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Weighted Cauchy-type problem of a functional differ-integral equation

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Abstract

In this work, we are concerned with a nonlinear weighted Cauchy type problem of a differ-integral equation of fractional order. We will prove some local and global existence theorems for this problem, also we will study the uniqueness and stability of its solution.

Key words: Fractional calculus; Weighted Cauchy-type problem; Stability.

1 Preliminaries

Let $L_1(I)$ be the class of Lebesgue integrable functions on the interval $I = [a, b]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ be the gamma function. Recall that the operator T is compact if it is continuous and maps bounded sets into relatively compact ones. The set of all compact operators from the subspace $U \subset X$ into the Banach space X is denoted by $C(U, X)$. Moreover, we set $B_r = \{u \in L_1(I) : ||u|| < r, r > 0\}$ and $||u||_1 = \int_0^1 e^{-Nt}|u(t)|dt$.
Definition 1.1 The fractional integral of the function $f(.) \in L_1(I)$ of order $\beta \in R^+$ is defined by (see [5] - [8])

$$I_a^\beta f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f(s)\, ds.$$

Definition 1.2 The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0, 1)$ is defined as (see [5] - [8])

$$D_a^\alpha f(t) = \frac{d}{dt} I_a^{1 - \alpha} f(t), \quad t \in [a, b].$$

2 Introduction

We deal with the nonlinear weighted Cauchy-type problem:

$$\left\{ \begin{aligned} D^\alpha u(t) &= f(t, u(\phi(t))), \\ t^{1-\alpha} u(t)|_{t=0} &= b. \end{aligned} \right. \tag{1}$$

This problem has been intensively studied by many authors (see [2], for instance). In comparison with earlier results of this type we get more general assumptions. In [2], the function $f(t,u)$ is assumed to be continuous on $R^+ \times R$, $|f(t,u)| \leq t^\mu e^{-\sigma t}\psi(t)|u|^m, \mu \geq 0, m > 1, \sigma > 0$, $\psi(t)$ is a continuous function on R^+ and $\phi(t) = t$.
In this work, we investigate the behavior of solutions for problem (1) with certain nonlinearities. Using the equivalence of the fractional differ-integral problem with the corresponding Volterra integral equation, we prove the existence of L_1-solution such that the function f satisfies the Caratheodory conditions and the growth condition. Moreover, we will study the uniqueness and the stability of the solution.

Now, let us recall some results which will be needed in the sequel.

Theorem 2.1 (Rothe Fixed Point Theorem) [4]

Let U be an open and bounded subset of a Banach space E, let $T \in C(\bar{U}, E)$. Then T has a fixed point if the following condition holds

$$T(\partial U) \subseteq \bar{U}.$$

Theorem 2.2 (Nonlinear alternative of Laray-Schauder type) [4]

Let U be an open subset of a convex set D in a Banach space E. Assume $0 \in U$ and $T \in C(\bar{U}, E)$. Then either

(A1) T has a fixed point in $\bar{U}$, or

(A2) there exists $\gamma \in (0,1)$ and $x \in \partial U$ such that $x = \gamma\, Tx$.

Theorem 2.3 (Kolmogorov compactness criterion) [3]

Let $\Omega \subseteq L^p\,(0,1),\ 1 \leq p < \infty$. If

(i) Ω is bounded in $L^p\,(0,1)$ and

(ii) $x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^p\,(0,1)$, where

$$x_h(t) = \frac{1}{h}\int_t^{t+h} x(s)\, ds.$$

3 Main results

We begin this section by proving the equivalence of problem (1) with the corresponding Volterra integral equation:

$$u(t) = b\, t^{\alpha-1} + \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(\phi(s)))\, ds, \quad t \in (0,1). \tag{2}$$

Indeed: Let $u(t)$ be a solution of (2), multiply both sides of (2) by $t^{1-\alpha}$, we get

$$t^{1-\alpha}u(t) = b + t^{1-\alpha}\, I^\alpha\, f(t, u(\phi(t))),$$

which gives

$$t^{1-\alpha}u(t)|_{t=0} \ = \ b.$$

Now, operating by $I^{1-\alpha}$ on both sides of (2), then

$$I^{1-\alpha}u(t) \ = \ b_1 \ + \ I \ f(t, u(\phi(t))).$$

Differentiating both sides we get

$$D^{\alpha} \ u(t) \ = \ f(t, u(\phi(t))).$$

Conversely, let $u(t)$ be a solution of (1), integrate both sides, then

$$I^{1-\alpha}u(t) \ - \ I^{1-\alpha}u(t)|_{t=0} \ = \ I \ f(t, u(\phi(t))),$$

operating by I^{α} on both sides of the last equation, then

$$Iu(t) \ - \ I^{\alpha} \ C \ = \ I^{1+\alpha} \ f(t, u(\phi(t))),$$

differentiate both sides, then

$$u(t) \ - \ C_1 \ t^{\alpha-1} \ = \ I^{\alpha} \ f(t, u(\phi(t))),$$

from the initial condition, we find that $C_1 = b$, then we obtain (2), i.e. Problem (1) and equation (2) are equivalent to each other.
Now define the operator T as

$$Tu(t) \ = \ b \ t^{\alpha-1} \ + \ \int_0^t \frac{(t \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ f(s, u(\phi(s))) \ ds, \quad t \in (0,1).$$

To solve equation (2) it is necessary to find a fixed point of the operator T.
Now, we present our main result by proving some local and global existence theorems for the integral equation (2) in L_1. To facilitate our discussion, let us first state the following assumptions:

(i) $f : (0,1) \times R \to R$ be a function with the following properties:

  (a) for each $t \in (0,1), f(t,.)$ is continuous,

  (b) for each $u \in R, f(.,u)$ is measurable,

  (c) there exist two real functions $t \to a(t), t \to b(t)$ such that

$$| \ f(t,u) \ | \ \leq \ a(t) \ + \ b(t) \ | \ u \ |, \quad \text{for each} \ \ t \in (0,1), \quad u \in R,$$

  where $a(.) \in L_1(0,1)$ and $b(.)$ is measurable and bounded.

(ii) $\phi : (0,1) \to (0,1)$ is nondecreasing and there is a constant $M > 0$ such that $\phi' \geq M$ a.e. on $(0,1)$.

Now, for the local existence of the solutions we have the following theorem:

Theorem 3.1

Let the assumptions (i) and (ii) are satisfied.

$$\text{If} \qquad \sup \mid b(t) \mid \; < \; M\,\Gamma(1+\alpha), \tag{3}$$

then the fractional order integral equation (2) has a solution $u \in B_r$, where

$$r \;\leq\; \frac{\frac{b}{\alpha} \;+\; \frac{1}{\Gamma(1+\alpha)} \mid\mid a \mid\mid}{1 \;-\; \frac{1}{\Gamma(1+\alpha)} \; \sup \mid b(t) \mid \frac{1}{M}}.$$

Proof. Let u be an arbitrary element in B_r. Then from the assumptions (i) - (ii), we have

$$\begin{aligned}
||Tu|| \;&=\; \int_0^1 \mid Tu(t) \mid dt \\
&\leq\; \int_0^1 \mid b\, t^{\alpha-1} \mid dt \;+\; \int_0^1 \mid \int_0^t \frac{(t\;-\;s)^{\alpha-1}}{\Gamma(\alpha)} \; f(s, u(\phi(s)))\; ds \mid dt \\
&\leq\; \left(\frac{b\, t^{\alpha}}{\alpha}\right)_0^1 \;+\; \int_0^1 \int_s^1 \frac{(t\;-\;s)^{\alpha-1}}{\Gamma(\alpha)} \; dt \mid f(s, u(\phi(s))) \mid ds \\
&\leq\; \frac{b}{\alpha} \;+\; \int_0^1 \frac{(t\;-\;s)^{\alpha}}{\Gamma(1+\alpha)} |_s^1 \; (\mid a(s) \mid \;+\; \mid b(s) \mid \mid u(\phi(s)) \mid)\; ds \\
&\leq\; \frac{b}{\alpha} \;+\; \int_0^1 \frac{(1\;-\;s)^{\alpha}}{\Gamma(1+\alpha)} \; (\mid a(s) \mid \;+\; \mid b(s) \mid \mid u(\phi(s)) \mid)\; ds \\
&\leq\; \frac{b}{\alpha} \;+\; \frac{1}{\Gamma(1+\alpha)} \int_0^1 \; (\mid a(s) \mid \;+\; \mid b(s) \mid \mid u(\phi(s)) \mid)\; ds \\
&\leq\; \frac{b}{\alpha} \;+\; \frac{1}{\Gamma(1+\alpha)} \mid\mid a \mid\mid \;+\; \frac{1}{\Gamma(1+\alpha)} \; \sup \mid b(t) \mid \int_0^1 \mid u(\phi(s)) \mid ds \\
&\leq\; \frac{b}{\alpha} \;+\; \frac{1}{\Gamma(1+\alpha)} \mid\mid a \mid\mid \;+\; \frac{1}{\Gamma(1+\alpha)} \; \sup \mid b(t) \mid . \frac{1}{M} \int_0^1 \mid u(\phi(s)) \mid \mid \phi' \mid ds \\
&\leq\; \frac{b}{\alpha} \;+\; \frac{1}{\Gamma(1+\alpha)} \mid\mid a \mid\mid \;+\; \frac{1}{\Gamma(1+\alpha)} \; \sup \mid b(t) \mid . \frac{1}{M} \int_{\phi(0)}^{\phi(1)} \mid u(x) \mid dx \\
&\leq\; \frac{b}{\alpha} \;+\; \frac{1}{\Gamma(1+\alpha)} \mid\mid a \mid\mid \;+\; \frac{1}{\Gamma(1+\alpha)} \; \sup \mid b(t) \mid . \frac{1}{M} \mid\mid u \mid\mid.
\end{aligned}$$

The last estimate shows that the operator T maps L_1 into itself. Now, let $u \in \partial B_r$, that is, $||u|| = r$, then the last inequality implies

$$||Tu|| \;\leq\; \frac{b}{\alpha} \;+\; \frac{1}{\Gamma(1+\alpha)} \mid\mid a \mid\mid \;+\; \frac{1}{\Gamma(1+\alpha)} \; \sup \mid b(t) \mid . \frac{1}{M} \; r.$$

Then $T(\partial B_r) \subset \bar{B}_r$ (closure of B_r) if

$$||Tu|| \;\leq\; \frac{b}{\alpha} \;+\; \frac{1}{\Gamma(1+\alpha)} \mid\mid a \mid\mid \;+\; \frac{1}{\Gamma(1+\alpha)} \; \sup \mid b(t) \mid . \frac{1}{M} \; r \;\leq\; r,$$

which implies that

$$\frac{b}{\alpha} + \frac{1}{\Gamma(1+\alpha)} \mid\mid a \mid\mid + \frac{1}{\Gamma(1+\alpha)} \sup \mid b(t) \mid . \frac{1}{M} r \leq r.$$

Therefore

$$r \leq \frac{\frac{b}{\alpha} + \frac{1}{\Gamma(1+\alpha)} \mid\mid a \mid\mid}{1 - \frac{1}{\Gamma(1+\alpha)} \sup \mid b(t) \mid \frac{1}{M}}.$$

Using inequality (3) we deduce that $r > 0$. Moreover, we have

$$\begin{aligned} ||f|| &= \int_0^1 \mid f(s, u(\phi(s))) \mid ds \\ &\leq \int_0^1 (\mid a(s) \mid + \mid b(s) \mid \mid u(\phi(s)) \mid) \; ds \\ &\leq ||a|| + \sup \mid b(t) \mid . \frac{1}{M} ||u||. \end{aligned}$$

This estimation shows that f in $L_1(0,1)$.
Further, f is continuous in u (assumption (a)) and I^α maps $L_1(0,1)$ continuously into itself, $I^\alpha f(t, u(\phi(t)))$ is continuous in u. Since u is an arbitrary element in B_r, T maps B_r continuously into $L_1(0,1)$.
Now, we will show that T is compact, to achieve this goal we will apply Theorem 2.3. So, let Ω be a bounded subset of B_r. Then $T(\Omega)$ is bounded in $L_1(0,1)$, i.e. condition (i) of Theorem 2.3 is satisfied. It remains to show that $(Tu)_h \to Tu$ in $L_1(0,1)$ as $h \to 0$, uniformly with respect to $Tu \in T\;\Omega$. We have the following estimation:

$$\begin{aligned} ||(Tu)_h - Tu|| &= \int_0^1 \mid (Tu)_h(t) - (Tu)(t) \mid dt \\ &= \int_0^1 \mid \frac{1}{h} \int_t^{t+h} (Tu)(s)\; ds - (Tu)(t) \mid dt \\ &\leq \int_0^1 \left(\frac{1}{h} \int_t^{t+h} \mid (Tu)(s) - (Tu)(t) \mid ds \right) dt \\ &\leq \int_0^1 \frac{1}{h} \int_t^{t+h} \mid b\; s^{\alpha-1} - b\; t^{\alpha-1} \mid ds\; dt \\ &+ \int_0^1 \frac{1}{h} \int_t^{t+h} \mid I^\alpha\; f(s, u(\phi(s))) - I^\alpha\; f(t, u(\phi(t))) \mid ds\; dt. \end{aligned}$$

Since $f \in L_1(0,1)$ we get that $I^\alpha f(.) \in L_1(0,1)$. Moreover $t^{\alpha-1} \in L_1(0,1)$. So, we have (see [1])

$$\frac{1}{h} \int_t^{t+h} \mid b\; s^{\alpha-1} - b\; t^{\alpha-1} \mid ds \to 0$$

and

$$\frac{1}{h} \int_t^{t+h} \mid I^\alpha\; f(s, u(\phi(s))) - I^\alpha\; f(t, u(\phi(t))) \mid ds \to 0$$

for a.e. $t \in (0,1)$. Therefore, by Theorem 2.3, we have that $T(\Omega)$ is relatively compact, that is, T is a compact operator.
Therefore, Theorem 2.1 with $U = B_r$ and $E = L_1(0,1)$ implies that T has a fixed point. This complete the proof.

Now we prove the existence of global solution:

Theorem 3.2

Let the conditions (i) - (ii) be satisfied in addition to the following condition:

(iii) Assume that every solution $u(.) \in L_1(0,1)$ to the equation

$$u(t) \; = \; \gamma \; \left(\; b \; t^{\alpha-1} \; + \; \int_0^t \; \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \; f(s, u(\phi(s))) \; ds \; \right) \quad \text{a.e. on } \; (0,1), \; 0 < \; \alpha \; < \; 1$$

satisfies $||u|| \neq r$ (r is arbitrary but fixed).

Then the fractional order integral equation (2) has at least one solution $u \in L_1(0,1)$.
Proof. Let u be an arbitrary element in the open set $B_r = \{u : ||u|| < r, r > 0\}$. Then from the assumptions (i) - (ii), we have

$$||Tu|| \; \leq \; \frac{b}{\alpha} \; + \; \frac{1}{\Gamma(1+\alpha)} \; || \; a \; || \; + \; \frac{1}{\Gamma(1+\alpha)} \; \sup | \; b(t) \; |.\frac{1}{M} \; || \; u \; ||.$$

The above inequality means that the operator T maps B_r into L_1. Moreover, we have

$$||f|| \; \leq \; ||a|| \; + \; \sup | \; b(t) \; |.\frac{1}{M} \; ||u||.$$

This estimation shows that f in $L_1(0,1)$.
As a consequence of Theorem 3.1 we get that T maps B_r continuously into $L_1(0,1)$ and T is compact.
Set $U = B_r$ and $D = E = L_1(0,1)$, then in the view of assumption (iii) the condition $A2$ of Theorem 2.2 does not hold. Therefore, Theorem 2.2 implies that T has a fixed point. This complete the proof.

4 Uniqueness of the solution

For the uniqueness of the solution we have the following theorem:

Theorem 4.1

Let the assumptions of Theorem 3.1 be satisfied, but instead of assumption (i) consider the following conditions:

$$| \; f(t,u) \; - \; f(t,v) \; | \; \leq \; L \; | \; u \; - \; v \; |$$

and

$$| \; f(t,0) \; | \; \leq \; a(t),$$

then the fractional order integral equation (2) has a unique solution.

Proof. Let $u_1(t)$ and $u_2(t)$ be any two solutions of equation (2), then

$$\begin{aligned} u_2(t) - u_1(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(f(s, u_2(\phi(s))) - f(s, u_1(\phi(s)))\right) ds, \\ e^{-Nt} | u_2(t) - u_1(t) | &\leq L\, e^{-Nt} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, | u_2(\phi(s)) - u_1(\phi(s)) | \, ds. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 e^{-Nt} | u_2(t) - u_1(t) | \, dt &\leq L \int_0^1 e^{-Nt} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u_2(\phi(s)) - u_1(\phi(s))| \, ds \, dt, \\ || u_2 - u_1 ||_1 &\leq L \int_0^1 \int_s^1 e^{-Nt} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt \, |u_2(\phi(s)) - u_1(\phi(s))| \, ds \\ &= L \int_0^1 \int_s^1 e^{-N(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt \, e^{-Ns} \, |u_2(\phi(s)) - u_1(\phi(s))| ds \\ &= L \int_0^1 \int_0^{N(1-s)} e^{-\theta} \frac{\theta^{\alpha-1}}{N^{\alpha-1}\Gamma(\alpha)} \frac{d\theta}{N} e^{-Ns} |u_2(\phi(s)) - u_1(\phi(s))| ds \\ &\leq \frac{L}{N^\alpha} \int_0^1 e^{-Ns} \, |u_2(\phi(s)) - u_1(\phi(s))| ds \\ &\leq \frac{L}{MN^\alpha} \int_0^1 e^{-Ns} \, |u_2(\phi(s)) - u_1(\phi(s))| \, |\phi'| ds \\ &= \frac{L}{MN^\alpha} \int_{\phi(0)}^{\phi(1)} e^{-N\phi^{-1}(x)} \, |u_2(x) - u_1(x)| \, dx \\ &\leq \frac{L}{MN^\alpha} \int_{\phi(0)}^{\phi(1)} e^{-Nx} \, |u_2(x) - u_1(x)| \, dx \\ &\leq \frac{L}{MN^\alpha} \, || u_2 - u_1 ||_1. \end{aligned}$$

We choose N such that $MN^\alpha > L$, therefore

$$|| u_2 - u_1 ||_1 < || u_2 - u_1 ||_1,$$

which implies that

$$u_1(t) = u_2(t).$$

5 Stability

Now we study the stability of the Weighted Cauchy-type problem (1).

Theorem 5.1

Let the assumptions of Theorem 4.1 be satisfied, then the solution of the Weighted Cauchy-type problem (1) is uniformly stable.
Proof. Let $u(t)$ be a solution of

$$u(t) \;=\; b\, t^{\alpha-1} \;+\; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s, u(\phi(s)))\, ds,$$

and let $\widetilde{u}(t)$ be a solution of the above equation such that $t^{1-\alpha}\, \widetilde{u}(t)|_{t=0} = \widetilde{b}$, then

$$\begin{aligned}
u(t) \;-\; \widetilde{u}(t) \;&=\; (b \;-\; \widetilde{b})\, t^{\alpha-1} \;+\; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, (f(s, u(\phi(s))) - f(s, \widetilde{u}(\phi(s))))\, ds,\\
e^{-Nt}|u(t) \;-\; \widetilde{u}(t)| \;&\leq\; e^{-Nt}|b \;-\; \widetilde{b}|\, t^{\alpha-1} \;+\; Le^{-Nt}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, |u(\phi(s)) - \widetilde{u}(\phi(s))|\, ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^1 e^{-Nt}|\, u(t) \;-\; \widetilde{u}(t)\,|\, dt \;&\leq\; \int_0^1 e^{-Nt}\,|\, b \;-\; \widetilde{b}\,|\, t^{\alpha-1}\, dt\\
&+\; L\int_0^1 e^{-Nt}\int_0^t \frac{(t \;-\; s)^{\alpha-1}}{\Gamma(\alpha)}\, |u(\phi(s)) \;-\; \widetilde{u}(\phi(s))|\, ds\, dt,\\
||\, u \;-\; \widetilde{u}\, ||_1 \;&\leq\; \int_0^N e^{-s}\,|\, b \;-\; \widetilde{b}\,|\, \frac{s^{\alpha-1}}{N^{\alpha-1}}\,\frac{ds}{N}\\
&+\; L\int_0^1\int_s^1 e^{-Nt}\,\frac{(t \;-\; s)^{\alpha-1}}{\Gamma(\alpha)}\, dt\, |u(\phi(s)) \;-\; \widetilde{u}(\phi(s))|\, ds\\
&\leq\; \frac{\Gamma(\alpha)}{N^\alpha}\,|\, b \;-\; \widetilde{b}\,|\\
&+\; L\int_0^1\int_s^1 e^{-N(t-s)}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt\; e^{-Ns}\, |u(\phi(s)) - \widetilde{u}(\phi(s))| ds\\
&=\; \frac{\Gamma(\alpha)}{N^\alpha}\,|\, b \;-\; \widetilde{b}\,|\\
&+\; L\int_0^1\int_0^{N(1-s)} e^{-\theta}\frac{\theta^{\alpha-1}}{N^{\alpha-1}\Gamma(\alpha)}\frac{d\theta}{N} e^{-Ns}|u(\phi(s)) - \widetilde{u}(\phi(s))| ds\\
&\leq\; \frac{\Gamma(\alpha)}{N^\alpha}\,|\, b \;-\; \widetilde{b}\,| \;+\; \frac{L}{N^\alpha}\int_0^1 e^{-Ns}\, |u(\phi(s)) \;-\; \widetilde{u}(\phi(s))| ds\\
&\leq\; \frac{\Gamma(\alpha)}{N^\alpha}\,|b \;-\; \widetilde{b}| \;+\; \frac{L}{MN^\alpha}\int_0^1 e^{-Ns}\, |u(\phi(s)) \;-\; \widetilde{u}(\phi(s))|\, |\phi'| ds\\
&=\; \frac{\Gamma(\alpha)}{N^\alpha}\,|b \;-\; \widetilde{b}| \;+\; \frac{L}{MN^\alpha}\int_{\phi(0)}^{\phi(1)} e^{-N\phi^{-1}(x)}\, |u(x) \;-\; \widetilde{u}(x)|\, dx
\end{aligned}$$

$$
\begin{aligned}
& \leq \frac{\Gamma(\alpha)}{N^\alpha}\,|b - \widetilde{b}| + \frac{L}{MN^\alpha}\int_{\phi(0)}^{\phi(1)} e^{-Nx}\,|u(x) - \widetilde{u}(x)|\,dx \\
& \leq \frac{\Gamma(\alpha)}{N^\alpha}\,|\,b - \widetilde{b}\,| + \frac{L}{MN^\alpha}\,||\,u - \widetilde{u}\,||_1 \\
\left(1 - \frac{L}{MN^\alpha}\right)\,||u - \widetilde{u}||_1 & \leq \frac{\Gamma(\alpha)}{N^\alpha}\,|\,b - \widetilde{b}\,| \\
\Rightarrow ||u - \widetilde{u}||_1 & \leq \left(1 - \frac{L}{MN^\alpha}\right)^{-1} . \frac{\Gamma(\alpha)}{N^\alpha}\,|\,b - \widetilde{b}\,| \\
& = \frac{M\ \Gamma(\alpha)}{M\ N^\alpha - L}\,|\,b - \widetilde{b}\,|.
\end{aligned}
$$

Therefore, if $|b - \widetilde{b}| < \delta(\varepsilon)$, then $||u - \widetilde{u}|| < \varepsilon$. Now from the equivalence we get that the solution of the Weighted Cauchy-type problem (1) is uniformly stable.

References

[1] Swartz, C. *Measure, Integration and Function spaces*, World Scientific, Singapore (1994).

[2] Furati, K. M. and Tatar N. E. Power-type estimates for a nonlinear fractional differential equation, *nonlinear analysis*, 62 (2005), 1025-1036.

[3] Dugundji, J. Granas, A. *Fixed Point theory*, Monografie Mathematyczne, PWN, Warsaw (1982).

[4] Deimling, K. *Nonlinear Functional Analysis*, Springer-Verlag (1985).

[5] Miller, K. S. and Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, New York (1993).

[6] Podlubny, I. and EL-Sayed, A. M. A. On two definitions of fractional calculus, *Preprint UEF 03-96 (ISBN 80-7099-252-2), Slovak Academy of Science-Institute of Experimental phys.* (1996).

[7] Podlubny, I. *Fractional Differential Equations*, Acad. press, San Diego-New York-London (1999).

[8] Samko, S., Kilbas, A. and Marichev, O. L. *Fractional Integrals and Derivatives*, Gordon and Breach Science Publisher, (1993).

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