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POSITIVE SOLUTIONS OF A BOUNDARY VALUE PROBLEM FOR A NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

ERIC R. KAUFMANN AND EBENE MBOUMI

ABSTRACT. In this paper we give sufficient conditions for the existence of at least one and at least three positive solutions to the nonlinear fractional boundary value problem

$$D^{\alpha}u + a(t)f(u) = 0, \quad 0 < t < 1, 1 < \alpha \le 2,$$
$$u(0) = 0, u'(1) = 0,$$

where D^{α} is the Riemann-Liouville differential operator of order α, $f : [0,\infty) \to [0,\infty)$ is a given continuous function and a is a positive and continuous function on $[0,1]$.

1. INTRODUCTION

We are interested in positive solutions of the nonlinear fractional boundary value problem

$$D^{\alpha}u + a(t)f(u) = 0, \quad 0 < t < 1,\ 1 < \alpha \le 2, \tag{1}$$
$$u(0) = 0,\ u'(1) = 0, \tag{2}$$

where D^{α} is the Riemann-Liouville differential operator of order α, $f : [0,\infty) \to [0,\infty)$ is a given continuous function and a is a positive and continuous function on $[0,1]$. We show that under certain growth conditions on the nonlinear term f, the fractional boundary value problem (1), (2) has at least one or at least three positive solutions. The main tools employed are two well known fixed point theorems for operators acting on cones in a Banach space.

The use of cone theoretic techniques in the study of solutions to boundary value problems has a rich and diverse history. Some authors have used fixed point theorems to show the existence of positive solutions to boundary value problems for ordinary differential equations, difference equations, and dynamic equations on time scales, see for example [1, 2, 5, 6, 11, 15, 18, 20, 21, 26, 27, 36] and references therein. In other papers, [21, 22, 35], authors have use fixed point theory to show the existence of solutions to singular boundary value problems. Still other papers have used cone theoretic techniques to compare the smallest eigenvalues of two operators, see [10, 12, 14, 19, 23, 33]. The texts by Agarwal, O'Regan and Wong [2] and by Guo and Lakshmikantham [13] are excellent resources for the use of fixed point theory in the study of existence of solutions to boundary value problems.

While much attention has focused on the Cauchy problem for fractional differential equations for both the Reimann-Liouville and Caputo differential operators, see [8, 17, 24, 28, 29, 30, 31, 32, 34] and references therein, there are few papers devoted to the study of fractional order boundary value problems, see for example

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[3, 4, 7, 9, 16, 25, 37, 38]. The motivation for this paper are the manuscripts [7], [9], [37] and [38]. The history, definitions, theory, and applications of fractional calculus are well laid out in the books by Miller and Ross [28], Oldham and Spanier [30], Podlubny [31], and Samko, Kilbas, and Marichev [32]. In particular, the book by Oldham and Spanier [30] has a chronological listing on major works in the study of fractional calculus starting with the correspondence between Leibnitz and L'Hospital in the late seventeenth century and continuing to 1974. In addition, the web site http://people.tuke.sk/igor.podlubny/fc.html, authored by I. Podlubny, is a very useful resource for those studying fractional calculus and its application.

In [37] Zhang used cone theory and the theory of upper and lower solutions to show the existence of at least one positive solution of the fractional order differential equation

$$D^{\alpha}u = f(t,u),\ 0 < t < 1, 0 < \alpha < 1.$$

Daftardar-Gejji [9] extended the results in [37] to show the existence of at least one positive solution of the system of fractional differential equations

$$D^{\alpha_i}u_i = f_i(t, u_1, u_2, \dots, u_n),\ u_i(0) = 0,\ 0 < \alpha_i < 1, 1 \le i \le n.$$

Recently, Bai and Lü [7] showed the existence of positive solutions of the fractional boundary value problem,

$$\begin{aligned} &D^{\alpha}u(t) + f(t, u(t)) = 0,\ 0 < t < 1, 1 < \alpha \le 2, \\ &u(0) = u(1) = 0. \end{aligned}$$

In [38], Zhang used the Leggett-Williams theorem to show the existence of triple positive solutions to the fractional boundary value problem

$$\begin{aligned} &D^{\alpha}_{0+}u(t) = f(t, u(t)),\ 0 < t < 1 \\ &u(0) + u'(0) = 0,\ u(1) + u'(1) = 0. \end{aligned}$$

Throughout this paper, we assume that f and a satisfies the following conditions:

(A1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous;
(A2) $a \in L^{\infty}[0,1]$;
(A3) there exists $m > 0$ such that $a(t) \ge m$ *a.e.* $t \in [0,1]$.

In Section 2 we present some basic definitions from fractional calculus. We also develop sign properties of the kernel $G(t,s)$. We conclude Section 2 with two well-known fixed point theorems. Using the framework developed in Section 2, we state and prove our main results in Section 3. In particular, we give sufficient conditions for the existence of at least one positive solution and at least three positive solutions of (1), (2).

2. Background and Preliminary Results

We begin with some definitions from the theory of fractional calculus. The fractional integral operator of order α for a function $u : (0,\infty) \to \mathbb{R}$ is defined to be

$$I^{\alpha}u(t) = \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}u(s)\,ds,$$

provided the integral converges. For a given function $u : (0, \infty) \to \mathbb{R}^+$, the Riemann-Louiville differential operator D^α of order α is defined to be

$$D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}\, ds$$

where $n = \lfloor \alpha \rfloor + 1$.

Remark: If $u \in C(0,1) \cap L(0,1)$, then $D^\alpha I^\alpha u(t) = u(t)$.

In order to rewrite (1), (2) as an integral equation, we need to know the action of the fractional integral operator I^α on $D^\alpha u$ for a given function u. To this end, we first note that if $\lambda > -1$, then

$$D^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}$$

$$D^\alpha t^{\alpha-k} = 0, \quad k = 1, 2, \ldots, n,$$

where $n = \lfloor \alpha \rfloor$.

The following two lemmas, found in [7], are crucial in finding an integral representation of the boundary value problem (1), (2).

Lemma 2.1. *Let $\alpha > 0$ and $u \in C(0,1) \cap L(0,1)$. Then the solution of $D^\alpha u(t) = 0$ is given by*

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

for some $C_i \in \mathbb{R}, i = 1, 2, \ldots, n$.

Lemma 2.2. *Suppose $u \in C(0,1) \cap L(0,1)$ is such that $D^\alpha u \in C(0,1) \cap L(0,1)$. Then*

$$I^\alpha D^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n} \tag{3}$$

for some $C_i \in \mathbb{R}, i = 1, 2, \ldots, n$.

The next step in inverting the boundary value problem is to find an integral representation of the solution of the linearized problem.

Lemma 2.3. *If $u \in C(0,1) \cap L(0,1)$ is a solution of*

$$D^\alpha u(t) + g(t) = 0, \ 0 < t < 1, \tag{4}$$

$$u(0) = u'(1) = 0, \tag{5}$$

then

$$u(t) = \int_0^1 G(t,s) g(s)\, ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \le s \le t \\ t^{\alpha-1}(1-s)^{\alpha-2}, & t < s < 1 \end{cases}.$$

Proof. Let g be continuous and $1 < \alpha \le 2$ and let $u \in C(0,1) \cap L(0,1)$ be a solution of (4), (5). By (3),

$$u(t) = \int_0^t \frac{-1}{\Gamma(\alpha)} (t-s)^{\alpha-1} g(s)\, ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}.$$

The boundary condition $u(0) = 0$ implies $C_2 = 0$. Thus,

$$u(t) = \int_0^t \frac{-1}{\Gamma(\alpha)}(t-s)^{\alpha-1} g(s)\, ds + C_1 t^{\alpha-1}. \tag{6}$$

Differentiate (6).

$$u'(t) = \frac{-(\alpha-1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} g(s)\, ds + C_1(\alpha-1)t^{\alpha-2}.$$

The boundary condition $u'(1) = 0$ implies that

$$C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} g(s)\, ds,$$

and the proof is complete.

□

Lemma 2.4. *Let $\beta \in (0,1)$ be fixed. The kernel, $G(t,s)$, satisfies the following properties.*

$$G(t,s) \geq 0, (t,s) \in [0,1] \times [0,1). \tag{7}$$

$$\max_{0 \leq t \leq 1} \int_0^1 G(t,s)\, ds = \frac{1}{\alpha(\alpha-1)\Gamma(\alpha)}. \tag{8}$$

$$\min_{\beta \leq t \leq 1} G(t,s) \geq \beta s G(s,s) \text{ for all } 0 \leq s < 1. \tag{9}$$

$$G(t,s) \leq G(s,s), \quad (t,s) \in [0,1] \times [0,1). \tag{10}$$

$$\int_0^1 s\, G(s,s)\, ds > 0. \tag{11}$$

Proof. Inequality (7) holds trivially.

To show (8), define $g_1(t,s) = t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}$, $0 \leq s \leq t$, and $g_2(t,s) = t^{\alpha-1}(1-s)^{\alpha-2}$, $t \leq s < 1$. Note,

$$\int_0^t g_1(t,s)\, ds = \frac{\alpha t^{\alpha-1} - (\alpha-1)t^{\alpha}}{\alpha(\alpha-1)} - \frac{t^{\alpha-1}(1-t)^{\alpha-1}}{\alpha-1},$$

and

$$\int_t^1 g_2(t,s)\, ds = \frac{t^{\alpha}(1-t)^{\alpha-1}}{\alpha-1}.$$

Hence,

$$\int_0^1 G(t,s)\, ds = \frac{\alpha t^{\alpha-1} - (\alpha-1)t^{\alpha}}{\alpha(\alpha-1)\Gamma(\alpha)},$$

from which (8) follows.

Now,

$$\frac{\partial g_1(t,s)}{\partial t} = (\alpha-1)t^{\alpha-2}\left[(1-s)^{\alpha-2} - \left(1 - \frac{s}{t}\right)^{\alpha-2}\right] \leq 0.$$

Hence, $g_1(t,s)$ is decreasing as a function of t. We have,

$$\begin{aligned} g_1(t,s) &\geq g_1(1,s) = \frac{1}{\Gamma(\alpha)}\big((1-s)^{\alpha-2} - (1-s)^{\alpha-1}\big) \\ &= \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-2}s \\ &\geq \beta s\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-2} \\ &\geq \beta s\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-2}s^{\alpha-1} = \beta s G(s,s). \end{aligned}$$

Clearly $g_2(t,s)$ is increasing as a function of t. Hence, for $\beta \leq t \leq s < 1$ we have,

$$\begin{aligned} g_2(t,s) &\geq g_2(\beta,s) \\ &\geq \frac{1}{\Gamma(\alpha)}\beta^{\alpha-1}(1-s)^{\alpha-2} \\ &\geq \frac{1}{\Gamma(\alpha)}\beta(1-s)^{\alpha-2} \\ &\geq \beta s\frac{1}{\Gamma(\alpha)}s^{\alpha-1}(1-s)^{\alpha-2} = \beta s G(s,s). \end{aligned}$$

And so, (9) holds.

From the monotonicity of g_1 and g_2, (10) follows.

Finally,

$$\int_0^1 sG(s,s)\,ds = \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha}(1-s)^{\alpha-2}\,ds > 0$$

Thus (11) is valid and the proof is complete.

□

Remark: By restricting the values of s to be in the interval $[\beta, 1)$ in inequality (9), we can prove the following inequality

$$\min_{\beta\leq t\leq 1} G(t,s) \geq \beta G(s,s), \qquad \text{for all } \beta \leq s < 1. \tag{12}$$

We will use inequality (12) in the proof of Theorem 3.2 and inequality (9) in the proof of Theorem 3.3.

We will use the following well-known cone expansion and compression theorem, see [26], to show the existence of at least one fixed point for T.

Theorem 2.5 (Krasnosel'skiĭ). *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T\colon \mathcal{K}\cap(\overline{\Omega}_2\setminus\Omega_1) \to \mathcal{K}$ be a completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|,\ u \in \mathcal{K}\cap\partial\Omega_1,$ *and* $\|Tu\| \geq \|u\|,\ u \in \mathcal{K}\cap\partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|,\ u \in \mathcal{K}\cap\partial\Omega_1,$ *and* $\|Tu\| \leq \|u\|,\ u \in \mathcal{K}\cap\partial\Omega_2$.

Then T has a fixed point in $\mathcal{K}\cap(\overline{\Omega}_2\setminus\Omega_1)$.

To show the existence of multiple solutions we will use the Leggett-Williams fixed point theorem [27]. To this end we need to define the following subsets of a cone $\mathcal{K}$.

$$\mathcal{K}_c = \{u \in \mathcal{K} : \|u\| < c\}.$$
$$\mathcal{K}(\alpha, b, d) = \{u \in \mathcal{K} : b \leq \alpha(u), \|u\| \leq d\}.$$

We say that the map α is a nonnegative continuous concave functional on a cone $\mathcal{K}$ of a real Banach space $\mathcal{B}$ provided that $\alpha\colon \mathcal{K} \to [0, \infty)$ is continuous and

$$\alpha(tu + (1-t)v) \;\leq\; t\alpha(u) + (1-t)\alpha(v)$$

for all $u, v \in \mathcal{K}$ and $0 \leq t \leq 1$.

Theorem 2.6 (Leggett-Williams)**.** *Suppose $T : \overline{\mathcal{K}}_c \to \overline{\mathcal{K}}_c$ is completely continuous and suppose there exists a concave positive functional α on $\mathcal{K}$ such that $\alpha(u) \leq \|u\|$ for $u \in \overline{\mathcal{K}}_c$. Suppose there exist constants $0 < a < b < d \leq c$ such that*

(B1) $\big\{u \in K(\alpha, b, d) :\ \alpha(u) > b\big\} \neq \emptyset$ *and* $\alpha(Tu) > b$ *if* $u \in K(\alpha, b, d)$;
(B2) $\|Tu\| < u$ *if* $u \in \mathcal{K}_a$, *and*
(B3) $\alpha(Tu) > b$ *for* $u \in \mathcal{K}(\alpha, b, c)$ *with* $\|Tu\| > d$.

Then T has at least three fixed points u_1, u_2, and u_3 such that $\|u_1\| < a$, $b < \alpha(u_2)$ and $\|u_3\| > a$ with $\alpha(u_3) < b$.

3. Main Results

Define $\mathcal{B} = \big(C[0,1], \|.\|\big)$ where $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Then $\mathcal{B}$ is a Banach space.
Define the cone $\mathcal{K} \subset \mathcal{B}$ by

$$\mathcal{K} = \{u \in \mathcal{B} : u(t) \geq 0, t \in [0,1]\},$$

and the operator $T\colon \mathcal{B} \to \mathcal{B}$ by

$$Tu(t) = \int_0^1 G(t,s)a(s)f(u(s))\,ds.$$

Note that fixed points of T are solutions of (1), (2). In order to use Theorems 2.5 and 2.6, we must show that $T : \mathcal{K} \to \mathcal{K}$ is completely continuous.

Lemma 3.1. *Let (A1)-(A3) hold. The operator $T : \mathcal{K} \to \mathcal{K}$ is completely continuous.*

Proof. Since $G(t,s) \geq 0$, then $Tu(t) \geq 0$ for all $u \in \mathcal{K}$. Hence if $u \in \mathcal{K}$ then $Tu \in \mathcal{K}$.
Fix $R > 0$ and let $\mathcal{M} = \{u \in \mathcal{B} : \|u\| < R\}$. Let $L = \max_{0 \leq s \leq R} f(s)$. Then for $u \in \mathcal{M}$,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &\leq \|a\|_\infty L \int_0^1 G(t,s)\,ds \\ &\leq \frac{L\|a\|_\infty}{\alpha(\alpha-1)\Gamma(\alpha)}. \end{aligned}$$

Hence,

$$\|Tu\| \leq \frac{L\|a\|_\infty}{\alpha(\alpha-1)\Gamma(\alpha)}$$

and so $T(\mathcal{M})$ is uniformly bounded.

Define $\delta = \left(\frac{(\alpha-1)\Gamma(\alpha)}{\|a\|_\infty L}\right)^{1/(\alpha-1)}$ and let $t_1, t_2 \in [0,1]$ be such that $t_1 < t_2$ and $t_2 - t_1 < \delta$. Then, for all $u \in \mathcal{M}$,

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &\leq \int_0^1 \big|G(t_2,s) - G(t_1,s)|a(s)f(u(s))\,ds \\
&\leq \frac{\|a\|_\infty L}{(\alpha-1)\Gamma(\alpha)}\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) \\
&\leq \frac{\|a\|_\infty L}{(\alpha-1)\Gamma(\alpha)}(t_2 - t_1)^{\alpha-1} \\
&\leq \varepsilon.
\end{aligned}$$

Thus T is equicontinuous on $\mathcal{M}$. An application of the Arzela-Ascoli Theorem shows that T is completely continuous and the proof is complete. □

In our first result, we show the existence of at least one positive solution of (1), (2).

Theorem 3.2. *Suppose that (A1)-(A3) is satisfied. Let $\beta \in (0,1)$, $M = \|a\|_\infty$, $0 < A \leq \frac{\alpha(\alpha-1)\Gamma(\alpha)}{M}$, and $B \geq (\beta m \int_\beta^1 G(s,s)ds)^{-1}$. Assume there exist positive constant r, R, where $r < R$ and $Br < AR$, such that f satisfies*

(H1) $f(x) \leq AR$ *for all* $x \in [0,R]$,
(H2) $f(x) \geq Br$ *for all* $x \in [0,r]$.

Then the boundary value problem (1), (2) *has at least one positive solution.*

Proof. We show that condition (ii) of Theorem 2.5 is satisfied. By Lemma 3.1, the operator $T : \mathcal{K} \to \mathcal{K}$ is completely continuous.

Define $\Omega_2 = \{u \in \mathcal{B} : \|u\| < R\}$. Let $u \in \mathcal{K} \cap \partial\Omega_2$. From (H1) and (8), we have

$$\begin{aligned}
\|Tu\| &= \max_{0\leq t\leq 1} \int_0^1 G(t,s)a(s)f(u(s))\,ds \\
&\leq AM \max_{0\leq t\leq 1} \int_0^1 G(t,s)\,ds\,R \\
&\leq A\frac{M}{\alpha(\alpha-1)\Gamma(\alpha)} R \\
&\leq \|u\|.
\end{aligned}$$

That is,

$$\|Tu\| \leq \|u\| \quad u \in \partial\mathcal{K} \cap \Omega_2. \tag{13}$$

Define $\Omega_1 = \{u \in \mathcal{B} : \|u\| < r\}$. Let $u \in \mathcal{K} \cap \partial\Omega_1$. From (H2) and (12) we have,

$$\begin{aligned} Tu(t) &\geq \int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &\geq Bmr \int_\beta^1 G(t,s)\,ds \\ &\geq Bmr\beta \int_\beta^1 G(s,s)\,ds \\ &\geq \|u\|. \end{aligned}$$

That is,

$$\|Tu\| \geq \|u\| \quad u \in \mathcal{K} \cap \partial\Omega_1. \tag{14}$$

Since $0 \in \overline{\Omega}_1 \subset \Omega_2$ and inequalities (13) and (14) hold, then by part (ii) of Theorem 2.5, there exists at least one fixed point of T in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. This fixed point is a solution of (1), (2) and the proof is complete. □

In the next result, we show the existence of at least three positive solutions of (1), (2).

Theorem 3.3. *Let $\beta \in (0,1)$, $M = \|a\|_\infty$, $0 < A \leq \frac{\alpha(\alpha-1)\Gamma(\alpha)}{M}$ and $B \geq (\beta m \int_\beta^1 sG(s,s)ds)^{-1}$. Let a, b and c be such that $0 < a < b < c$. Assume that the following hypotheses are satisfied*

(H3) *$f(u) < Aa$ for all $(t,u) \in [0,1] \times [0,a]$,*
(H4) *$f(u) > Bb$ for all $(t,u) \in [\beta,1] \times [b,c]$,*
(H5) *$f(u) \leq Ac$ for all $(t,u) \in [0,1] \times [0,c]$.*

Then the boundary value problem (1), (2) has at least three positive solutions $u_1, u_2, u_3 \in \mathcal{K}$ satisfying

$$\begin{gathered} \|u_1\| < a, \\ b < \alpha(u_2), \\ a < \|u_3\| \text{ with } \alpha(u_3) < b. \end{gathered}$$

Proof. Define a nonnegative functional on $\mathcal{B}$ by $\alpha(u) = \min_{\beta \leq t \leq 1} |u(t)|$. We show that the conditions of Theorem 2.6 are satisfied.

Let $u \in \mathcal{K}_c$. Then $\|u\| \leq c$ and by (H5) and (8),

$$\begin{aligned} \|Tu\| &= \max_{0 \leq t \leq 1} \int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &< \frac{MA}{\alpha(\alpha-1)\Gamma(\alpha)} ds\, c \\ &\leq c. \end{aligned}$$

Hence $T : \mathcal{K}_c \to \mathcal{K}_c$ and by Lemma 3.1, T is completely continuous.

Using an analogous argument, it follows from condition (H3) that if $u \in \mathcal{K}_a$ then $\|Tu\| < a$. Condition (B2) of Theorem 2.6 holds.

Let d be a fixed constant such that $b < d \leq c$. Then $\alpha(d) = d > b$ and $\|d\| = d$. As such, $\mathcal{K}(\alpha, b, d) \neq \emptyset$. Let $u \in \mathcal{K}(\alpha, b, d)$ then $\|u\| \leq d \leq c$ and $\min_{\beta \leq t \leq 1} u(s) \geq b$. By assumption (H4), and (9),

$$\begin{aligned} \alpha(Tu) &= \min_{\beta \leq t \leq 1} \int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &> m \int_0^1 s\,G(s,s)\,ds\,Bb \\ &> b. \end{aligned}$$

That is for all $u \in \mathcal{K}(\alpha, b, d), \alpha(Tu) > b$. Condition (B1) of Theorem 2.6 holds.

Finally, if $u \in \mathcal{K}(\alpha, b, c)$ with $\|Tu\| > d$ then $\|u\| \leq c$ and $\min_{\beta \leq t \leq 1} u(s) \geq b$ and from assumption (H4) we can show $\alpha(Tu) > b$. Condition (B3) of Theorem 2.6 holds.

As a consequence of Theorem 2.6, T has at least three fixed point u_1, u_2, u_3 such that $\|u_1\| < a, b < \alpha(u_2), a < \|u_3\|$ with $\alpha(u_3) < b$. These fixed points are solutions of (1), (2) and the proof is complete. □

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Department of Mathematics & Statistics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA
E-mail address: erkaufmann@ualr.edu
E-mail address: exmboumi@ualr.edu