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New results on the positive pseudo almost periodic solutions for a generalized model of hematopoiesis

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Abstract. The main purpose of this paper is to study the existence and global exponential stability of the positive pseudo almost periodic solutions for a generalized model of hematopoiesis with multiple time-varying delays. By using the exponential dichotomy theory and fixed point theorem, some sufficient conditions are given to ensure that all solutions of this model converge exponentially to the positive pseudo almost periodic solution, which improve and extend some known relevant results. Moreover, an example and its numerical simulation are given to illustrate the theoretical results.

Keywords: positive pseudo almost periodic solution, global exponential stability, model of hematopoiesis.

2010 Mathematics Subject Classification: 34C25,34K13.

1 Introduction

In a classic study of population dynamics, the following delay differential equation model

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{K} \frac{b_i(t)x^m(t-\tau_i(t))}{1+x^n(t-\tau_i(t))}, \tag{1.1}$$

has been used by [4, 9] to describe the dynamics of hematopoiesis (blood cell production). Here $0 \le m \le n$, and

$$a, b_i, \tau_i \colon \mathbb{R} \to (0,+\infty) \text{ are continuous functions for } i = 1,2,\dots,K.$$

In medical terms, $x(t)$ denotes the density of mature cells in blood circulation, the cells are lost from the circulation with a rate $a(t)$ at time t, the flux $f(x(t-\tau_i(t))) = \frac{b_i(t)x^m(t-\tau_i(t))}{1+x^n(t-\tau_i(t))}$ of the cells into the circulation from the stem cell compartment depends on $x(t-\tau_i(t))$ at time $t-\tau_i(t)$, and $\tau_i(t)$ is the time delay between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstreams at time t. As we know, the existence of almost periodic, asymptotically almost periodic, pseudo almost periodic solutions

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are among the most attractive topics in qualitative theory of differential equations due to their applications, especially in biology, economics and physics [3, 17]. The concept of pseudo almost periodicity, which is the central subject in this paper, was introduced by Zhang [14, 15, 16] in the early nineties. Besides, equation (1.1) belongs to a class of biological systems and it (or its analogue equation) has been attracted more attention on problem of almost periodic solutions and pseudo almost periodic solutions because of its extensively realistic significance. For example, some criteria ensuring the existence and stability of positive almost periodic solutions were established in [1, 2, 8, 10, 18] and the references cited therein. Most recently, the author in [10] also obtained a new sufficient condition of the existence and uniqueness of positive pseudo almost periodic solution for equation (1.1) with $m = 0$, which can be described as follows:

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{K} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))}.$$

However, to the best of our knowledge, there exist few results on the existence and exponential stability of the positive pseudo almost periodic solutions of (1.1) without $m = 0$. On the other hand, since the exponential convergence rate can be unveiled, the global exponential convergence plays a key role in characterizing the behavior of dynamical system (see [6, 7, 11]). Thus, it is worthwhile to continue to investigate the existence and global exponential stability of positive pseudo almost periodic solutions of (1.1) without $m = 0$.

Motivated by the above discussions, in this paper we consider the existence, uniqueness and global exponential stability of positive pseudo almost periodic solutions of (1.1). Here in this present paper, a new approach will be developed to obtain a delay-independent condition for the global exponential stability of the positive pseudo almost periodic solutions of (1.1), and the exponential convergence rate can be unveiled.

Throughout this paper, for $i = 1, 2, \dots, K$, it will be assumed that $a\colon \mathbb{R} \to (0, +\infty)$ is an almost periodic function, $b_i, \tau_i\colon \mathbb{R} \to [0, +\infty)$ are pseudo almost periodic functions, and

$$a^- = \inf_{t\in\mathbb{R}} a(t) > 0, \quad a^+ = \sup_{t\in\mathbb{R}} a(t), \quad b_i^- = \inf_{t\in\mathbb{R}} b_i(t) > 0, \quad b_i^+ = \sup_{t\in\mathbb{R}} b_i(t), \tag{1.2}$$

$$0 \le m \le 1, \quad r = \max_{1\le i\le K}\{\sup_{t\in\mathbb{R}} \tau_i(t)\} > 0, \quad \sum_{i=1}^{K} b_i^- > a^+. \tag{1.3}$$

Let $\mathbb{R}_+$ denote the space of nonnegative real numbers, $C = C([-r,\ 0], \mathbb{R})$ be the space of continuous functions equipped with the usual supremum norm $\|\cdot\|$, and let $C_+ = C([-r, 0], \mathbb{R}_+)$. If $x(t)$ is defined on $[-r + t_0,\ \sigma)$ with $t_0, \sigma \in \mathbb{R}$, then we define $x_t \in C$ where $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0]$.

The initial conditions associated with (1.1) are defined as follows:

$$x_{t_0} = \varphi, \quad \varphi \in C_+ \text{ and } \varphi(0) > 0. \tag{1.4}$$

We denote by $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ an admissible solution of admissible initial value problem (1.1) and (1.4). Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of the existence of $x_t(t_0, \varphi)$.

Remark 1.1. Let $f(u) = \frac{u^m}{1+u^n}$, one can get

$$\left.\begin{aligned} f'(u) &= \tfrac{u^{m-1}(m-(n-m)u^n)}{(1+u^n)^2} > 0, \text{ for all } u \in \left(0, \sqrt[n]{\tfrac{m}{n-m}}\right) \\ f'(u) &= \tfrac{u^{m-1}(m-(n-m)u^n)}{(1+u^n)^2} < 0, \text{ for all } u \in \left(\sqrt[n]{\tfrac{m}{n-m}},\ +\infty\right) \end{aligned}\right\}, \text{ where } m < n. \tag{1.5}$$

Since

$$\lim_{\alpha\to 0^+}\frac{\alpha^{m-1}}{1+\alpha^n}=\begin{cases}1, & m=1,\\ +\infty, & m<1,\end{cases}$$

we can choose a positive constant κ such that

$$\frac{\alpha^{m-1}}{1+\alpha^n}>\frac{a^+}{\sum\limits_{i=1}^{K} b_i^-}\quad\text{for all}\quad \alpha\in(0,\ \kappa],$$

and

$$\kappa<\sqrt[n]{\frac{m}{n-m}},\quad\text{if}\quad m<n. \tag{1.6}$$

Moreover, from (1.5), (1.6) implies there exists a constant $\tilde{\kappa}$ such that

$$\kappa<\sqrt[n]{\frac{m}{n-m}}<\tilde{\kappa},\quad \frac{\kappa^m}{1+\kappa^n}=\frac{\tilde{\kappa}^m}{1+\tilde{\kappa}^n},\quad\text{if}\quad m<n. \tag{1.7}$$

2 Preliminary results

In this section, some lemmas and definitions will be presented, which are of importance in proving our main results in Section 3.

In this paper, $BC(\mathbb{R},\mathbb{R})$ denotes the set of bounded continued functions from $\mathbb{R}$ to $\mathbb{R}$. Note that $(BC(\mathbb{R},\mathbb{R}),\|\cdot\|_\infty)$ is a Banach space where $\|\cdot\|_\infty$ denotes the supremum $\|f\|_\infty := \sup\limits_{t\in\mathbb{R}}|f(t)|$.

Definition 2.1 ([3, 17])**.** Let $u(t)\in BC(\mathbb{R},\mathbb{R})$. The function $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon>0$, the set $T(u,\varepsilon)=\{\delta : |u(t+\delta)-u(t)|<\varepsilon \ \text{ for all }\ t\in\mathbb{R}\}$ is relatively dense, i.e., for any $\varepsilon>0$, it is possible to find a real number $l=l(\varepsilon)>0$, with the property that for any interval with length $l(\varepsilon)$, there exists a number $\delta=\delta(\varepsilon)$ in this interval such that $|u(t+\delta)-u(t)|<\varepsilon$, for all $t\in\mathbb{R}$.

We denote by $AP(\mathbb{R},\mathbb{R})$ the set of the almost periodic functions from $\mathbb{R}$ to $\mathbb{R}$. Define the class of functions $PAP_0(\mathbb{R},\mathbb{R})$ as follows:

$$\left\{f\in BC(\mathbb{R},\mathbb{R}) \mid \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|dt=0\right\}.$$

Definition 2.2 ([17])**.** A function $f\in BC(\mathbb{R},\mathbb{R})$ is called pseudo almost periodic if it can be expressed as

$$f=h+\varphi,$$

where $h\in AP(\mathbb{R},\mathbb{R})$ and $\varphi\in PAP_0(\mathbb{R},\mathbb{R})$. The collection of such functions will be denoted by $PAP(\mathbb{R},\mathbb{R})$.

Definition 2.3 ([3, 17])**.** Let $x\in\mathbb{R}^l$ and $Q(t)$ be an $l\times l$ continuous matrix defined on $\mathbb{R}$. The linear system

$$x'(t)=Q(t)x(t) \tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (2.1) satisfying

$$\|X(t)PX^{-1}(s)\|\le ke^{-\alpha(t-s)}\quad\text{for } t\ge s,$$
$$\|X(t)(I-P)X^{-1}(s)\|\le ke^{-\alpha(s-t)}\quad\text{for } t\le s.$$

Lemma 2.4 ([17]). *Assume that $Q(t)$ is an almost periodic matrix function and $g(t) \in PAP(\mathbb{R}, \mathbb{R}^l)$. If the linear system (2.1) admits an exponential dichotomy, then the pseudo almost periodic system*

$$x'(t) = Q(t)x + g(t) \tag{2.2}$$

has a unique pseudo almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)\,ds - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)g(s)\,ds. \tag{2.3}$$

Lemma 2.5 ([3, 17]). *Let $c_i(t)$ be an almost periodic function on $\mathbb{R}$ and*

$$M[c_i] = \lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} c_i(s)\,ds > 0, \quad i = 1, 2, \ldots, l.$$

Then the linear system

$$x'(t) = \mathrm{diag}(-c_1(t), -c_2(t), \ldots, -c_l(t))x(t)$$

admits an exponential dichotomy on $\mathbb{R}$.

Lemma 2.6 ([8, Lemma 2.1]). *Every solution $x(t; t_0, \varphi)$ of (1.1) and (1.4) is positive and bounded on $[t_0, \eta(\varphi))$, and $\eta(\varphi) = +\infty$.*

Lemma 2.7 ([8, Lemma 2.2]). *Suppose that there exists a positive constant $M > \kappa$ such that*

$$\sup_{t\in\mathbb{R}} \left\{ -a(t)M + \sum_{i=1}^{K} b_i(t) \right\} < 0, \quad \text{and} \quad \sqrt[n]{\frac{m}{n-m}} < M \le \tilde{\kappa} \ \text{ if } \ m < n. \tag{2.4}$$

Then, there exists $t_\varphi > t_0$ such that

$$\kappa < x(t; t_0, \varphi) < M \ \text{ for all } \ t \ge t_\varphi. \tag{2.5}$$

Set

$$B^* = \{\varphi \mid \varphi \in PAP(\mathbb{R}, \mathbb{R}) \ \text{ is uniformly continuous on } \mathbb{R},\ K_1 \le \varphi(t) \le K_2, \ \text{ for all } \ t \in \mathbb{R}\}.$$

Then, we get the following lemma.

Lemma 2.8. *B^* is a closed subset of $PAP(\mathbb{R}, \mathbb{R})$.*

Proof. Suppose that $\{x_p\}_{p=1}^{+\infty} \subseteq B^*$ satisfies

$$\lim_{p\to+\infty} \|x_p - x\|_\infty = 0. \tag{2.6}$$

Obviously, $x \in PAP(\mathbb{R}, \mathbb{R})$, and $K_1 \le x(t) \le K_2$, for all $t \in \mathbb{R}$. We next show that x is uniformly continuous on $\mathbb{R}$. In fact, for any $\varepsilon > 0$, from (2.6), we can choose $p > 0$ such that

$$\|x_p - x\|_\infty < \frac{\varepsilon}{3}. \tag{2.7}$$

Note that x_p is uniformly continuous on $\mathbb{R}$. Then, there exists $\delta = \delta(\varepsilon)$ such that

$$|x_p(t_1) - x_p(t_2)| < \frac{\varepsilon}{3}, \text{ where } t_1, t_2 \in \mathbb{R} \text{ and } |t_1 - t_2| < \delta,$$

which, together with (2.7), implies that

$$\begin{aligned} |x(t_1) - x(t_2)| &\le |x(t_1) - x_p(t_1)| + |x_p(t_1) - x_p(t_2)| + |x_p(t_2) - x(t_2)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon\,, \text{ where } t_1, t_2 \in \mathbb{R} \text{ and } |t_1 - t_2| < \delta, \end{aligned}$$

i.e., x is uniformly continuous on $\mathbb{R}$ and $x \in B^*$. Hence, B^* is a closed subset of $PAP(\mathbb{R}, \mathbb{R})$. This completes the proof of Lemma 2.8. $\square$

3 Main results

Theorem 3.1. *Suppose that* (2.4) *holds, and*

$$\sup_{t\in\mathbb{R}}\left\{-a(t)+\sum_{i=1}^{K}b_i(t)\left[M^m\frac{n}{4\kappa}+\frac{1}{1+\kappa^n}m\kappa^{m-1}\right]\right\}<0. \tag{3.1}$$

Then, equation (1.1) *has at least one positive pseudo almost periodic solution* $x^*(t)$.

Proof. Consider $\Upsilon\colon [0;1]\to\mathbb{R}$ defined by

$$\Upsilon(u)=\sup_{t\in\mathbb{R}}\left\{-a(t)+\sum_{i=1}^{K}b_i(t)\left[M^m\frac{n}{4\kappa}+\frac{1}{1+\kappa^n}m\kappa^{m-1}\right]e^u\right\},\quad u\in[0,1].$$

Then, from (3.1), we have

$$\Upsilon(0)=\sup_{t\in\mathbb{R}}\left\{-a(t)+\sum_{i=1}^{K}b_i(t)\left[M^m\frac{n}{4\kappa}+\frac{1}{1+\kappa^n}m\kappa^{m-1}\right]\right\}<0,$$

which implies that there exists a constant $\varsigma\in(0,1]$ such that

$$\Upsilon(\varsigma)=\sup_{t\in\mathbb{R}}\left\{-a(t)+\sum_{i=1}^{K}b_i(t)\left[M^m\frac{n}{4\kappa}+\frac{1}{1+\kappa^n}m\kappa^{m-1}\right]e^{\varsigma}\right\}<0. \tag{3.2}$$

Set

$$B=\{\varphi\mid\varphi\in PAP(\mathbb{R},\mathbb{R})\ \text{ is uniformly continuous on }\mathbb{R},\ \kappa\le\varphi(t)\le M,\ \text{ for all }\ t\in\mathbb{R}\}.$$

It follows from Lemma 2.8 that B is a closed subset of $PAP(\mathbb{R},\mathbb{R})$. Let $\phi\in B$ and $f(t,z)=\phi(t-z)$. From Theorem 5.3 in [17, p. 58] and Definition 5.7 in [17, p. 59], the uniform continuity of ϕ implies that $f\in PAP(\mathbb{R}\times\Omega)$ and f is continuous in $z\in L$ and uniformly in $t\in\mathbb{R}$ for all compact subset L of $\Omega\subset\mathbb{R}$. This, together with $\tau_i\in PAP(\mathbb{R},\mathbb{R})$ and Theorem 5.11 in [17, p. 60], yields

$$\phi(t-\tau_i(t))\in PAP(\mathbb{R},\mathbb{R}),\quad i=1,2,\dots,K.$$

According to Corollary 5.4 in [17, p. 58] and the composition theorem of pseudo almost periodic functions, we have

$$\sum_{i=1}^{K}\frac{b_i(t)\phi^m(t-\tau_i(t))}{1+\phi^n(t-\tau_i(t))}\in PAP(\mathbb{R},\mathbb{R}).$$

We next consider an auxiliary equation

$$x'(t)=-a(t)x(t)+\sum_{i=1}^{K}\frac{b_i(t)\phi^m(t-\tau_i(t))}{1+\phi^n(t-\tau_i(t))}. \tag{3.3}$$

Notice that $M[a]>0$, it follows from Lemma 2.4 that the linear equation

$$x'(t)=-a(t)x(t)$$

admits an exponential dichotomy on $\mathbb{R}$. Thus, by Lemma 2.4, we obtain that the system (3.3) has exactly one pseudo almost periodic solution

$$x^{\phi}(t)=\int_{-\infty}^{t}e^{-\int_s^t a(u)\,du}\left[\sum_{i=1}^{K}\frac{b_i(s)\phi^m(s-\tau_i(s))}{1+\phi^n(s-\tau_i(s))}\right]ds, \tag{3.4}$$

Define a mapping $T\colon PAP(\mathbb{R},\mathbb{R}) \longrightarrow PAP(\mathbb{R},\mathbb{R})$ by setting

$$T(\phi(t)) = x^{\phi}(t),\ \forall\ \phi \in PAP(\mathbb{R},\mathbb{R}).$$

For any $\phi \in B$, from (2.4), we have

$$x^{\phi}(t) \le \int_{-\infty}^{t} e^{-\int_s^t a(u)\,du} \left[\sum_{i=1}^{K} b_i(s)\right] ds \le \int_{-\infty}^{t} e^{-\int_s^t a(u)\,du} a(s)M\,ds = M \ \text{ for all } \ t\in\mathbb{R}. \tag{3.5}$$

Moreover, from Remark 1.1, we get

$$\left.\begin{array}{l} \frac{\kappa^m}{1+\kappa^n} = \frac{\tilde{\kappa}^m}{1+\tilde{\kappa}^n},\ \ \kappa < \sqrt[n]{\frac{m}{n-m}} < M \le \tilde{\kappa} \\ f'(u) = \left(\frac{u^m}{1+u^n}\right)' = \frac{u^{m-1}(m-(n-m)u^n)}{(1+u^n)^2} > 0,\ \forall u \in \left(0,\ \sqrt[n]{\frac{m}{n-m}}\right) \\ f'(u) = \left(\frac{u^m}{1+u^n}\right)' = \frac{u^{m-1}(m-(n-m)u^n)}{(1+u^n)^2} < 0,\ \forall u \in \left(\sqrt[n]{\frac{m}{n-m}},\ +\infty\right) \end{array}\right\},\ \text{where } \ m < n, \tag{3.6}$$

$$\left(\frac{u^n}{1+u^n}\right)' = \frac{nu^{n-1}}{(1+u^n)^2} > 0\ ,\ \text{for all } \ u \in (0,\ +\infty), \tag{3.7}$$

and

$$\frac{\kappa^{m-1}}{1+\kappa^n} > \frac{a^+}{\sum\limits_{i=1}^{K} b_i^-}, \tag{3.8}$$

which yield

$$\begin{aligned} x^{\phi}(t) &\ge \int_{-\infty}^{t} e^{-\int_s^t a(u)\,du} \left[\sum_{i=1}^{K} \frac{b_i(s)\kappa^m}{1+\kappa^n}\right] ds \\ &= \int_{-\infty}^{t} e^{-\int_s^t a(u)\,du} \left[\sum_{i=1}^{K} \frac{b_i(s)\kappa^{m-1}}{1+\kappa^n}\right] \kappa\,ds \\ &\ge \int_{-\infty}^{t} e^{-\int_s^t a(u)\,du} a(s)\kappa\,ds = \kappa, \ \ \text{for all } \ t\in\mathbb{R}. \end{aligned} \tag{3.9}$$

This implies that the mapping T is a self-mapping from B to B. Now, we prove that the mapping T is a contraction mapping on B. In fact, for $\varphi, \psi \in B$, we get

$$\begin{aligned} &\|T(\varphi) - T(\psi)\|_\infty \\ &\quad = \sup_{t\in\mathbb{R}} |T(\varphi)(t) - T(\psi)(t)| \\ &\quad = \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{t} e^{-\int_s^t a(u)\,du} \sum_{i=1}^{K} b_i(s) \left[\frac{\varphi^m(s-\tau_i(s))}{1+\varphi^n(s-\tau_i(s))} - \frac{\psi^m(s-\tau_i(s))}{1+\psi^n(s-\tau_i(s))}\right] ds \right| \\ &\quad \le \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \left[e^{-\int_s^t a(u)\,du} \sum_{i=1}^{K} b_i(s) \left[\left| \frac{\varphi^m(s-\tau_i(s))}{1+\varphi^n(s-\tau_i(s))} - \frac{\varphi^m(s-\tau_i(s))}{1+\psi^n(s-\tau_i(s))} \right| \right.\right. \\ &\qquad\qquad \left.\left. + \left| \frac{\varphi^m(s-\tau_i(s))}{1+\psi^n(s-\tau_i(s))} - \frac{\psi^m(s-\tau_i(s))}{1+\psi^n(s-\tau_i(s))} \right| \right] \right] ds \\ &\quad \le \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \left[e^{-\int_s^t a(u)\,du} \sum_{i=1}^{K} b_i(s) \left[\varphi^m(s-\tau_i(s)) \left| \frac{1}{1+\varphi^n(s-\tau_i(s))} - \frac{1}{1+\psi^n(s-\tau_i(s))} \right| \right.\right. \\ &\qquad\qquad \left.\left. + \frac{1}{1+\psi^n(s-\tau_i(s))} \left| \varphi^m(s-\tau_i(s)) - \psi^m(s-\tau_i(s)) \right| \right] \right] ds. \end{aligned} \tag{3.10}$$

From the differential mid-value theorem, we have

$$\left|\frac{1}{1+x^n}-\frac{1}{1+y^n}\right|=\left|\frac{-n\theta^{n-1}}{(1+\theta^n)^2}\right||x-y|\le\frac{n\theta^{n-1}}{(2\sqrt{\theta^n})^2}|x-y|\le\frac{n}{4\kappa}|x-y|, \tag{3.11}$$

and

$$|x^m-y^m|\le m\kappa^{m-1}|x-y|, \tag{3.12}$$

where $x,y\in[\kappa,\ M]$, θ lies between x and y. Then, (3.10), (3.11) and (3.12) yield

$$\begin{aligned}
&\|T(\varphi)-T(\psi)\|_\infty\\
&\quad\le\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\int_s^t a(u)\,du}\sum_{i=1}^{K}b_i(s)\left[M^m\frac{n}{4\kappa}+\frac{1}{1+\kappa^n}m\kappa^{m-1}\right]|\varphi(s-\tau_i(s))-\psi(s-\tau_i(s))|\,ds\\
&\quad\le\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\int_s^t a(u)\,du}a(s)e^{-\varsigma}|\varphi(s-\tau_i(s))-\psi(s-\tau_i(s))|\,ds\\
&\quad\le e^{-\varsigma}\|\varphi-\psi\|_\infty.
\end{aligned}$$

Noting that $e^{-\varsigma}<1$, it is clear that the mapping T is a contraction on B. Using Theorem 0.3.1 of [5], we obtain that the mapping T possesses a unique fixed point $\varphi^*\in B$ with $T\varphi^*=\varphi^*$. By (3.3), φ^* satisfies (1.1). So φ^* is a positive pseudo almost periodic solution of (1.1) in B. The proof of Theorem 3.1 is completed. □

By using Lemma 2.7, the proof of global exponential stability of $x^*(t)$ is similar to that of Theorem 3.2 in [8], and we obtain the following theorem.

Theorem 3.2. *Under the assumptions of Theorem 3.1, the pseudo almost periodic solution $x^*(t)$ of equation (1.1) is globally exponentially stable.*

Remark 3.3. Most recently, B. Liu [8] considered the almost periodic solution problem of (1.1) with almost periodic coefficients and delays under the assumption of (2.4). Noting that the pseudo almost periodic functions is a natural generalization of the concept of almost periodicity, it is obvious that the main results in [8] are special cases of our results.

4 An example

In this section, we present an example to check the validity of our results we obtained in the previous sections.

Example 4.1. Consider the following model of hematopoiesis with multiple time-varying delays:

$$\begin{aligned}
x'(t)=&-1.5x(t)+\frac{1}{2}\left(2+\frac{1}{2}\left|\cos\sqrt{2}t\right|\right)\frac{x^{\frac14}(t-2e^{-t^4\sin^2 t})}{1+x^{\frac12}(t-2e^{-t^4\sin^2 t})}\\
&+\frac{1}{2}\left(2+\frac{1}{2}\left|\sin\sqrt{3}t\right|\right)\frac{x^{\frac14}(t-2e^{-t^6\sin^2 t})}{1+x^{\frac12}(t-2e^{-t^6\sin^2 t})}.
\end{aligned}\tag{4.1}$$

Obviously,

$$a^+=a^-=1.5,\quad b_1^-=b_2^-=1,\quad b_1^+=b_2^+=1.25,\quad n=\frac{1}{2},\quad m=\frac{1}{4},\quad r=2e.$$

Let $\kappa = 0.5$ and $M = 2$. Then

$$\frac{\left(\frac{1}{2}\right)^{\frac{1}{4}}}{1+\left(\frac{1}{2}\right)^{\frac{1}{2}}} = \frac{2^{\frac{1}{4}}}{1+2^{\frac{1}{2}}},$$

and

$$M = \tilde{\kappa} = 2, \qquad -a^{-}M + b_1^{+} + b_2^{+} = -0.5 < 0,$$

$$\inf_{\alpha\in[0,\kappa]} \frac{\alpha^{m-1}}{1+\alpha^{n}} \geq \frac{\left(\frac{1}{2}\right)^{\frac{1}{4}-1}}{1+\left(\frac{1}{2}\right)^{\frac{1}{2}}} \approx 0.9852 > 0.75 = \frac{a^{+}}{\sum\limits_{i=1}^{K} b_i^{-}},$$

$$-a^{-} + (b_1^{+} + b_2^{+})\left[M^{m}\frac{n}{4\kappa} + \frac{1}{1+\kappa^{n}} m\kappa^{m-1}\right]$$

$$= -1.5 + 2.5 \times \left[2^{\frac{1}{4}} \times \frac{1}{4} + \frac{1}{1+\left(\frac{1}{2}\right)^{\frac{1}{2}}} \times \frac{1}{4} \times 2^{\frac{3}{4}}\right] \approx -0.141 < 0,$$

which imply that (4.1) satisfies the assumptions of Theorem 3.2. Therefore, equation (4.1) has a unique positive pseudo almost periodic solution $x^*(t)$, which is globally exponentially stable with the exponential convergence rate $\lambda \approx 0.01$. The numerical simulation in Figure 4.1 strongly supports the conclusion.

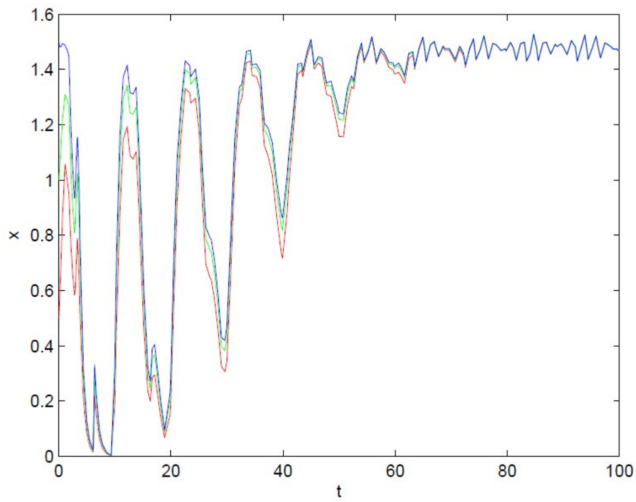

Figure 4.1: Numerical solution $x(t)$ of equation (4.1) for initial value $\varphi(s) \equiv 0.5, 1.0, 1.5$ $s \in [-2e, 0]$.

Remark 4.2. It is easy to check that the results in [12, 13, 18] are invalid for the global exponential stability of positive pseudo almost periodic solution of (4.1) since

$$\tau_1(t) = 2e^{-t^4 \sin^2 t} \quad \text{and} \quad \tau_2(t) = 2e^{-t^6 \sin^2 t}$$

are pseudo almost periodic functions, not almost periodic. This implies that the results of this paper can be applied to the case not covered in the existing works. As pointed out in [8], it is difficult to establish the criteria ensuring global exponential stability of the positive pseudo almost periodic solution for equation (1.1) with $m > 1$. Whether or not our results and method in this paper are available for this case, it is an open interesting problem and we leave it as our work in the future.

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