# Sufficient condition for existence of solutions for higher-order resonance boundary value problem with one-dimensional p-Laplacian ${ }^{1}$ 

Liu Yang ${ }^{1}$ Chunfang Shen ${ }^{1}$ Xiping Liu ${ }^{2}$<br>(1 Department of Mathematics, Hefei, Teacher's College, Hefei Anhui Province, 236032, PR China<br>2 College of Science, University of Shanghai for Science and Technology, Shanghai, 200093, PR China)


#### Abstract

By using coincidence degree theory of Mawhin, existence results for some higher order resonance multipoint boundary value problems with one dimensional p-Laplacian operator are obtained.


Keywords: boundary value problem; one-dimensional p-Laplacian; resonance; coincidence degree.

## 1. Introduction

In this paper we consider higher-order multi-point boundary value problem with one-dimensional p-Laplacian

$$
\begin{equation*}
\left(\varphi_{p}\left(x^{(i)}(t)\right)\right)^{(n-i)}=f\left(t, x(t), x^{\prime}(t), \cdots, x^{(n-1)}(t)\right)+e(t), t \in(0,1) \tag{1.1}
\end{equation*}
$$

subject to one of the following boundary conditions:
$x(1)=\sum_{j=1}^{m-2} \alpha_{j} x\left(\xi_{j}\right), x^{\prime \prime}(0)=\cdots=x^{(i-1)}(0)=x^{(i+1)}(0)=\cdots=x^{(n-1)}(0)=0, x^{(i-1)}(1)=x^{(i-1)}(\xi), x^{(i)}(1)=x^{(i)}(\eta)$,
where $p>1$ is a constant; $\varphi_{p}: R \rightarrow R, \varphi_{p}(u)=|u|^{p-2} u ; f:[0,1] \times R^{n} \rightarrow R$ is a continuous function and $1 \leq i \leq n-1$ is a fixed integer, $e(t) \in L^{1}[0,1], \alpha_{j}(1 \leq j \leq m-2) \in R, \eta, \xi, \xi_{j} \in(0,1), j=1, \cdots, m-2,0<\xi_{1}<\cdots<\xi_{m-2}<1$.

We notice that the operator $\varphi_{p}(u)=|u|^{p-2} u$ is called the (one-dimensional) p-Laplacian and it appears in many contexts. For example, it is used extensively in non-Newtonian fluids, in some reaction-diffusion problems, in flow through porous media, in nonlinear elasticity, glaceology and petroleum extraction.

The boundary value problem (1.1), (1.2) is said to be at resonance in the sense that the associate homogeneous problem

$$
\left(\varphi_{p}\left(x^{(i)}(t)\right)\right)^{(n-i)}=0,0 \leq t \leq 1
$$

subject to boundary condition (1.2) has nontrival solutions.
The study on multi-point nonlocal boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1,2]. Since then some existence results have been obtained for general

[^0]nonlinear boundary value problems by several authors. We refer the reader to some recent results, such as [3-7] at non-resonance and [8-12] at resonance. For resonance case, by using Leray-Schauder continuation theorem, nonlinear-alternative of Leray-Schauder and coincidence degree theorem, the main technique of these works is to convert the problem into the abstract form $L x=N x$, where $L$ is a non-invertible linear operator. For problem (1.1) with some resonance conditions, if $p=2$, some existence results are established by [10-12].

But as far as we know, the existence results for high order resonance problems with p-Laplacian operator such as (1.1), (1.2) with $p \neq 2$ have never been studied before. This is mainly due to the facts that in this situation, above methods are not applicable directly since the p-Laplacian operator $\left(\varphi_{p}\left(x^{i}(t)\right)\right)^{(n-i)}$ is not linear with respect to $x$. Inspired by $[13,14]$, the goal of this paper is to fill the gap in this area. By using Mawhin continuation theorem the existence results for above problem are established.

## 2.Preliminaries

First we recall briefly some notations and an abstract existence results.
Let $X, Y$ be real Banach spaces and let $L: \operatorname{dom} L \subset X \rightarrow Y$ be a Fredholm operator with index zero, here dom L denotes the domain of L . This means that $\operatorname{ImL}$ is closed in $Y$ and $\operatorname{dim} \operatorname{Ker} L=\operatorname{dim}(Y / \operatorname{ImL})<+\infty$. Consider the supplementary subspaces $X_{1}$ and $Y_{1}$ such that $X=\operatorname{Ker} L \oplus X_{1}$ and $Y=\operatorname{ImL} \oplus Y_{1}$ and let $P: X \rightarrow K e r L$ and $Q: Y \rightarrow Y_{1}$ be the natural projections. Clearly, $\operatorname{Ker} L \cap\left(\operatorname{dom} L \cap X_{1}\right)=\{0\}$, thus the restrictions $L_{p}:=\left.L\right|_{\text {domL } \cap X_{1}}$ is invertible. Denote by K the inverse of $L_{p}$.

Let $\Omega$ be an open bounded subset of $X$ with $\operatorname{domL} \cap \Omega \neq \emptyset$. A map $N: \bar{\Omega} \rightarrow Y$ is said to be L-compact in $\bar{\Omega}$ if $Q N(\bar{\Omega})$ is bounded and the operator $K(I-Q) N: \bar{\Omega} \rightarrow X$ is compact. We first give the famous Mawhin continuation theorem.

Lemma 2.1(Mawhin [15]). Suppose that X and Y are Banach spaces, and $L: \operatorname{domL} \subset X \rightarrow Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N: \bar{\Omega} \rightarrow Y$ is L-compact on $\bar{\Omega}$.If
(1) $L x \neq \lambda N x, \forall x \in \partial \Omega \cap \operatorname{dom} L, \lambda \in(0,1)$;
(2) $N x \notin \operatorname{ImL}, \forall x \in \partial \Omega \cap K e r L$;
(3) $\operatorname{deg}\{J Q N, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, where $J: \operatorname{Ker} L \rightarrow \operatorname{Im} Q$ is an isomorphism,
then the equation $L x=N x$ has a solution in $\bar{\Omega} \cap \operatorname{dom} L$.
3.Existence results for problem (1.1), (1.2)

In order to eliminate the dilemma that L isn't linear for the case $p \neq 2$, we set

$$
\left\{\begin{array}{l}
x_{1}(t)=x(t)  \tag{3.1}\\
x_{2}(t)=\varphi_{p}\left(x^{(i)}(t)\right)
\end{array}\right.
$$

then problem (1.1), (1.2) is equivalent to system

$$
\left\{\begin{array}{l}
x_{1}^{(i)}(t)=\varphi_{q}\left(x_{2}\right)  \tag{3.2}\\
x_{2}^{(n-i)}(t)=f\left(t, x_{1}, \cdots, x_{1}^{(i-1)}, \varphi_{q}\left(x_{2}\right), \cdots,\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right)+e(t)
\end{array}\right.
$$

with boundary conditions
$x_{1}(1)=\sum_{j=1}^{m-2} \alpha_{j} x_{1}\left(\xi_{j}\right), x_{1}^{\prime \prime}(0)=\cdots=x_{1}^{(i-1)}(0)=x_{2}^{\prime}(0)=\cdots=x_{2}^{(n-i-1)}(0)=0, x_{1}^{(i-1)}(1)=x_{1}^{(i-1)}(\xi), x_{2}(1)=x_{2}(\eta)$ where $\varphi_{q}$ is the inverse function of $\varphi_{p}, \varphi_{q}(u)=|u|^{q-2} u$, where $1 / p+1 / q=1$. Clearly if $x(t)=\left(x_{1}(t), x_{2}(t)\right)$ is a solution for system (3.2), then $x_{1}(t)$ must be a solution for problem (1.1),(1.2).

Define
$X=\left\{u(t)=\left(u_{1}(t), u_{2}(t)\right) \mid u_{1}(t) \in C^{i}[0,1] u_{2}(t) \in C^{n-i}[0,1] m\right\}$ with the norm

$$
\|u\|=\max \left\{\left|u_{1}\right|_{\infty},\left|u_{1}^{\prime}\right|_{\infty}, \cdots,\left|u_{1}^{(i-1)}\right|_{\infty},\left|\varphi_{q}\left(u_{2}\right)\right|_{\infty}, \cdots,\left|\varphi_{q}\left(u_{2}\right)^{(n-i-1)}\right|_{\infty}\right\}
$$

$Y=\left\{v(t)=\left(v_{1}(t), v_{2}(t)\right) \mid v_{i}(t) \in L^{1}[0,1], i=1,2\right\} \quad$ with the norm $\|v\|=\max \left\{\left|v_{1}\right|_{1},\left|\varphi_{q}\left(v_{2}\right)\right|_{1}\right\}$,
where $|u|_{\infty}=\max _{0 \leq t \leq 1}|u(t)|,|u|_{1}=\int_{0}^{1}|u(t)| d t$. Clearly X and Y are Banach spaces. We will use the Sobolev space $W^{(i, n-i)}(0,1)$ defined as
$W^{(i, n-i)}(0,1)=\left\{u=\left(u_{1}, u_{2}\right):(0,1) \rightarrow R: u_{1}, u_{2}\right.$ are absolutely continuous on $[0,1]$ and $\left.u_{1}^{(i)}, u_{2}^{(n-i)} \in L^{1}[0,1]\right\}$.
Define $L: \operatorname{dom} L \subset X \rightarrow Y$ by

$$
L x:=\left(x_{1}^{(i)}(t), x_{2}^{(n-i)}(t)\right)
$$

where $\operatorname{dom} L=\left\{x \in W^{(i, n-i)}(0,1): x_{1}(1)=\sum_{j=1}^{m-2} \alpha_{j} x_{1}\left(\xi_{j}\right)\right.$,

$$
\left.x_{1}^{\prime \prime}(0)=\cdots=x_{1}^{(i-1)}(0)=x_{2}^{\prime}(0)=\cdots=x_{2}^{(n-i-1)}(0)=0, x_{1}^{(i-1)}(1)=x_{1}^{(i-1)}(\xi), x_{2}(1)=x_{2}(\eta)\right\}
$$

and $N: X \rightarrow Y$ by

$$
N x:=\left(\varphi_{q}\left(x_{2}\right), f\left(t, x_{1}, \cdots, x_{1}^{(i-1)}, \varphi_{q}\left(x_{2}\right), \cdots,\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right)+e(t)\right)
$$

Then system (3.2) can be written as $L x=N x$, here $L$ is a linear operator.
In this section we shall prove existence results for system (3.2) under the case $\sum_{j=1}^{m-2} \alpha_{j}=1, \sum_{j=1}^{m-2} \alpha_{j} \xi_{j} \neq 1$.

Lemma 3.1 If $\sum_{j=1}^{m-2} \alpha_{j}=1, \sum_{j=1}^{m-2} \alpha_{j} \xi_{j} \neq 1$, then
(1) $\operatorname{ImL}=\left\{\left(y_{1}, y_{2}\right) \in Y: \int_{\xi}^{1} y_{1}(t) d t=0, \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} y_{2}\left(s_{1}\right) d s_{1} \cdots d s_{n-i}=0\right\}$.
(2) $L: \operatorname{dom} L \subset X \rightarrow Y$ is a Fredholm operator with index zero,
(3) Define projector operator $P: X \rightarrow K e r L$ as

$$
P x=\left(x_{1}(0), x_{2}(0)\right)
$$

then the generalized inverse of operator $\mathrm{L}, K_{P}: \operatorname{ImL} \rightarrow \operatorname{domL} \cap \operatorname{Ker} P$ can be written as $K_{P}(y)=$
$\left(-\frac{\sum_{j=1}^{m-2} \alpha_{j}}{1-\sum_{j=1}^{m-2} \alpha_{j} \xi_{j}} \int_{\xi_{j}}^{1} \int_{0}^{s_{i}} \cdots \int_{0}^{s_{2}} y_{1}\left(s_{1}\right) d s_{1} \cdots d s_{i}+\int_{0}^{t} \cdots \int_{0}^{s_{2}} y_{1}\left(s_{1}\right) d s_{1} \cdots d s_{i}, \int_{0}^{t} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} y_{2}\left(s_{1}\right) d s_{1} \cdots d s_{n-i}\right)$ satisfying $\left\|K_{P}(y(t))\right\| \leq \triangle\|y\|$, where $\triangle=1+\frac{\sum_{j=1}^{m-2}\left|\alpha_{j}\right|\left(1-\xi_{j}\right)}{\left|1-\sum_{j=1}^{m-2} \alpha_{j} \xi_{j}\right|}$ is a constant.
Proof: (1):First we show

$$
\operatorname{ImL}=\left\{\left(y_{1}, y_{2}\right) \in Y: \int_{\xi}^{1} y_{1}(t) d t=0, \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} y_{2}\left(s_{1}\right) d s_{1} \cdots d s_{n-i}=0\right\}
$$

First suppose $y(t)=\left(y_{1}(t), y_{2}(t)\right) \in \operatorname{Im} L$, then there exists $x(t)=\left(x_{1}(t), x_{2}(t)\right) \in d o m L$ such that $L x=y$. That is

$$
\begin{aligned}
& x_{1}(t)=\int_{0}^{t} \int_{0}^{s_{i}} \cdots \int_{0}^{s_{2}} y_{1}\left(s_{1}\right) d s_{1} \cdots d s_{i}+a_{i-1} t^{i-1}+\cdots+a_{1} t+a_{0} \\
& x_{2}(t)=\int_{0}^{t} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} y_{2}\left(s_{1}\right) d s_{1} \cdots d s_{n-i}+b_{n-i-1} t^{n-i-1}+\cdots+b_{1} t+b_{0}
\end{aligned}
$$

Then boundary condition

$$
x_{1}(1)=\sum_{j=1}^{m-2} \alpha_{j} x_{1}\left(\xi_{j}\right), x_{1}^{\prime \prime}(0)=\cdots=x_{1}^{(i-1)}(0)=x_{2}^{\prime}(0)=\cdots=x_{2}^{(n-i-1)}(0)=0, x_{1}^{(i-1)}(1)=x_{1}^{(i-1)}(\xi), x_{2}(1)=x_{2}(\eta)
$$

imply that

$$
\int_{\xi}^{1} y_{1}(t) d t=0, \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} y_{2}\left(s_{1}\right) d s_{1} \cdots d s_{n-i}=0
$$

Next we suppose $y(t) \in\left\{\left(y_{1}, y_{2}\right) \in Y: \int_{\xi}^{1} y_{1}(t) d t=0, \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} y_{2}\left(s_{1}\right) d s_{1} \cdots d s_{n-i}=0\right\}$.
Let $x(t)=\left(x_{1}(t), x_{2}(t)\right)$, where

$$
x_{1}(t)=-\frac{\sum_{j=1}^{m-2} \alpha_{j}}{1-\sum_{j=1}^{m-2} \alpha_{j} \xi_{j}} \int_{\xi_{j}}^{1} \int_{0}^{s_{i}} \cdots \int_{0}^{s_{2}} y_{1}\left(s_{1}\right) d s_{1} \cdots d s_{i}+\int_{0}^{t} \cdots \int_{0}^{s_{2}} y_{1}\left(s_{1}\right) d s_{1} \cdots d s_{i}
$$

$$
x_{2}(t)=\int_{0}^{t} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} y_{2}\left(s_{1}\right) d s_{1} \cdots d s_{n-i}
$$

then $L x=\left(x_{1}^{(i)}(t), x_{2}^{(n-i)}(t)\right)=\left(y_{1}(t), y_{2}(t)\right)$. Furthermore consider

$$
\int_{\xi}^{1} y_{1}(t) d t=0, \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} y_{2}\left(s_{1}\right) d s_{1} \cdots d s_{n-i}=0
$$

by a simple computation,
$x_{1}(1)=\sum_{j=1}^{m-2} \alpha_{j} x_{1}\left(\xi_{j}\right), x_{1}^{\prime \prime}(0)=\cdots=x_{1}^{(i-1)}(0)=x_{2}^{\prime}(0)=\cdots=x_{2}^{(n-i-1)}(0)=0, x_{1}^{(i-1)}(1)=x_{1}^{(i-1)}(\xi), x_{2}(1)=x_{2}(\eta)$
Then $x(t) \in \operatorname{dom} L$, thus $y(t) \in \operatorname{Im} L$. Sum up all above we obtain that

$$
\operatorname{ImL}=\left\{\left(y_{1}, y_{2}\right) \in Y: \int_{\xi}^{1} y_{1}(t) d t=0, \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} y_{2}\left(s_{1}\right) d s_{1} \cdots d s_{n-i}=0\right\}
$$

(2):Following we claim that L is a Fredholm operator with index zero. It's easy to see that $\operatorname{Ker} L=(a, b), a, b \in R$. Suppose $y(t) \in Y$, define the projector operator $Q$ as

$$
Q(y)=\left(Q\left(y_{1}(t)\right), Q\left(y_{2}(t)\right)\right)=\left(\frac{\int_{\xi}^{1} y_{1}(t) d t}{1-\xi}, \frac{(n-1)!}{1-\eta^{n-i}} \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} y_{2}\left(s_{1}\right) d s_{1} \cdots d s_{n-i}\right)
$$

Let $y^{*}=y(t)-Q(y(t))=\left(y_{1}-Q\left(y_{1}\right), y_{2}-Q\left(y_{2}\right)\right)$, it's easy to see that $y^{*} \in \operatorname{ImL}$. Hence $Y=\operatorname{ImL}+\operatorname{Ker} L$, furthermore considering $\operatorname{Im} L \cap \operatorname{Ker} L=\{0\}$, we have $Y=\operatorname{ImL} \oplus \operatorname{Ker} L$. Thus

$$
\operatorname{dim} \operatorname{Ker} L=\operatorname{codim} \operatorname{ImL},
$$

which means $L$ is a Fredholm operator with index zero.
(3):Define the projector operator $P: X \rightarrow \operatorname{Ker} L$ as

$$
P x=\left(x_{1}(0), x_{2}(0)\right),
$$

for $y(t) \in I m L$, we have

$$
\left(L K_{P}\right)(y(t))=y(t)
$$

and for $x(t) \in \operatorname{domL} \cap \operatorname{KerP}$, following facts

$$
\begin{aligned}
- & \frac{\sum_{j=1}^{m-2} \alpha_{j}}{1-\sum_{j=1}^{m-2} \alpha_{j} \xi_{j}} \int_{\xi_{j}}^{1} \int_{0}^{s_{i}} \cdots \int_{0}^{s_{2}} x_{1}^{(i)}\left(s_{1}\right) d s_{1} \cdots d s_{i}+\int_{0}^{t} \cdots \int_{0}^{s_{2}} x_{1}^{(i)}\left(s_{1}\right) d s_{1} \cdots d s_{i}=x_{1}(t)-x_{1}(0)=x_{1}(t), \\
& \int_{0}^{t} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}} x_{2}^{(n-i)}\left(s_{1}\right) d s_{1} d s_{n-i}=x_{2}(t)-x_{2}(0)=x_{2}(t)
\end{aligned}
$$

show that $K_{P}=\left(L_{d o m L \cap K e r P}\right)^{-1}$. Furthermore from the definition of the norms in the $X, Y$, we have

$$
\left\|K_{P}(y(t))\right\| \leq \triangle\|y\| .
$$

The above arguments complete the proof of Lemma 3.1.
Theorem 3.1: Let $f:[0,1] \times R^{n} \rightarrow R$ be a continuous function. Assume there exists $m_{1} \in 1,2, \cdots, m-3$ such that $\alpha_{j}>0$ for $1 \leq j \leq m_{1}$ and $\alpha_{j}<0$ for $m_{1}+1 \leq j \leq m-2$, furthermore following conditions are satisfied: $\left(C_{1}\right)$ There exist functions $a_{k}(t) \in L^{1}[0,1], k=1,2, \cdots, n$ and constant $\theta \in[0,1)$ such that for all $\left(x_{1}, x_{2}, \cdots, x_{n}\right) \in$ $R^{n}, t \in[0,1]$,one of following conditions is satisfied:

$$
\begin{align*}
\left|f\left(t, x_{1}, x_{2}, \cdots, x_{n}\right)+e(t)\right| \leq & \left(\sum_{k=1}^{n} a_{k}(t)\left|x_{k}\right|+b(t)\left|x_{n}\right|^{\theta}+r(t)\right)^{p-1}  \tag{3.3}\\
\left|f\left(t, x_{1}, x_{2}, \cdots, x_{n}\right)+e(t)\right| \leq & \left(\sum_{k=1}^{n} a_{k}(t)\left|x_{k}\right|+b(t)\left|x_{n-1}\right|^{\theta}+r(t)\right)^{p-1}  \tag{3.4}\\
& \cdots \quad \cdots  \tag{3.5}\\
\left|f\left(t, x_{1}, x_{2}, \cdots, x_{n}\right)+e(t)\right| \leq & \left(\sum_{k=1}^{n} a_{k}(t)\left|x_{k}\right|+b(t)\left|x_{1}\right|^{\theta}+r(t)\right)^{p-1}
\end{align*}
$$

$\left(C_{2}\right)$ There exists a constant $M>0$ such that for $x \in \operatorname{dom} L$, if $\left|x_{1}(t)\right|>M$,

$$
\begin{equation*}
\int_{\xi}^{1} \varphi_{q}\left(\int_{\sigma}^{t} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, x_{1}, \cdots, x_{n}\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}\right) d t \neq 0 \tag{3.6}
\end{equation*}
$$

for all $x_{2}, \cdots, x_{n} \in R^{n-1}, \sigma \in(0,1), t \in(0,1) \backslash\{\sigma\}$.
$\left(C_{3}\right)$ There exist $M^{*}>0$ such that for any $c_{1} \in R$, if $\left|c_{1}\right|>M^{*}$, for all $c_{2} \in R$, then either

$$
\begin{equation*}
c_{2} \times \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, c_{1}, 0, \cdots, 0, \varphi_{q}\left(c_{2}\right), 0, \cdots, 0\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}<0 \tag{3.7}
\end{equation*}
$$

or else

$$
\begin{equation*}
c_{2} \times \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, c_{1}, 0, \cdots, 0, \varphi_{q}\left(c_{2}\right), 0, \cdots, 0\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}>0 \tag{3.8}
\end{equation*}
$$

Then for each $e \in L^{1}[0,1]$, the resonance problem (1.1), (1.2) with $\sum_{j=1}^{m-2} \alpha_{j}=1, \sum_{j=1}^{m-2} \alpha_{j} \xi_{j} \neq 1$ has at least one solution in $C^{n-1}[0,1]$ provided that

$$
\sum_{k=1}^{n}\left|a_{k}\right|_{1}<\frac{1}{1+\triangle}
$$

Proof : We divide the proof into the following steps.

## Step 1.Let

$$
\Omega_{1}=\{x \in \operatorname{dom} L \backslash \operatorname{Ker} L: L x=\lambda N x\} \text { for some } \lambda \in[0,1] .
$$

Then $\Omega_{1}$ is bounded.
Suppose that $x \in \Omega_{1}, L x=\lambda N x$, thus $\lambda \neq 0, N x \in \operatorname{ImL}=\operatorname{Ker} Q$, hence

$$
\int_{\xi}^{1} \varphi_{q}\left(x_{2}(t)\right) d t=0, \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, x_{1}, \cdots, x_{1}^{(i-1)}, \varphi_{q}\left(x_{2}\right), \cdots,\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}=0
$$

For $x_{1}^{(i-1)}(1)=x_{1}^{(i-1)}(\xi)$, there exist $\sigma_{1} \in(\xi, 1)$ such that $x_{1}^{(i)}\left(\sigma_{1}\right)=0$. Integrate both sides of (1.1), we have

$$
\begin{equation*}
x_{2}(t)=\int_{\sigma_{1}}^{t} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, x_{1}, \cdots, x_{1}^{(i-1)}, \varphi_{q}\left(x_{2}\right), \cdots,\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}=0 \tag{3.9}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\int_{\xi}^{1} \varphi_{q}\left(\int_{\sigma_{1}}^{t} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, x_{1}, \cdots, x_{1}^{(i-1)}, \varphi_{q}\left(x_{2}\right), \cdots,\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}\right) d t=0 \tag{3.10}
\end{equation*}
$$

Then (3.10) together with condition $\left(C_{2}\right)$ imply that there exists $t_{0} \in[0,1]$ such that $\left|x_{1}\left(t_{0}\right)\right|<M$. In view of $x_{1}(t)=x_{1}\left(t_{0}\right)+\int_{t_{0}}^{t} x_{1}^{\prime}(s) d s$, we obtain that

$$
\begin{equation*}
|x(t)|<M+\left|x_{1}^{\prime}\right|_{1} \tag{3.11}
\end{equation*}
$$

Furthermore, for $\alpha_{j}>0,1 \leq j \leq m_{1}$ and $\alpha_{j}<0, m_{1}+1 \leq j \leq m-2$ and $x_{1}(1)=\sum_{j=1}^{m-2} \alpha_{j} x_{1}\left(\xi_{j}\right)$, we have

$$
x_{1}(1)-\sum_{j=m_{1}+1}^{m-2} \alpha_{j} x_{1}\left(\xi_{j}\right)=\sum_{j=1}^{m_{1}} \alpha_{j} x_{1}\left(\xi_{j}\right),
$$

then there exists $t_{1} \in\left[\xi_{m_{1}+1}, 1\right], t_{2} \in\left[0, \xi_{m_{1}}\right]$ such that

$$
x_{1}\left(t_{1}\right)=\frac{x_{1}(1)-\sum_{j=m_{1}+1}^{m-2} \alpha_{j} x_{1}\left(\xi_{j}\right)}{1-\sum_{m_{1}+1}^{m-2} \alpha_{j}}, x_{1}\left(t_{2}\right)=\frac{\sum_{j=1}^{m_{1}} \alpha_{j} x_{1}\left(\xi_{j}\right)}{\sum_{j=1}^{m_{1}} \alpha_{j}}
$$

thus in view of $\sum_{j=1}^{m-2} \alpha_{j}=1$, we obtain that $x_{1}\left(t_{1}\right)=x_{1}\left(t_{2}\right)$, and $t_{1} \neq t_{2}$. This implies that there exists $t_{3} \in\left(t_{1}, t_{2}\right)$ such that $x_{1}^{\prime}\left(t_{3}\right)=0$. Then from $x_{1}^{\prime}(t)=x_{1}^{\prime}\left(t_{3}\right)+\int_{t_{3}}^{t} x_{1}^{\prime \prime}(s) d s$, we obtain

$$
\begin{equation*}
\left|x_{1}^{\prime}\right| \leq\left|x_{1}^{\prime \prime}\right|_{1} \tag{3.12}
\end{equation*}
$$

Consider the boundary condition $x_{1}^{\prime \prime}(0)=x_{1}^{\prime \prime \prime}(0)=\cdots=x_{1}^{(i-1)}(0)=x_{2}^{\prime}(0)=\cdots=x_{2}^{(n-i-1)}(0)=0$ together with $x_{2}\left(\sigma_{1}\right)=0$, it's easy to get

$$
\begin{equation*}
\left|x_{1}^{\prime \prime}\right|_{\infty} \leq\left|x_{1}^{\prime \prime \prime}\right|_{\infty} \leq \cdots\left|x_{1}^{(i)}\right|_{\infty}=\left|\varphi_{q}\left(x_{2}\right)\right|_{\infty} \cdots \leq\left|\left(\varphi_{q}\left(x_{2}\right)^{(n-i-1)}\right)\right|_{\infty} \tag{3.13}
\end{equation*}
$$

Consider (3.11), (3.12), (3.13) we have

$$
\begin{align*}
\|P x\| & \leq \max \left\{\left|x_{1}(0)\right|,\left|\varphi_{q}\left(x_{2}(0)\right)\right|\right\} \\
& \leq \max \left\{M+\left|\varphi_{q}\left(x_{2}\right)\right|_{1}, \mid \varphi_{q}\left(\left.f\left(t, x_{1}, \cdots, x_{1}^{(i-1)}, \varphi_{q}\left(x_{2}\right), \cdots,\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}+e(t)\right)\right|_{1}\right\}\right. \tag{3.14}
\end{align*}
$$

Again for $x \in \Omega_{1}, x \in \operatorname{dom} L \backslash \operatorname{Ker} L$, then $(I-P) x \in \operatorname{domL} \cap \operatorname{Ker} P, L P x=0$, thus from Lemma 3.1, we have

$$
\|(I-P) x\|=\left\|K_{P} L(I-P) x\right\| \leq \triangle\|L(I-P) x\|_{1}=\triangle\|L x\| \leq \triangle\|N x\|
$$

$$
\begin{equation*}
\leq \triangle \max \left\{\left|\varphi_{q}\left(x_{2}\right)\right|_{1},\left|\varphi_{q}\left(f\left(t, x_{1}, \cdots, x_{1}^{(i-1)}, \varphi_{q}\left(x_{2}\right), \cdots,\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right)+e(t)\right)\right|_{1}\right\} \tag{3.15}
\end{equation*}
$$

From (3.14),(3.15) we have
$\|x\| \leq\|P x\|+\|(I-P) x\| \leq M+(1+\triangle)\left|\varphi_{q}\left(f\left(t, x_{1}, \cdots, x_{1}^{(i-1)}, \varphi_{q}\left(x_{2}\right), \cdots,\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right)+e(t)\right)\right|_{1}$.
If assumption (3.3) holds, we obtain that

$$
\begin{aligned}
\|x\| & \leq M+(1+\triangle)\left|\varphi_{q}\left(f\left(t, x_{1}, \cdots, x_{1}^{(i-1)}, \varphi_{q}\left(x_{2}\right), \cdots,\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right)+e(t)\right)\right|_{1} \\
& \leq(1+\triangle)\left(|a|_{1}\left|x_{1}\right|_{\infty}+\cdots+\left|a_{i}\right|_{1}\left|x_{1}^{(i-1)}\right|_{\infty}+\left|a_{i+1}\right|_{1}\left|\varphi_{q}\left(x_{2}\right)\right|_{\infty}+\cdots\right. \\
& \left.+\left|a_{n}\right|_{1}\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty}+|b|_{1}\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty}^{\theta}+C\right)
\end{aligned}
$$

where $C=|r|_{1}+|e|_{1}+\frac{M}{1+\triangle}$.
From $\left|x_{1}\right|_{\infty} \leq\|x\|$, we obtain
$\left|x_{1}\right|_{\infty} \leq \frac{1+\triangle}{1-(1+\triangle)\left|a_{1}\right|_{1}}\left[\left|a_{2}\right|_{1}\left|x_{1}^{\prime}\right|_{\infty}+\cdots+\left|a_{i+1}\right|_{1}\left|\varphi_{q}\left(x_{2}\right)\right|_{\infty}+\cdots+\left|a_{n}\right|_{1}\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty}+|b|_{1}\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty}^{\theta}+C\right]$
From $\left|x_{1}^{\prime}\right| \leq\|x\|$, we obtain
$\left|x_{1}^{\prime}\right|_{\infty} \leq \frac{1+\triangle}{1-(1+\triangle)\left(\left|a_{1}\right|_{1}+\left|a_{2}\right|_{1}\right)}\left[\left|a_{3}\right|_{1}\left|x_{1}^{\prime \prime}\right|_{\infty}+\cdots\right.$

$$
\left.+\left|a_{i+1}\right|_{1}\left|\varphi_{q}\left(x_{2}\right)\right|_{\infty}+\cdots+\left|a_{n}\right|_{1}\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty}+|b|_{1}\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty}^{\theta}+C\right] .
$$

$$
\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty} \leq \frac{1+\triangle}{1-(1+\triangle) \sum_{k=1}^{n-1}\left|a_{k}\right|_{1}}\left[\left|a_{n}\right|_{1}\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty}+|b|_{1}\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty}^{\theta}+C\right]
$$

then

$$
\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty} \leq \frac{(1+\triangle)|b|_{1}}{1-(1+\triangle) \sum_{k=1}^{n-1}\left|a_{k}\right|_{1}}\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty}^{\theta}+\frac{2 C}{1-2 \sum_{k=1}^{n-1}\left|a_{k}\right|_{1}} .
$$

Consider $\theta \in[0,1)$ together with $\sum_{k=1}^{n}\left|a_{k}\right|_{1}<\frac{1}{1+\triangle}$, we claim that there exists constant $M_{1}>0$ such that

$$
\begin{equation*}
\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-i-1)}\right|_{\infty} \leq M_{1} \tag{3.17}
\end{equation*}
$$

Then there exist constants $M_{k}>0, k=2, \cdots, i, M_{j}>0, j=i+1, \cdots, n$ such that

$$
\left|x_{1}^{(k)}\right|_{\infty}<M_{k},\left|\left(\varphi_{q}\left(x_{2}\right)\right)^{(n-j)}\right|_{\infty}<M_{j},
$$

thus there exists $N>0$ such that $\|x\|<N$,therefor we show that $\Omega_{1}$ is bounded.
Step 2.The set $\Omega_{2}=\{x \in \operatorname{Ker} L: N x \in \operatorname{Im} L\}$ is bounded.
The fact $x \in \Omega_{2}$ implies that $x=\left(c_{1}, c_{2}\right)$ and

$$
N(x)=\left(\varphi_{q}\left(c_{2}\right), f\left(t, c_{1}, 0, \cdots, 0, \varphi_{q}\left(c_{2}\right), \cdots, 0\right)+e(t)\right)
$$

From $Q N x=0$, we have

$$
\int_{\xi}^{1} \varphi_{q}\left(c_{2}\right) d t=0, \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, c_{1}, 0, \cdots, 0, \varphi_{q}\left(c_{2}\right), \cdots, 0\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}=0
$$

which implies $c_{2}=0$ and

$$
\int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, c_{1}, 0, \cdots, 0\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}=0
$$

Consider condition (C3), we obtain that $\left|c_{1}\right| \leq M^{*}$, then the set $\Omega_{2}$ is bounded.
Step 3. If the first part of condition (C3) is satisfied,there exists $M^{*}>0$ such that for any $c \in R$, if $c_{1}>M^{*}$, then

$$
c_{2} \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, c_{1}, 0, \cdots, 0, \varphi_{q}\left(c_{2}\right), \cdots, 0\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}<0
$$

Let $\Omega_{3}=\{x \in \operatorname{Ker} L:-\lambda x+(1-\lambda) J Q N x=0, \lambda \in[0,1]$, here $J: \operatorname{Im} Q \rightarrow K e r L$ is the linear isomorphism given by $J\left(c_{1}, c_{2}\right)=\left(c_{1}, c_{2}\right)$,we obtain

$$
\begin{aligned}
& \lambda c_{1}=(1-\lambda) \varphi_{q}\left(c_{2}\right) \\
& \lambda c_{2}=(1-\lambda) \frac{(n-i)!}{1-\eta^{n-i}} \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, c_{1}, 0, \cdots, \varphi_{q}\left(c_{2}\right), 0, \cdots, 0\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}
\end{aligned}
$$

If $\lambda=1$,it's easy to see $c_{1}=c_{2}=0$.If $\lambda=0, \varphi_{q}\left(c_{2}\right)=0$ implies $c_{2}=0$, then

$$
\int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, c_{1}, 0, \cdots, 0\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}=0
$$

Considering condition $C 3,\left|c_{1}\right|<M^{*}$.
For $\lambda \neq 0, \lambda \neq 1$, if $\left|c_{1}\right| \geq M^{*}$, we obtain that

$$
\lambda c_{2}^{2}=c_{2}(1-\lambda) \frac{(n-i)!}{1-\eta^{n-i}} \int_{\eta}^{1} \int_{0}^{s_{n-i}} \cdots \int_{0}^{s_{2}}\left(f\left(s_{1}, c_{1}, 0, \cdots, \varphi_{q}\left(c_{2}\right), 0, \cdots, 0\right)+e\left(s_{1}\right)\right) d s_{1} \cdots d s_{n-i}<0
$$

which contradicts to $\lambda c_{2}^{2} \geq 0$.Thus $\left|c_{1}\right|<M^{*}$. From $\lambda c_{1}=(1-\lambda) \varphi_{q}\left(c_{2}\right)$ and $\lambda \neq 0, \lambda \neq 1,\left|c_{2}\right|<\left(\frac{\lambda}{1-\lambda} M^{*}\right)^{p-1}$. Thus the set $\Omega_{3}$ is bounded.
Step 4.If the second part of condition (C3) is satisfied, similar with above argument, the set $\Omega_{4}=\{x \in \operatorname{KerL}$ : $\lambda x+(1-\lambda) J Q N x=0, \lambda \in[0,1]\}$ is bounded too.
Now we show all the conditions of Lemma 2.1 are satisfied.
Let $\Omega$ be a bounded open set of Y such that $\bigcup_{i=1}^{3} \overline{\Omega_{i}} \subset \Omega$. By the Ascoli-Arezela theorem, we can show that
$K_{P}(I-Q) N: \bar{\Omega} \rightarrow Y$ is compact, thus $N$ is L-compact on $\bar{\Omega}$. Then by the above arguments, we have
(i) $L x \neq N x$, for every $(x, \lambda) \in[(\operatorname{dom} L \backslash \operatorname{Ker} L) \bigcap \partial \Omega] \times(0,1)$;
(ii) $N x \neq I m L$, for every $x \in \operatorname{Ker} L \bigcap \partial \Omega$;
(iii)If the first part of condition (C3) holds, we let

$$
H(x, \lambda)=-\lambda x+(1-\lambda) J Q N x
$$

According to the above argument, we know that $H(x, \lambda) \neq 0$, for $x \in \operatorname{Ker} L \bigcap \partial \Omega$, by the homotopy property of degree, we get

$$
\begin{aligned}
\operatorname{deg}\left(\left.J Q N\right|_{\text {Ker } L}, \Omega \cap \operatorname{Ker} L, 0\right)= & \operatorname{deg}(H(x, 0), \Omega \cap \operatorname{Ker} L, 0) \\
& =\operatorname{deg}(H(x, 1), \Omega \cap \operatorname{Ker} L, 0) \\
& =\operatorname{deg}(-I, \Omega \cap K e r L, 0) \neq 0 .
\end{aligned}
$$

If the second part of condition C3 holds, we let

$$
H(x, \lambda)=\lambda x+(1-\lambda) J Q N x
$$

Similar to argument above, we have $\operatorname{deg}\left(\left.J Q N\right|_{\text {Ker } L}, \Omega \cap \operatorname{Ker} L, 0\right)=\operatorname{deg}(I, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.
Then by Lemma 2.1, $L x=N x$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$, so that problem (1.1), (1.2) has at least one solution in $C^{n-1}[0,1]$. The proof of Theorem 3.1 is completed.

## References

1 V.A.Il'in and E.I.Moiseev, Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator. Differential Equation, 1987, 23(8):979-987.

2 V.A.Il'in and E.I.Moiseev, Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects. Differential Equation, 1987, 23(7):803-810

3 R.Y.Ma, Positive solutions for a nonlinear three-point boundary value problem, Electronic Journal of Differential Equations. 34(1999), 1-8.

4 B.Liu, Positive solutions of a nonlinear four-point boundary value problems, Appl.Math.Comput, 155(2004)179-203.

5 R.Y.Ma, N.Cataneda, Existence of solution for nonlinear m-point boundary value problem, J.Math.Anal.Appl, 256(2001)556-567.

6 Yanping Guo and Weigao Ge, Positive solutions for three-point boundary-value problems with dependence on the first order derivative, J.Math.Anal.Appl, 290(2004):291-301.

7 Z.B.Bai, W.G.Ge and Y.F.Wang, Multiplicity results for some second-order four-point boundary-value problems, Nonlinear Analysis. 60(2004), 491-500.

8 Feng W and Webb J R L, Solvability of three-point boundary value problems at resonance. Nonlinear Analysis, 30(6), 1997:3227-3238.

9 B.Liu, Solvability of multi-point boundary value problem at resonance(II), Appl.Math.Comput, 136(2003), 353-377.

10 S.P.Lu and W.G.Ge, On the existence of m-point boundary value problem at resonance for higher order differential equation, J.Math.Anal.Appl, 287(2003), 522-539.

11 X.J.Lin, Z.J.Du and W.G.Ge, Solvability of multipoint boundary value problems at resonance for higher order ordinary differential equations, Comput.Math.Appl, 49(2005), 1-11.

12 Z.J.Du, X.J.Lin and W.G.Ge, Some higher order multi-point boundary value problems at resonance, J.Math. Anal.Appl, 177(2005), 55-65.

13 W. Ge and J. Ren, An extension of Mawhin's continuation theorem and its application to boundary value problems with a p-Laplacian. Nonlinear Analysis TMA 58 (2004), no. 3-4, 477-488.

14 W.S. Cheung and J. Ren, Periodic solutions for p-Laplacian differential equations with multiple deviating arguments, Nonlinear Analysis TMA, 62(2005), no.4, 727-742.

15 J.Mawhin, Topological degree methods in nonlinear boundary value problems, in: NSFCBMS Regional Conference Series in Math, American Mathematics Society, Providence, RI 1979.
(Received April 27, 2007)


[^0]:    ${ }^{1}$ The work is sponsored by the Natural Science Foundation of Anhui Educational Department(Kj2007b055) and Youth Project Foundation of Anhui Educational Department(2007jqL101,2007jqL102)

