset{*}{\rightharpoonup} u_\varepsilon \quad \text{weak-star in} \quad L^\infty(0,T; L^2(\Omega));\\
\gamma u_{\varepsilon m} \overset{*}{\rightharpoonup} \gamma u_\varepsilon \quad \text{weak-star in} \quad L^\infty(0,T; L^2(\Gamma_1));\\
u_{\varepsilon m} \rightharpoonup u_\varepsilon \quad \text{weak} \quad \text{in} \quad L^p(0,T; V_0);\\
\gamma u_{\varepsilon m} \rightharpoonup \gamma u_\varepsilon \quad \text{weak in} \quad L^p(0,T; W^{\frac{1}{p'},p}(\Gamma_1));\\
\gamma u_{\varepsilon m} \rightharpoonup \gamma u_\varepsilon \quad \text{weak in} \quad L^{\rho+2}(0,T; L^{\rho+2}(\Gamma_1)) \equiv L^{\rho+2}(\Sigma_1);\\
u_{\varepsilon m}(T) \rightharpoonup \chi \quad \text{weak in} \quad L^2(\Omega);\\
\gamma u_{\varepsilon m}(T) \rightharpoonup \varsigma \quad \text{weak in} \quad L^2(\Gamma_1)\\
|\gamma u_{\varepsilon m}|^\rho \gamma u_{\varepsilon m} \rightharpoonup \eta \quad \text{weak in} \quad L^{\frac{\rho+2}{\rho+1}}(0,T; L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)) \equiv L^{\frac{\rho+2}{\rho+1}}(\Sigma_1)\\
u'_{\varepsilon m} \rightharpoonup u'_\varepsilon \quad \text{weak in} \quad L^{p'}(0,T; V'_0);\\
\gamma u'_{\varepsilon m} \rightharpoonup \gamma u'_\varepsilon \quad \text{weak in} \quad L^{p'}(0,T; W^{-\frac{1}{p'},p'}(\Gamma_1))
\end{array}\right. \tag{16}$$

Remark 2. *Note that by the convergence* $(16)_1$, $(16)_9$ *and* $(16)_2, (16)_{10}$ *it makes sense to calculate* $u_\varepsilon(0)$, $u_\varepsilon(T)$ *and* $\gamma u_\varepsilon(0)$, $\gamma u_\varepsilon(T)$ *respectively.*

Let $V = L^p(0,T;V_0)$ and $\mathcal{B}$ the operator from V given by

$$(\mathcal{B}(u), v)_{V' \times V} = \int_0^T a(u(t), v(t))\, dt, \qquad \forall \ u, v \in V, \tag{17}$$

hence, $\mathcal{B}$ is hemicontinuous, monotonic operator and $\|\mathcal{B}u\|_{V'} \leq C\|u\|_V^{p-1}, \quad \forall\, u \in V$. Thus, from estimative $(16)_3$ we have $(\mathcal{B}u_{\varepsilon m})_m$ is bounded in V', hence

$$\mathcal{B}u_{\varepsilon m} \rightharpoonup \zeta \quad \text{is} \quad V'. \tag{18}$$

From convergence $(16)_9$ we obtain $< u'_{\varepsilon m}, \varphi > \to < u'_\varepsilon, \varphi >, \quad \forall\, \varphi \in L^p(0,T;V_0)$, that is,

$$\int_0^T (u'_{\varepsilon m}, v)_\Omega \theta dt \to \int_0^T (u'_\varepsilon, v)_\Omega \theta dt, \quad v \in V_{0m} \subset V_0, \ \ \forall\, \theta \in \mathcal{D}(0,T) \subset L^p(0,T),$$

or

$$(u'_{\varepsilon m}, v)_\Omega \to (u'_\varepsilon, v)_\Omega \quad v \in V_{0m}, \quad \text{in} \quad \mathcal{D}'(0,T). \tag{19}$$

From convergence $(16)_{10}$ we have $< \gamma u'_{\varepsilon m}, \gamma\varphi > \to < \gamma u'_\varepsilon, \gamma\varphi >, \quad \forall\ \varphi \in L^p(0,T;V_0)$, that is,

$$(\gamma u'_{\varepsilon m}, \gamma v)_{\Gamma_1} \to (\gamma u'_\varepsilon, \gamma v)_{\Gamma_1} \quad v \in V_{0m}, \quad \text{in} \quad \mathcal{D}'(0,T). \tag{20}$$

Analogously, we have

$$(\mathcal{B}u_{\varepsilon m}, v) \to (\zeta, v) \quad v \in V_{0m}, \quad \text{in} \quad \mathcal{D}'(0,T), \tag{21}$$

and

$$(|\gamma u_{\varepsilon m}|^\rho \gamma u_{\varepsilon m}, \gamma v)_{\Gamma_1} \to (\eta, \gamma v)_{\Gamma_1} \quad v \in V_{0m}, \quad \text{in} \quad \mathcal{D}'(0,T). \tag{22}$$

Thus, taking the limit as $m \to \infty$ in the approximated equation $(11)_1$, using the convergence $(19)-(22)$ and the density of V_{0m} in V_0, we obtain:

$$\begin{aligned} &\varepsilon(u'_\varepsilon, v)_\Omega + (\gamma u'_\varepsilon, \gamma v)_{\Gamma_1} + (\zeta, v)_\Omega \ + \ (\eta, \gamma v)_{\Gamma_1} = (f(t), \gamma v)_{\Gamma_1} \\ &\qquad \forall\ v \in V_0 \quad \text{in} \quad \mathcal{D}'(0,T). \end{aligned} \tag{23}$$

To follow we will proof that: $|\gamma u_\varepsilon|^\rho \gamma u_\varepsilon = \eta$ and $\mathcal{B}u_\varepsilon = \zeta$.

Proof. In fact, from the estimate above we have

$$\begin{aligned} &(\gamma u_{\varepsilon m}) \quad \text{is bounded in} \quad L^p(0,T;W^{\frac{1}{p'},p}(\Gamma_1)); \\ &(\gamma u'_{\varepsilon m}) \quad \text{is bounded in} \quad L^{p'}(0,T;W^{-\frac{1}{p'},p'}(\Gamma_1)). \end{aligned}$$

As $W^{\frac{1}{p'},p}(\Gamma_1) \overset{c}{\hookrightarrow} L^p(\Gamma_1) \hookrightarrow L^2(\Gamma_1) \hookrightarrow L^{p'}(\Gamma_1) \hookrightarrow W^{-\frac{1}{p'},p'}(\Gamma_1)$, we have $W^{\frac{1}{p'},p}(\Gamma_1) \overset{c}{\hookrightarrow} L^p(\Gamma_1) \hookrightarrow W^{-\frac{1}{p'},p'}(\Gamma_1)$. Thus, of the Aubin-Lions's Theorem we obtain a subsequence, still denoted by $(\gamma u_{\varepsilon m})$, such that $\gamma u_{\varepsilon m} \to \gamma u_\varepsilon$ in $L^p(\Sigma_1)$, where still we can extract other subsequence which we insist in denote by $(\gamma u_{\varepsilon m})$, such that: $\gamma u_{\varepsilon m} \to \gamma u_\varepsilon$ a.e Σ_1, thus,

$$|\gamma u_{\varepsilon m}|^\rho \gamma u_{\varepsilon m} \to |\gamma u_\varepsilon|^\rho \gamma u_\varepsilon \quad \text{a.e} \quad \Sigma_1. \tag{24}$$

From estimative (14) we obtain:

$$\| |\gamma u_{\varepsilon m}|^\rho \gamma u_{\varepsilon m} \|_{L^{\frac{\rho+2}{\rho+1}}(\Sigma_1)} \le C \tag{25}$$

Hence, from (24), (25) and the Lions's Lema, we obtain

$$|\gamma u_{\varepsilon m}|^\rho \gamma u_{\varepsilon m} \rightharpoonup |\gamma u_\varepsilon|^\rho \gamma u_\varepsilon \quad \text{weak} \quad \text{in} \quad L^{\frac{\rho+2}{\rho+1}}(\Sigma_1).$$

Thus $|\gamma u_\varepsilon|^\rho \gamma u_\varepsilon = \eta$. □

Now we will prove that $\mathcal{B} u_\varepsilon = \zeta$.

Proof. In fact for this purpose we needed prove that: (i) $u_\varepsilon(0) = w_0$, (ii) $\chi = u_\varepsilon(T)$, (iii) $\gamma u_\varepsilon(0) = u_0$ and (iv) $\gamma u_\varepsilon(T) = \varsigma$.

In fact, to prove (i) we use the convergence $(16)_1$ and $(16)_9$ which yield

$$\int_0^T (u_{\varepsilon m}, v)\varphi' \, dt \to \int_0^T (u_\varepsilon, v)\varphi' \, dt \quad \forall \; v \in L^2(\Omega),$$

$$\varphi \in C^1([0,T]), \; \varphi(0) = 1, \; \varphi(T) = 0$$

and

$$\int_0^T \frac{d}{dt}(u_{\varepsilon m}, v)\varphi \, dt \to \int_0^T \frac{d}{dt}(u_\varepsilon, v)\varphi \, dt \quad \forall \; v \in L^2(\Omega) \subset V_0',$$

$$\varphi \in C^1([0,T]),\ \varphi(0)=1,\ \varphi(T)=0,$$

where

$$\int_0^T \frac{d}{dt}\{(u_{\varepsilon m},v)\varphi\}\ dt \to \int_0^T \frac{d}{dt}\{(u_\varepsilon,v)\varphi\}\ dt \quad \forall\ \ v\in L^2(\Omega),$$

$$\varphi \in C^1([0,T]),\ \varphi(0)=1,\ \varphi(T)=0,$$

hence,

$$(u_{\varepsilon m}(0),v)\to(u_\varepsilon(0),v), \quad \forall\ \ v\in L^2(\Omega),$$

that is,

$$u_{\varepsilon m}(0)\rightharpoonup u_\varepsilon(0) \quad \text{in} \quad L^2(\Omega).$$

As $u_{\varepsilon m}(0)\to w_0$ in $V_0 \hookrightarrow L^2(\Omega)$, we have $u_{\varepsilon m}(0)\to w_0$ in $L^2(\Omega)$, where $u_{\varepsilon m}(0)\rightharpoonup w_0$ in $L^2(\Omega)$. Hence

$$u_\varepsilon(0)=w_0.$$

Analogously, working as $\varphi(0)=0$ and $\varphi(T)=1$ we obtain

$$u_\varepsilon(T)=\chi.$$

In fact, to prove (iii) we use the convergence $(16)_2$ and $(16)_{10}$ which yield

$$(\gamma u_{\varepsilon m}(0),v)\to(\gamma u_\varepsilon(0),v), \quad \forall\ \ v\in L^2(\Gamma_1),$$

where

$$\gamma u_{\varepsilon m}(0)\rightharpoonup \gamma u_\varepsilon(0) \quad \text{em} \quad L^2(\Gamma_1).$$

As $u_{\varepsilon m}(0)\to w_0$ in V_0 and γ continuous from V_0 in $W^{\frac{1}{p'},p}(\Gamma_1)$, we have that $\gamma u_{\varepsilon m}(0)\to\gamma w_0$ in $W^{\frac{1}{p'},p}(\Gamma_1)$. Being $W^{\frac{1}{p'},p}(\Gamma_1)\hookrightarrow L^2(\Gamma_1)$, for $p>2>\dfrac{2n}{2n+1}$, by Fractionary Sobolev's Theorem, we have $\gamma u_{\varepsilon m}(0)\to\gamma w_0$ in $L^2(\Gamma_1)$. Therefore $\gamma u_\varepsilon(0)=\gamma w_0=u_0$, by remark 1. Analogously, we have $\gamma u_\varepsilon(T)=$ ς □

We will show that: $\mathcal{B}u_\varepsilon = \zeta$. In fact, being the operator $\mathcal{B}: V \rightarrow V'$ mononotonic, we obtain:

$$(\mathcal{B}u_{\varepsilon m}, u_{\varepsilon m}) - (\mathcal{B}u_{\varepsilon m}, v) - (\mathcal{B}v, u_{\varepsilon m} - v) \geq 0. \tag{26}$$

Taking $v = u_{\varepsilon m}$ and integrating of 0 the T in the approximated equation $(11)_1$ we obtain:

$$(\mathcal{B}u_{\varepsilon m}, u_{\varepsilon m}) = \int_0^T a(u_{\varepsilon m}, u_{\varepsilon m})dt = \int_0^T (f, u_{\varepsilon m})_{\Gamma_1} dt - \varepsilon \int_0^T (u'_{\varepsilon m}, u_{\varepsilon m})_\Omega dt -$$
$$\int_0^T (\gamma u'_{\varepsilon m}, \gamma u_{\varepsilon m})_{\Gamma_1} dt - \int_0^T (|\gamma u_{\varepsilon m}|^\rho \gamma u_{\varepsilon m}, \gamma u_{\varepsilon m})_{\Gamma_1} dt.$$

Thus, substituting in (26), we have

$$0 \leq \int_0^T (f, u_{\varepsilon m})_{\Gamma_1} dt - \frac{\varepsilon}{2}|u_{\varepsilon m}(T)|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}|u_{\varepsilon m}(0)|^2_{L^2(\Omega)} - \frac{1}{2}|\gamma u_{\varepsilon m}(T)|^2_{L^2(\Gamma_1)} +$$
$$\frac{1}{2}|\gamma u_{\varepsilon m}(0)|^2_{L^2(\Gamma_1)} - \int_0^T \|\gamma u_{\varepsilon m}\|^{\rho+2}_{L^{\rho+2}(\Gamma_1)} dt - (\mathcal{B}u_{\varepsilon m}, v) - (\mathcal{B}v, u_{\varepsilon m} - v).$$

Using the convergence obtained and applying the $\liminf_{m\rightarrow\infty}$ in both sides of the inequality above we have:

$$\begin{aligned} 0 \leq \int_0^T (f, u_\varepsilon)_{\Gamma_1} dt - \frac{\varepsilon}{2}|u_\varepsilon(T)|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}|w_0|^2_{L^2(\Omega)} - \frac{1}{2}|\gamma u_\varepsilon(T)|^2_{L^2(\Gamma_1)} + \\ \frac{1}{2}|u_0|^2_{L^2(\Gamma_1)} - \int_0^T \|\gamma u_\varepsilon\|^{\rho+2}_{L^{\rho+2}(\Gamma_1)} dt - (\zeta, v) - (\mathcal{B}v, u_\varepsilon - v). \end{aligned} \tag{27}$$

Taking $v = u_\varepsilon$ and integrating of 0 the T in the equation (23) we obtain:

$$\int_0^T (f, u_\varepsilon)_{\Gamma_1} dt = (\zeta, u_\varepsilon) + \frac{\varepsilon}{2}|u_\varepsilon(T)|^2_{L^2(\Omega)} - \frac{\varepsilon}{2}|w_0|^2_{L^2(\Omega)} +$$
$$\frac{1}{2}|\gamma u_\varepsilon(T)|^2_{L^2(\Gamma_1)} - \frac{1}{2}|u_0|^2_{L^2(\Gamma_1)} + \int_0^T \|\gamma u_\varepsilon\|^{\rho+2}_{L^{\rho+2}(\Gamma_1)} dt.$$

If we substitute this expression in (27), we obtain

$$0 \leq (\zeta - \mathcal{B}v, u_\varepsilon - v), \quad \forall \quad v \in V.$$

Consider $u_\varepsilon - v = \lambda w, \ \ \lambda > 0$. Thus, using the hemicontinuity of the operator $\mathcal{B}$, we obtain $0 \leq (\zeta - \mathcal{B}u_\varepsilon, w), \quad \forall \quad w \in V$. Working with $\lambda < 0$ we have:

$(\zeta - \mathcal{B}u_\varepsilon, w) \leq 0, \quad \forall \;\; w \in V.$

Therefore $(\zeta - \mathcal{B}u_\varepsilon, w) = 0, \quad \forall \;\; w \in V$, thus $\mathcal{B}u_\varepsilon = \zeta$.

Note that $(\mathcal{B}u_{\varepsilon m}, w) \to (\mathcal{B}u_\varepsilon, w), \quad \forall \;\; w \in V = L^p(0,T;V_0)$, hence,

$$\int_0^T a(u_{\varepsilon m}, v)\theta dt \to \int_0^T a(u_\varepsilon, v)\theta dt, \;\; \forall \;\; v \in V_0, \quad \forall \;\; \theta \in \mathcal{D}(0,T) \subset L^p(0,T).$$

Thus $a(u_{\varepsilon m}, v) \to a(u_\varepsilon, v) \quad \forall \;\; v \in V_0$ in $\mathcal{D}'(0,T)$. Therefore,

$$\left| \begin{array}{l} \varepsilon(u'_\varepsilon, v)_\Omega + (\gamma u'_\varepsilon, \gamma v)_{\Gamma_1} + a(u_\varepsilon, v) + \\ \\ (|\gamma u_\varepsilon|^\rho \gamma u_\varepsilon, \gamma v)_{\Gamma_1} = (f(t), \gamma v)_{\Gamma_1} \quad \forall \, v \in V_0 \quad \text{in} \quad \mathcal{D}'(0,T), \end{array} \right. \tag{28}$$

Uniqueness of the Solution

To obtain the uniqueness of the solution, we suppose that there exists two solutions such that u_ε, $\widehat{u_\varepsilon}$ in the conditions of the Theorem 1. It following that $w_\varepsilon = u_\varepsilon - \widehat{u_\varepsilon}$ satisfy:

$$w_\varepsilon \;\in\; L^p(0,T;V_0) \cap L^\infty(0,T;L^2(\Omega)), \tag{29}$$

$$\gamma w_\varepsilon \;\in\; L^{\rho+2}(0,T;L^{\rho+2}(\Gamma_1)) \cap L^\infty(0,T;L^2(\Gamma_1)), \tag{30}$$

$$w'_\varepsilon \;\in\; L^{p'}(0,T;W^{-1,p'}(\Omega)), \tag{31}$$

$$\gamma w'_\varepsilon \;\in\; L^{p'}(0,T;W^{-\frac{1}{p'},p'}(\Gamma_1)), \tag{32}$$

$$\begin{array}{c} \varepsilon(w'_\varepsilon, v)_\Omega + (\gamma w'_\varepsilon, \gamma v)_{\Gamma_1} + a(u_\varepsilon, v) - a(\widehat{u_\varepsilon}, v) + \\ \\ (|\gamma u_\varepsilon|^\rho \gamma u_\varepsilon - |\gamma \widehat{u_\varepsilon}|^\rho \gamma \widehat{u_\varepsilon}, \gamma v)_{\Gamma_1} = 0, \;\; \forall \, v \in V_0. \end{array} \tag{33}$$

Taking $v = w_\varepsilon$ in (33) and integrating from 0 the $t \leq T$ we obtain:

$$\frac{\varepsilon}{2}|w_\varepsilon(t)|^2_{L^2(\Omega)} + \frac{1}{2}|\gamma w_\varepsilon(t)|^2_{L^2(\Gamma_1)} + \int_0^t (a(u_\varepsilon, w) - a(\widehat{u_\varepsilon}, w))dt +$$
$$\int_0^t (|\gamma u_\varepsilon|^\rho \gamma u_\varepsilon - |\gamma \widehat{u_\varepsilon}|^\rho \gamma \widehat{u_\varepsilon}, \gamma u_\varepsilon - \gamma \widehat{u_\varepsilon})dt = 0.$$

Using the monotoneity of the function $h(s) = |s|^\rho s$ and $a(u_\varepsilon, w) - a(\widehat{u_\varepsilon}, w) \geq 0$, we have

$$\frac{\varepsilon}{2}|w_\varepsilon(t)|^2_{L^2(\Omega)} + \frac{1}{2}|\gamma w_\varepsilon(t)|^2_{L^2(\Gamma_1)} \leq 0.$$

Therefore, we have that $w_\varepsilon(t) = 0 \quad \forall \, t \in [0,T]$. Thus the Theorem is proved.

■

5 Main Result

In this Section we will prove the following result

Theorem 2. *When $\varepsilon \to 0$ we have*

$$u_\varepsilon \rightharpoonup u \quad in \quad L^p(0,T;V_0), \tag{34}$$

being u the solution of the problem $(*)$.

Proof: Making $v = u_\varepsilon(t)$ in (8) and proceeding as in the previous Theorem we obtain

$$\left| \begin{array}{ll} (\sqrt{\varepsilon} u_\varepsilon) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)); \\ (\gamma u_\varepsilon) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Gamma_1)); \\ (u_\varepsilon) \quad \text{is bounded in} \quad L^p(0,T;V_0); \\ (\gamma u_\varepsilon) \quad \text{is bounded in} \quad L^p(0,T;W^{\frac{1}{p'},p}(\Gamma_1)); \\ (\gamma u_\varepsilon) \quad \text{is bounded in} \quad L^{\rho+2}(0,T;L^{\rho+2}(\Gamma_1)); \\ (u_\varepsilon(T)) \quad \text{is bounded in} \quad L^2(\Omega); \\ (\gamma u_\varepsilon(T)) \quad \text{is bounded in} \quad L^2(\Gamma_1). \end{array} \right. \tag{35}$$

and

$$\left| \begin{array}{ll} (\varepsilon u'_\varepsilon) \quad \text{is bounded in} \quad L^{p'}(0,T;V'_0); \\ \left(|\gamma u_\varepsilon|^\rho \gamma u_\varepsilon\right) \quad \text{is bounded in} \quad L^{\frac{\rho+2}{\rho+1}}(0,T;L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)) \\ (\gamma u_\varepsilon) \quad \text{is bounded in} \quad L^p(0,T;W^{\frac{1}{p'},p}(\Gamma_1)); \\ (\gamma u'_\varepsilon) \quad \text{is bounded in} \quad L^{p'}(0,T;W^{-\frac{1}{p'},p'}(\Gamma_1)) \\ (\mathcal{B} u_\varepsilon) \quad \text{is bounded in} \quad V' = L^{p'}(0,T;V'_0). \end{array} \right. \tag{36}$$

Hence there exists an subsequence, still represented by (u_ε), such that, when $\varepsilon \to 0$

$$\left|\begin{array}{l}
\sqrt{\varepsilon} u_\varepsilon \overset{*}{\rightharpoonup} 0 \quad \text{weak-star in} \quad L^\infty(0,T;L^2(\Omega));\\
\gamma u_\varepsilon \overset{*}{\rightharpoonup} \gamma u \quad \text{weak-star in} \quad L^\infty(0,T;L^2(\Gamma_1));\\
u_\varepsilon \rightharpoonup u \quad \text{weak in} \quad V = L^p(0,T;V_0);\\
\gamma u_\varepsilon \rightharpoonup \gamma u \quad \text{weak in} \quad L^p(0,T;W^{\frac{1}{p'},p}(\Gamma_1));\\
\gamma u_\varepsilon \rightharpoonup \gamma u \quad \text{weak in} \quad L^{\rho+2}(0,T;L^{\rho+2}(\Gamma_1)) \equiv L^{\rho+2}(\Sigma_1);\\
\gamma u_\varepsilon(T) \rightharpoonup \chi \quad \text{weak in} \quad L^2(\Gamma_1)\\
\varepsilon u'_\varepsilon \rightharpoonup 0 \quad \text{weak in} \quad V' = L^{p'}(0,T;V'_0);\\
|\gamma u_\varepsilon|^\rho \gamma u_\varepsilon \rightharpoonup \eta \quad \text{weak in} \quad L^{\frac{\rho+2}{\rho+1}}(0,T;L^{\frac{\rho+2}{\rho+1}}(\Gamma_1))\\
\gamma u'_\varepsilon \rightharpoonup \gamma u' \quad \text{weak in} \quad L^{p'}(0,T;W^{-\frac{1}{p'},p'}(\Gamma_1));\\
\mathcal{B} u_\varepsilon \rightharpoonup \zeta \quad \text{weak in} \quad V' = L^{p'}(0,T;V'_0).
\end{array}\right. \tag{37}$$

Analogously to the Theorem 1 we can to show that $|\gamma u|^\rho \gamma u = \eta$ and $\mathcal{B}u = \zeta$. Using the convergence (37) in (28) we obtain the variational formulation of the problem $(*)$, when $\varepsilon \to 0$

$$(\gamma u', \gamma v)_{\Gamma_1} + a(u,v) + (|\gamma u|^\rho \gamma u, \gamma v)_{\Gamma_1} = (f(t), \gamma v)_{\Gamma_1} \quad \forall\, v \in V_0 \quad \text{in} \quad \mathcal{D}'(0,T).$$

On the other hand, with a analogously analysis as in the Remark 2, we have as in the Theorem 1: $\gamma u_\varepsilon(0) \rightharpoonup \gamma u(0)$ in $L^2(\Gamma_1)$. Therefore

$$\gamma u(0) = u_0 \quad \text{on} \quad \Gamma_1.$$

In this sense, we have the solution of the problem $(*)$ as limit of the perturbed problem $(**)$. ■

6 Boundary Stabilization

The aim of section is study the algebric decay for the energy $E(t)$ associated to weak solution of the problem $(*)$. To asymptotic behavior, we use the Nakao's method [9]. This energy is given by

$$E(t) = \frac{1}{2}|\gamma u(t)|^2_{L^2(\Gamma_1)} \tag{38}$$

Remark 3. *Note that, the solution of the problem* $(*)$ *we can be extend to* $[0, \infty)$, *when* $f = 0$.

Theorem 3. *Let* $E(t)$ *a energy associated the weak solution of problem* $(*)$. *Then, there exists a constant* $\delta > 0$ *such that the energy satisfies*

$$E(t) \le C\frac{1}{(1+t)^{\frac{1}{\delta}}},\ \forall t \ge 0.$$

Considering (28) with $f = 0$ and $v = u_\epsilon$ we get:

$$\frac{1}{2}\frac{d}{dt}|\sqrt{\epsilon}u_\epsilon(t)|^2 + \frac{1}{2}\frac{d}{dt}|\gamma u_\epsilon(t)|^2 = -\|u_\epsilon(t)\|^p_{V_0} - \|\gamma u_\epsilon(t)\|^{\rho+2}_{L^{\rho+2}(\Omega)}$$

Let $E_\varepsilon(t) = \frac{1}{2}|\sqrt{\epsilon}u_\epsilon(t)|^2 + \frac{1}{2}|\gamma u_\epsilon(t)|^2$, then

$$\frac{d}{dt}E_\varepsilon(t) \le 0,\ \forall t \ge 0 \tag{39}$$

and

$$E_\varepsilon(t) \le E_\varepsilon(0) = \frac{1}{2}|\sqrt{\epsilon}u_\epsilon(0)|^2 + \frac{1}{2}|\gamma u_\epsilon(0)|^2 \le C(\gamma u_0)$$

because, $\frac{1}{2}\sqrt{\varepsilon}u_\varepsilon(0) \to 0$ in $L^2(\Omega)$ and $\gamma u_\varepsilon(0) \to \gamma u(0)$ in $L^2(\Gamma_1)$ when $\varepsilon \to 0$. Therefore, $E_\varepsilon(t)$ is increasing and bounded.

We have that

$$\frac{d}{dt}E_\varepsilon(t) = -\|u_\varepsilon(t)\|^p_{V_0} - \|\gamma u_\varepsilon(t)\|^{\rho+2}_{L^{\rho+2}(\Omega)} \tag{40}$$

then

$$E_\varepsilon(t+1) - E_\varepsilon(t) = -\int_t^{t+1} \|u_\varepsilon(t)\|^p_{V_0} - \int_t^{t+1} \|\gamma u_\varepsilon(t)\|^{\rho+2}_{L^{\rho+2}(\Omega)}.$$

Therefore

$$\int_t^{t+1} \|u_\varepsilon(t)\|^p_{V_0} + \int_t^{t+1} \|\gamma u_\varepsilon(t)\|^{\rho+2}_{L^{\rho+2}(\Omega)} = E_\varepsilon(t) - E_\varepsilon(t+1) \tag{41}$$

Since $V_0 \hookrightarrow L^2(\Omega)$ and $L^{\rho+2}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$, we have

$$\int_t^{t+1} |u_\varepsilon(t)|^p_{L^2(\Omega)} + \int_t^{t+1} |\gamma u_\varepsilon(t)|^p_{L^2(\Gamma_1)} \leq C_1(E_\varepsilon(t) - E_\varepsilon(t+1)) \tag{42}$$

Therefore exist $t^* \in [t, t+1]$, such that

$$|u_\varepsilon(t^*)|^p_{L^2(\Omega)} + |\gamma u_\varepsilon(t^*)|^p_{L^2(\Gamma_1)} \leq C_1[E_\varepsilon(t) - E_\varepsilon(t+1)],$$

where,

$$\frac{1}{2}|u_\varepsilon(t^*)|^2 \leq C_2[E_\varepsilon(t) - E_\varepsilon(t+1)]^{\frac{2}{p}}$$
$$\frac{1}{2}|\gamma u_\varepsilon(t^*)|^2 \leq C_2[E_\varepsilon(t) - E_\varepsilon(t+1)]^{\frac{2}{p}}.$$

Thus

$$\frac{\varepsilon}{2}|u_\varepsilon(t^*)|^2 \leq \varepsilon C_2[E_\varepsilon(t) - E_\varepsilon(t+1)]^{\frac{2}{p}}$$
$$\frac{1}{2}|\gamma u_\varepsilon(t^*)|^2 \leq C_2[E_\varepsilon(t) - E_\varepsilon(t+1)]^{\frac{2}{p}}$$

we obtain

$$\frac{1}{2}|\sqrt{\varepsilon} u_\varepsilon(t^*)|^2 + \frac{1}{2}|\gamma u_\varepsilon(t^*)|^2 \leq (1+\varepsilon)C_2[E_\varepsilon(t) - E_\varepsilon(t+1)]^{\frac{2}{p}}$$

Therefore,

$$E_\varepsilon(t^*) \leq (1+\varepsilon)C_2[E_\varepsilon(t) - E_\varepsilon(t+1)]^{\frac{2}{p}}. \tag{43}$$

Integrating from (40) of t to t^*, we obtain

$$E_\varepsilon(t) = E_\varepsilon(t^*) + \int_t^{t^*} \|u_\varepsilon(t)\|^p_{V_0} + \int_t^{t^*} \|\gamma u_\varepsilon(t)\|^p_{L^p(\Gamma_1)} \leq$$
$$((1+\varepsilon)C_2)(E_\varepsilon(t) - E_\varepsilon(t+1))^{\frac{2}{p}} + (E_\varepsilon(t) - E_\varepsilon(t+1))$$

Therefore,

$$E_\varepsilon(t)^{\frac{p}{2}} \leq \max\{1, (1+\varepsilon)C_2\}^{\frac{p}{2}}(E_\varepsilon(t) - E_\varepsilon(t+1)) + (E_\varepsilon(t) - E_\varepsilon(t+1))^{\frac{p}{2}} \leq$$
$$\max\{1, (1+\varepsilon)C_2\}^{\frac{p}{2}}\left[E_\varepsilon(t) - E_\varepsilon(t+1)][1 + (E_\varepsilon(t) - E_\varepsilon(t+1))^{\frac{p-2}{2}}\right].$$

Being, $E_\varepsilon(t)$ limited for all $\varepsilon > 0$, follows that

$$E_\varepsilon(t)^{\frac{p}{2}} \leq C_3 \max\{1, (1+\varepsilon)C_2\}^{\frac{p}{2}}(E_\varepsilon(t) - E_\varepsilon(t+1))$$

or

$$E_\varepsilon(t)^{\frac{p}{2}} \le C_{1\varepsilon}(E_\varepsilon(t) - E_\varepsilon(t+1)),$$

where $C_{1\varepsilon} = C_3 \max\{1, (1+\varepsilon)C_2\}^{\frac{p}{2}}$.

Thus, by Nakao's Lemma, there exists $\delta > 0$ such that

$$E_{1\varepsilon}(t) \le C_{1\varepsilon}\frac{1}{(1+t)^{\frac{1}{\delta}}}$$

Note that $C_{1\varepsilon} \to \max\{1, C_2\}^{\frac{p}{2}}$, $\gamma u_\varepsilon(t) \rightharpoonup \gamma u(t)$ weak in $L^2(\Gamma_1)$ and $\sqrt{\varepsilon}u_\varepsilon(t) \rightharpoonup 0$ weak in $L^2(\Omega)$.

From

$$\liminf_{\varepsilon\to 0} E_{1\varepsilon}(t) \le \liminf_{\varepsilon\to 0} C_{1\varepsilon}\frac{1}{(1+t)^{\frac{1}{\delta}}}, \ \forall t \ge 0$$

implies that

$$\liminf_{\varepsilon\to 0}\left\{\frac{1}{2}|\sqrt{\varepsilon}u_\varepsilon(t)|^2_{L^2(\Omega)} + \frac{1}{2}|\gamma u_\varepsilon(t)|^2_{L^2(\Gamma_1)}\right\} \le \max\{1, C_2\}^{\frac{p}{2}}\frac{1}{(1+t)^{\frac{1}{\delta}}}, \ \forall t \ge 0$$

Thus,

$$\frac{1}{2}|\gamma u(t)|^2_{L^2(\Gamma_1)} \le C_1\frac{1}{(1+t)^{\frac{1}{\delta}}}, \ \forall t \ge 0,$$

where $C_1 = \max\{1, C_2\}^{\frac{p}{2}}$.

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