Global solutions and exponential decay for a nonlinear coupled system of beam equations of Kirchhoff type with memory in a domain with moving boundary

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Abstract

In this paper we prove the exponential decay in the case $n > 2$, as time goes to infinity, of regular solutions for a nonlinear coupled system of beam equations of Kirchhoff type with memory and weak damping

$$
\begin{aligned}
&u_{tt} + \Delta^2 u - M(||\nabla u||^2_{L^2(\Omega_t)} + ||\nabla v||^2_{L^2(\Omega_t)})\Delta u \\
&+ \int_0^t g_1(t-s)\Delta u(s)ds + \alpha u_t + h(u-v) = 0 \quad \text{in} \quad \hat{Q}, \\
&v_{tt} + \Delta^2 v - M(||\nabla u||^2_{L^2(\Omega_t)} + ||\nabla v||^2_{L^2(\Omega_t)})\Delta v \\
&+ \int_0^t g_2(t-s)\Delta v(s)ds + \alpha v_t - h(u-v) = 0 \quad \text{in} \quad \hat{Q}
\end{aligned}
$$

in a non cylindrical domain of $\mathbb{R}^{n+1}$ $(n \geq 1)$ under suitable hypothesis on the scalar functions M, h, g_1 and g_2, and where α is a positive constant. We show that such dissipation is strong enough to produce uniform rate of decay. Besides, the coupling is nonlinear which brings up some additional difficulties, which plays the problem interesting. We establish existence and uniqueness of regular solutions for any $n \geq 1$.

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1. INTRODUCTION

Let Ω be an open bounded domain of $\mathbb{R}^n$ containing the origin and having C^2 boundary. Let $\gamma : [0, \infty[\to \mathbb{R}$ be a continuously differentiable function. See hypothesis (1.15)-(1.17) on γ. Let us consider the family of subdomains $\{\Omega_t\}_{0\leq t<\infty}$ of $\mathbb{R}^n$ given by

$$\Omega_t = T(\Omega), \quad T : y \in \Omega \mapsto x = \gamma(t)y$$

whose boundaries are denote by Γ_t and $\hat{Q}$ the non cylindrical domain of $\mathbb{R}^{n+1}$

$$\hat{Q} = \bigcup_{0\leq t<\infty} \Omega_t \times \{t\}$$

with lateral boundary

$$\hat{\sum} = \bigcup_{0\leq t<\infty} \Gamma_t \times \{t\}.$$

Let us consider the Hilbert space $L^2(\Omega)$ endowed with the inner product

$$(u, v) = \int_\Omega u(x)v(x)dx$$

and corresponding norm

$$\|u\|^2_{L^2(\Omega)} = (u, u).$$

We also consider the Sobolev space $H^1(\Omega)$ endowed with the scalar product

$$(u, v)_{H^1(\Omega)} = (u, v) + (\nabla u, \nabla v).$$

We define the subspace of $H^1(\Omega)$, denoted by $H^1_0(\Omega)$, as the closure of $C^\infty_0(\Omega)$ in the strong topology of $H^1(\Omega)$. By $H^{-1}(\Omega)$ we denote the dual space of $H^1_0(\Omega)$. This space endowed with the norm induced by the scalar product

$$((u, v))_{H^1_0(\Omega)} = (\nabla u, \nabla v)$$

is, owing to the Poincaré inequality

$$\|u\|^2_{L^2(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega)},$$

a Hilbert space. We define for all $1 \leq p < \infty$

$$\|u\|^p_{L^p(\Omega)} = \int_\Omega |u(x)|^p dx,$$

and if $p = \infty$

$$\|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} ess|u(x)|.$$

In this work we study the existence of strong solutions as well the exponential decay of the energy of the nonlinear coupled system of beam equations of Kirchhoff type with memory given by

$$u_{tt}+\Delta^2 u - M(||\nabla u||^2_{L^2(\Omega_t)}+||\nabla v||^2_{L^2(\Omega_t)})\Delta u$$
$$+\int_0^t g_1(t-s)\Delta u(s)ds+\alpha u_t+h(u-v)=0 \quad \text{in} \quad \hat{Q}, \tag{1.1}$$
$$v_{tt}+\Delta^2 v - M(||\nabla u||^2_{L^2(\Omega_t)}+||\nabla v||^2_{L^2(\Omega_t)})\Delta v$$
$$+\int_0^t g_2(t-s)\Delta v(s)ds+\alpha v_t-h(u-v)=0 \quad \text{in} \quad \hat{Q}, \tag{1.2}$$
$$u=v=\frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0 \quad \text{on} \quad \hat{\sum}, \tag{1.3}$$
$$(u(x,0),v(x,0))=(u_0(x),v_0(x)), \quad (u_t(x,0),v_t(x,0))=(u_1(x),v_1(x)) \quad \text{in} \quad \Omega_0, \tag{1.4}$$

where $\nu=\nu(\sigma,t)$ is the unit normal at $(\sigma,t)\in\hat{\Sigma}$ directed towards the exterior of $\hat{Q}$. If we denote by η the outer normal to the boundary Γ of Ω, we have, using a parametrization of Γ

$$\nu(\sigma,t)=\frac{1}{r}(\eta(\xi),-\gamma'^2(t)\xi\cdot\eta(\xi)), \quad \xi=\frac{\sigma}{\gamma(t)} \tag{1.5}$$

where

$$r=(1+\gamma'^2(t)|\xi\cdot\eta(\xi)|^2)^{\frac{1}{2}}.$$

In fact, fix $(\sigma,t)\in\hat{\Sigma}$. Let $\varphi=0$ be a parametrization of a part U of Γ, U containing $\xi=\frac{\sigma}{\gamma(t)}$. The parametrization of a part V of $\hat{\Sigma}$ is $\psi(\sigma,t)=\varphi(\frac{\sigma}{\gamma(t)})=\varphi(\xi)=0$. We have

$$\nabla\psi(\sigma,t)=\frac{1}{\gamma(t)}(\nabla\varphi(\xi),-\gamma'(t)\xi\cdot\nabla\varphi(\xi)).$$

From this and observing that $\eta(\xi)=\nabla\varphi(\xi)/|\nabla\varphi(\xi)|$, the formula (1.5) follows.

Let $\overline{\nu}(\cdot,t)$ be the x-component of the unit normal $\nu(\cdot,\cdot)$, $|\overline{\nu}|\le 1$. Then by the relation (1.5), one has

$$\overline{\nu}(\sigma,t)=\eta(\frac{\sigma}{\gamma(t)}). \tag{1.6}$$

In this paper we deal with nonlinear coupled system of beam equations of Kirchhoff type with memory over a non cylindrical domain. We show the existence and uniqueness of strong solutions to the initial boundary value problem (1.1)-(1.4). The method we use to prove the result of existence and uniqueness is based on the transformation of our problem into another initial boundary value problem defined over a cylindrical domain whose sections are not time-dependent. This is done using a suitable change of variable. Then we show the existence and uniqueness for this new problem. Our existence result on non cylindrical domain will follow using the inverse transformation. That is, using the diffeomorphism $\tau:\hat{Q}\to Q$ defined by

$$\tau:\hat{Q}\to Q, \quad (x,t)\in\Omega_t\mapsto(y,t)=(\frac{x}{\gamma(t)},t) \tag{1.7}$$

and $\tau^{-1}: Q \to \hat{Q}$ defined by

$$\tau^{-1}(y,t) = (x,t) = (\gamma(t)y,t). \tag{1.8}$$

Denoting by ϕ and φ the functions

$$\phi(y,t) = u \circ \tau^{-1}(y,t) = u(\gamma(t)y,t), \quad \varphi(y,t) = v \circ \tau^{-1}(y,t) = v(\gamma(t)y,t) \tag{1.9}$$

the initial boundary value problem (1.1)-(1.4) becomes

$$\begin{aligned}
&\phi_{tt} + \gamma^{-4}\Delta^2\phi - \gamma^{-2}M(\gamma^{n-2}(||\nabla\phi||^2_{L^2(\Omega)} + ||\nabla\varphi||^2_{L^2(\Omega)}))\Delta\phi \\
&+ \int_0^t g_1(t-s)\gamma^{-2}(s)\Delta\phi(s)ds + \alpha\phi_t - A(t)\phi + a_1\cdot\nabla\partial_t\phi \\
&+ a_2\cdot\nabla\phi + h(\phi-\varphi) = 0 \quad \text{in} \quad Q, && (1.10)\\
&\varphi_{tt} + \gamma^{-4}\Delta^2\varphi - \gamma^{-2}M(\gamma^{n-2}(||\nabla\phi||^2_{L^2(\Omega)} + ||\nabla\varphi||^2_{L^2(\Omega)}))\Delta\varphi \\
&+ \int_0^t g_2(t-s)\gamma^{-2}(s)\Delta\varphi(s)ds + \alpha\varphi_t - A(t)\varphi + a_1\cdot\nabla\partial_t\varphi \\
&+ a_2\cdot\nabla\varphi - h(\phi-\varphi) = 0 \quad \text{in} \quad Q, && (1.11)\\
&\phi|_\Gamma = \varphi|_\Gamma = \frac{\partial\phi}{\partial\nu}|_\Gamma = \frac{\partial\varphi}{\partial\nu}|_\Gamma = 0, && (1.12)\\
&(\phi,\varphi)|_{t=0} = (\phi_0,\varphi_0) \quad (\phi_t,\varphi_t)|_{t=0} = (\phi_1,\varphi_1) \quad \text{in} \quad \Omega, && (1.13)
\end{aligned}$$

where

$$A(t)\phi = \sum_{i,j=1}^n \partial_{y_i}(a_{ij}\partial_{y_j}\phi), \quad A(t)\varphi = \sum_{i,j=1}^n \partial_{y_i}(a_{ij}\partial_{y_j}\varphi)$$

and

$$\begin{cases} a_{ij}(y,t) = -(\gamma'\gamma^{-1})^2 y_i y_j \quad (i,j = 1,\ldots,n), \\ a_1(y,t) = -2\gamma'\gamma^{-1}y, \\ a_2(y,t) = -\gamma^{-2}y(\gamma''\gamma + \gamma'(\alpha\gamma + (n-1)\gamma')). \end{cases} \tag{1.14}$$

To show the existence of a strong solution we will use the following hypotheses:

$$\gamma' \le 0 \quad n > 2, \quad \gamma' \ge 0 \quad \text{if} \quad n \le 2, \tag{1.15}$$

$$\gamma \in L^\infty(0,\infty), \quad \inf_{0\le t<\infty} \gamma(t) = \gamma_0 > 0, \tag{1.16}$$

$$\gamma' \in W^{2,\infty}(0,\infty) \cap W^{2,1}(0,\infty). \tag{1.17}$$

Note that the assumption (1.15) means that $\hat{Q}$ is decreasing if $n > 2$ and increasing if $n \le 2$ in the sense that when $t > t'$ and $n > 2$ then the projection of $\Omega_{t'}$ on the subspace $t = 0$ contain the projection of Ω_t on the same subspace and contrary in the case $n \le 2$. The above method was introduced by Dal Passo and Ughi [21] to study certain class of parabolic equations in non cylindrical domain. We assume that $h \in C^1(\Re)$ satisfies

$$h(s)s \geq 0, \quad \forall s \in \Re.$$

Additionally, we suppose that h is superlinear, that is

$$h(s)s \geq H(s), \quad H(z) = \int_0^z h(s)ds, \quad \forall s \in \Re,$$

with the following growth conditions

$$|h(t) - h(s)| \leq C(1 + |t|^{\rho-1} + |s|^{\rho-1})|t - s|, \quad \forall t, s \in \Re,$$

for some $C > 0$ and $\rho \geq 1$ such that $(n-2)\rho \leq n$. Concerning the function $M \in C^1([0, \infty[)$, we assume that

$$M(\tau) \geq -m_0, \quad M(\tau)\tau \geq \widehat{M}(\tau), \quad \forall \tau \geq 0, \tag{1.18}$$

where $\widehat{M}(\tau) = \int_0^\tau M(s)ds$ and

$$0 < m_0 < \lambda_1 ||\gamma||_{L^\infty}^{-2} \tag{1.19}$$

where λ_1 is the first eigenvalue of the spectral Dirichlet problem

$$\Delta^2 w = \lambda_1 w \quad \text{in} \quad \Omega,$$
$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \Gamma.$$

We recall also the classical inequality

$$||\Delta w||_{L^2(\Omega)} \geq \sqrt{\lambda_1} ||\nabla w||_{L^2(\Omega)}. \tag{1.20}$$

Unlike the existing papers on stability for hyperbolic equations in non cylindrical domain, we do not use the penalty method introduced by J. L. Lions [16], but work directly in our non cylindrical domain $\hat{Q}$. To see the dissipative properties of the system we have to construct a suitable functional whose derivative is negative and is equivalent to the first order energy. This functional is obtained using the multiplicative technique following Komornik [8] or Rivera [20]. We only obtained the exponential decay of solution for our problem for the case $n > 2$. The main difficulty in obtaining the decay for $n \leq 2$ is due to the geometry of the non cylindrical domain because it affects substantially the problem, since we work directly in $\hat{Q}$. Therefore the case $n \leq 2$ is an important open problem. From the physics point of view, the system (1.1)-(1.4) describes the transverse deflection of a streched viscoelastic beam fixed in a moving boundary device. The viscoelasticity property of the material is characterized by the memory terms

$$\int_0^t g_1(t-s)\Delta u(s)ds, \quad \int_0^t g_2(t-s)\Delta v(s)ds.$$

The uniform stabilization of plates equations with linear or nonlinear boundary feedback was investigated by several authors, see for example [7, 9, 10, 11, 13, 22] among others. In a fixed domain, it is well-known, the relaxation function g decays to zero implies that the energy of the system also decays to zero, see [2, 12, 19, 23]. But in a moving domain the transverse deflection $u(x,t)$ and $v(x,t)$ of a beam which charges its configuration at each instant of time, increasing its deformation and hence increasing its tension. Moreover, the horizontal movement of the boundary yields nonlinear terms involving derivatives in the space variable. To control these nonlinearities, we add in the system a frictional damping, characterized by u_t and v_t. This term will play an important role in the dissipative nature of the problem. A quite complete discussion in the modelling of transverse deflection and transverse vibrations, respectively, of purely for the nonlinear beam equation and elastic membranes can be found in J. Ferreira et al. [6], J. Límaco et al. [17] and L. A. Medeiros et al. [18]. This model was proposed by Woinowsky [24] for the case of cylindrical domain, without the terms $-\Delta u$ and $\int_0^t g_1(t-s)\Delta u(s)ds$ but with the term

$$-M(\int_\Omega |\nabla u|^2)\Delta u.$$

See also Eisley [5] and Burgreen [1] for physics justification and background of the model. We use the standard notations which can be found in Lion's book [15, 16]. In the sequel by C (sometimes $C_1, C_2, \ldots$) we denote various positive constants which do not depend on t or on the initial data. This paper is organized as follows. In section 2 we prove a basic result on the existence, regularity and uniqueness of regular solutions. We use Galerkin approximation, Aubin-Lions theorem, energy method introduced by Lions [16] and some technical ideas to show existence regularity and uniqueness of regular solution for the problem (1.1)-(1.4). Finally, in section 3, we establish a result on the exponential decay of the regular solution to the problem (1.1)-(1.4). We use the technique of the multipliers introduced by Komornik [8], Lions [16] and Rivera [20] coupled with some technical lemmas and some technical ideas.

2. EXISTENCE AND REGULARITY OF GLOBAL SOLUTIONS

In this section we shall study the existence and regularity of solutions for the system (1.1)-(1.4). For this we assume that the kernels $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ is in $C^1(0,\infty)$, and satisfy

$$g_i, -g_i' \geq 0, \quad \gamma_1^{-2} - \int_0^\infty g_i(s)\gamma^{-2}(s)ds = \beta_i > 0, \quad i = 1, 2, \tag{2.1}$$

where

$$\gamma_1 = \sup_{0 \leq t < \infty} \gamma(t).$$

To simplify our analysis, we define the binary operator

$$g\Box\frac{\Phi(t)}{\gamma(t)}=\int_\Omega\int_0^t g(t-s)\gamma^{-2}(s)|\Phi(t)-\Phi(s)|^2dsdx.$$

With this notation we have the following lemma.

Lemma 2.1 *For $\Phi\in C^1(0,T:H_0^2(\Omega))$ and $g\in C^1(\Re_+)$ we have*

$$\begin{aligned}\int_\Omega\int_0^t g(t-s)\gamma^{-2}(s)\nabla\Phi(s)\cdot\nabla\Phi_t dsdx &= -\frac{1}{2}\frac{g(t)}{\gamma^2(0)}\int_\Omega|\nabla\Phi|^2dx+\frac{1}{2}g'\Box\frac{\nabla\Phi}{\gamma}\\ &\quad -\frac{1}{2}\frac{d}{dt}\left[g\Box\frac{\nabla\Phi}{\gamma}-(\int_0^t\frac{g(s)}{\gamma^2(s)}ds)\int_\Omega|\nabla\Phi|^2dx\right],\end{aligned}$$

and

$$\begin{aligned}\int_\Omega\int_0^t g(t-s)\gamma^{-2}(s)\Delta\Phi(s)\Delta\Phi_t dsdx &= -\frac{1}{2}\frac{g(t)}{\gamma^2(0)}\int_\Omega|\Delta\Phi|^2dx+\frac{1}{2}g'\Box\frac{\Delta\Phi}{\gamma}\\ &\quad -\frac{1}{2}\frac{d}{dt}\left[g\Box\frac{\Delta\Phi}{\gamma}-(\int_0^t\frac{g(s)}{\gamma^2(s)}ds)\int_\Omega|\Delta\Phi|^2dx\right].\end{aligned}$$

The proof of this lemma follows by differentiating the terms $g\Box\frac{\nabla\Phi(t)}{\gamma(t)}$ and $g\Box\frac{\Delta\Phi(t)}{\gamma(t)}$. The well-posedness of system (1.10)-(1.13) is given by the following theorem.

Theorem 2.1 *Let us take $(\phi_0,\varphi_0)\in(H_0^2(\Omega)\cap H^4(\Omega))^2$, $(\phi_1,\varphi_1)\in(H_0^2(\Omega))^2$ and let us suppose that assumptions (1.15)-(1.20) and (2.1) hold. Then there exists a unique solution (ϕ,φ) of the problem (1.10)-(1.13) satisfying*

$$\phi,\varphi\in L^\infty(0,\infty:H_0^2(\Omega)\cap H^4(\Omega)),$$

$$\phi_t,\varphi_t\in L^\infty(0,\infty:H_0^1(\Omega)),$$

$$\phi_{tt},\varphi_{tt}\in L^\infty(0,\infty:L^2(\Omega)).$$

Proof. Let us denote by B the operator

$$Bw=-\Delta^2w,\quad D(B)=H_0^2(\Omega)\cap H^4(\Omega).$$

It is well know that B is a positive self adjoint operator in the Hilbert space $L^2(\Omega)$ for which there exist sequences $\{w_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of eigenfunctions and eigenvalues of B such that the set of linear combinations of $\{w_n\}_{n\in\mathbb{N}}$ is dense in $D(B)$ and $\lambda_1<\lambda_2\le\ldots\le\lambda_n\to\infty$ as $n\to\infty$. Let us denote by

$$\phi_0^m=\sum_{j=1}^m(\phi_0,w_j)w_j,\quad \varphi_0^m=\sum_{j=1}^m(\varphi_0,w_j)w_j,$$

$$\phi_1^m=\sum_{j=1}^m(\phi_1,w_j)w_j\quad \varphi_1^m=\sum_{j=1}^m(\varphi_1,w_j)w_j.$$

Note that for any $\{(\phi_0, \phi_1), (\varphi_0, \varphi_1)\} \in (D(B) \times H_0^2(\Omega))^2$, we have $\phi_0^m \to \phi_0$ strong in $D(B)$, $\varphi_0^m \to \varphi_0$ strong in $D(B)$, $\phi_1^m \to \phi_1$ strong in $H_0^2(\Omega)$ and $\varphi_1^m \to \varphi_1$ strong in $H_0^2(\Omega)$.

Let us denote by V_m the space generated by $w_1, \ldots, w_m$. Standard results on ordinary differential equations imply the existence of a local solution (ϕ^m, φ^m) of the form

$$(\phi^m(t), \varphi^m(t)) = \sum_{j=1}^{m} (g_{jm}(t), f_{jm}(t)) w_j,$$

to the system

$$\begin{aligned}
&\int_\Omega \phi_{tt}^m w_j dy + \alpha \int_\Omega \phi_t^m w_j dy + \int_\Omega \gamma^{-4} \Delta^2 \phi^m w_j dy \\
&-\gamma^{-2} M(\gamma^{n-2}(||\nabla \phi^m||^2_{L^2(\Omega)} + ||\nabla \varphi^m||^2_{L^2(\Omega)})) \int_\Omega \Delta \phi^m w_j dy \\
&+ \int_\Omega \int_0^t g_1(t-s) \gamma^{-2}(s) \nabla \phi^m(s) \cdot \nabla w_j ds dy + \int_\Omega A(t) \phi^m w_j dy \\
&+ \int_\Omega a_1 \cdot \nabla \phi_t^m w_j dy + \int_\Omega a_2 \cdot \nabla \phi^m w_j dy + \int_\Omega h(\phi - \varphi) w_j dy = 0, \qquad (2.2) \\
&(j = 1, \ldots, m), \\
&\int_\Omega \varphi_{tt}^m w_j dy + \alpha \int_\Omega \varphi_t^m w_j dy + \int_\Omega \gamma^{-4} \Delta^2 \varphi^m w_j dy \\
&-\gamma^{-2} M(\gamma^{n-2}(||\nabla \phi^m||^2_{L^2(\Omega)} + ||\nabla \varphi^m||^2_{L^2(\Omega)})) \int_\Omega \Delta \varphi^m w_j dy \\
&+ \int_\Omega \int_0^t g_2(t-s) \gamma^{-2}(s) \nabla \varphi^m(s) \cdot \nabla w_j ds dy + \int_\Omega A(t) \varphi^m w_j dy \\
&+ \int_\Omega a_1 \cdot \nabla \varphi_t^m w_j dy + \int_\Omega a_2 \cdot \nabla \varphi^m w_j dy - \int_\Omega h(\phi - \varphi) w_j dy = 0, \qquad (2.3) \\
&(j = 1, \ldots, m), \\
&(\phi^m(y,0), \varphi^m(y,0)) = (\phi_0^m, \varphi_0^m), \quad (\phi_t^m(y,0), \varphi_0^m(y,0)) = (\phi_1^m, \varphi_1^m). \qquad (2.4)
\end{aligned}$$

The extension of the solution to the whole interval $[0, \infty)$ is a consequence of the first estimate which we are going to prove below.

A Priori estimate I

Multiplying the equations (2.2) by $g'_{jm}(t)$ and (2.3) by $f'_{jm}(t)$, summing up the product result in $j = 1, 2, \ldots, m$ and using the Lemma 2.1 we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\pounds_1^m(t,\phi^m,\varphi^m) + \alpha||\phi_t^m||^2_{L^2(\Omega)} + \int_\Omega A(t)\phi^m\phi_t^m dy + \int_\Omega a_1\cdot\nabla\phi_t^m\phi_t^m dy\\
&+\int_\Omega a_2\cdot\nabla\phi^m\phi_t^m dy + \alpha||\varphi_t^m||^2_{L^2(\Omega)} + \int_\Omega A(t)\varphi^m\varphi_t^m dy + \int_\Omega a_1\cdot\nabla\varphi_t^m\varphi_t^m dy\\
&-\frac{(n-2)\gamma'}{2\gamma^{n+1}}\Big[\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)})M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)})\\
&-\widehat{M}(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)})\Big]\\
&= -\frac{1}{2}\frac{g_1(t)}{\gamma^2(0)}||\nabla\phi^m||^2_{L^2(\Omega)} + \frac{1}{2}g_1'\Box\frac{\nabla\phi^m}{\gamma} - 4\frac{\gamma'}{\gamma^5}\|\Delta\phi^m\|^2_{L^2(\Omega)}\\
&-\frac{1}{2}\frac{g_2(t)}{\gamma^2(0)}||\nabla\varphi^m||^2_{L^2(\Omega)} + \frac{1}{2}g_2'\Box\frac{\nabla\varphi^m}{\gamma} - 4\frac{\gamma'}{\gamma^5}\|\Delta\varphi^m\|^2_{L^2(\Omega)},
\end{aligned}
$$

where

$$
\begin{aligned}
\pounds_1^m(t,\phi^m,\varphi^m) \;=\;& ||\phi_t^m||^2_{L^2(\Omega)} + \left(\frac{1}{\gamma^2(t)} - \int_0^t g_1(s)\gamma^{-2}(s)ds\right)||\nabla\phi^m||^2_{L^2(\Omega)}\\
&+\gamma^{-n}\widehat{M}(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)}))\\
&+\gamma^{-4}\|\Delta\phi^m\|^2_{L^2(\Omega)} + g_1\Box\frac{\nabla\phi^m}{\gamma} + ||\varphi_t^m||^2_{L^2(\Omega)}\\
&+\left(\frac{1}{\gamma^2(t)} - \int_0^t g_2(s)\gamma^{-2}(s)ds\right)||\nabla\varphi^m||^2_{L^2(\Omega)}\\
&+\gamma^{-4}\|\Delta\varphi^m\|^2_{L^2(\Omega)} + g_2\Box\frac{\nabla\varphi^m}{\gamma} + 2\int_\Omega H(\phi-\varphi)dy.
\end{aligned}
$$

From (1.16)-(1.17) and (1.19)-(1.20) it follows that

$$
\begin{aligned}
&\gamma^{-n}\widehat{M}(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)}) + \gamma^{-4}(\|\Delta\phi^m\|^2_{L^2(\Omega)} + \|\Delta\varphi^m\|^2_{L^2(\Omega)})\\
&\geq \frac{m_1}{||\gamma||^2_\infty}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)}).
\end{aligned}
\tag{2.5}
$$

where $m_1 = (\frac{\lambda_1}{||\gamma||^2_\infty} - m_0)$. Taking into account (1.16), (1.17), (2.1) and the last equality we obtain

$$
\frac{1}{2}\frac{d}{dt}\pounds_1^m(t,\phi^m,\varphi^m) + \alpha||\phi_t^m||^2_{L^2(\Omega)} + \alpha||\varphi_t^m||^2_{L^2(\Omega)} \leq C(|\gamma'| + |\gamma''|)\pounds_1^m(t,\phi^m,\varphi^m). \tag{2.6}
$$

Integrating the inequality (2.6), using Gronwall's Lemma and taking account (1.17) we get

$$
\pounds_1^m(t,\phi^m,\varphi^m) + \int_0^t ||\phi_s^m(s)||^2_{L^2(\Omega)} + ||\varphi_s^m(s)||^2_{L^2(\Omega)}ds \leq C, \quad \forall m\in\mathbb{N}, \quad \forall t\in[0,T]. \tag{2.7}
$$

A priori estimate II

Now, if we multiply the equations (2.2) by $\sqrt{\lambda_j}g'_{jm}$ and (2.3) by $\sqrt{\lambda_j}f'_{jm}$ and summing up in

$j = 1, \ldots, m$ we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\phi_t^m\|_{L^2(\Omega)}^2 + \alpha\|\nabla\phi_t^m\|_{L^2(\Omega)}^2 + \frac{\gamma^{-4}}{2}\frac{d}{dt}\|\nabla\Delta\phi^m\|_{L^2(\Omega)}^2 \\
&+\frac{\gamma^{-2}}{2}M(\gamma^{n-2}(||\nabla\phi^m||_{L^2(\Omega)}^2 + ||\nabla\varphi^m||_{L^2(\Omega)}^2)\frac{d}{dt}\|\Delta\phi^m\|_{L^2(\Omega)}^2 \\
&-\int_\Omega\int_0^t g_1(t-s)\gamma^{-2}(s)\Delta\phi^m(s)\Delta\phi_t^m ds dy \\
&-\int_\Omega A(t)\phi^m\Delta\phi_t^m dy - \int_\Omega a_1\cdot\nabla\phi_t^m\Delta\phi_t^m dy \\
&-\int_\Omega a_2\cdot\nabla\phi^m\Delta\phi_t^m dy + \int_\Omega h(\phi^m - \varphi^m)\Delta\phi_t^m dy = 0, \\
&\frac{1}{2}\frac{d}{dt}\|\nabla\varphi_t^m\|_{L^2(\Omega)}^2 + \alpha\|\nabla\varphi_t^m\|_{L^2(\Omega)}^2 + \frac{\gamma^{-4}}{2}\frac{d}{dt}\|\nabla\Delta\varphi^m\|_{L^2(\Omega)}^2 \\
&+\frac{\gamma^{-2}}{2}M(\gamma^{n-2}(||\nabla\phi^m||_{L^2(\Omega)}^2 + ||\nabla\varphi^m||_{L^2(\Omega)}^2)\frac{d}{dt}\|\Delta\varphi^m\|_{L^2(\Omega)}^2 \\
&-\int_\Omega\int_0^t g_2(t-s)\gamma^{-2}(s)\Delta\varphi^m(s)\Delta\varphi_t^m ds dy \\
&-\int_\Omega A(t)\varphi^m\Delta\varphi_t^m dy - \int_\Omega a_1\cdot\nabla\varphi_t^m\Delta\varphi_t^m dy \\
&-\int_\Omega a_2\cdot\nabla\varphi^m\Delta\varphi_t^m dy - \int_\Omega h(\phi^m - \varphi^m)\Delta\varphi_t^m dy = 0.
\end{aligned}
$$

Summing the last two equalities and using the lemma 2.1 we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{L}_2^m(t) + \alpha(\|\nabla\phi_t^m\|_{L^2(\Omega)}^2 + \|\nabla\varphi_t^m\|_{L^2(\Omega)}^2) = -\frac{1}{2}\frac{g_1(t)}{\gamma^2(0)}\|\Delta\phi^m\|_{L^2(\Omega)}^2 \\
&+\frac{1}{2}g_1'\Box\frac{\Delta\phi^m}{\gamma} - \frac{2\gamma'}{\gamma^5}(\|\nabla\Delta\phi^m\|_{L^2(\Omega)}^2 + \|\nabla\Delta\varphi^m\|_{L^2(\Omega)}^2) \\
&+\frac{d}{dt}\left(\gamma^{-2}M(\gamma^{n-2}((||\nabla\phi^m||_{L^2(\Omega)}^2 + ||\nabla\varphi^m||_{L^2(\Omega)}^2))\right)(\|\Delta\phi^m\|_{L^2(\Omega)}^2 + \|\Delta\varphi^m\|_{L^2(\Omega)}^2) \\
&+\int_\Omega A(t)\phi^m\Delta\phi_t^m dy + \int_\Omega a_1\cdot\nabla\phi_t^m\Delta\phi_t^m dy + \int_\Omega a_2\cdot\nabla\phi^m\Delta\phi_t^m dy \\
&-\frac{1}{2}\frac{g_2(t)}{\gamma^2(0)}\|\Delta\varphi^m\|_{L^2(\Omega)}^2 + \frac{1}{2}g_2'\Box\frac{\Delta\varphi^m}{\gamma} + \int_\Omega A(t)\varphi^m\Delta\varphi_t^m dy \\
&+\int_\Omega a_1\cdot\nabla\varphi_t^m\Delta\varphi_t^m dy + \int_\Omega a_2\cdot\nabla\varphi^m\Delta\varphi_t^m dy \\
&-\int_\Omega h(\phi^m - \varphi^m)\Delta\phi_t^m dy + \int_\Omega h(\phi^m - \varphi^m)\Delta\varphi_t^m dy
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}_2^m(t) \;=\;& \|\nabla\phi_t^m\|_{L^2(\Omega)}^2 + \left(\frac{1}{\gamma^2(t)} - \int_0^t g_1(s)\gamma^{-2}(s)ds\right)||\Delta\phi^m||_{L^2(\Omega)}^2 \\
&+\gamma^{-4}(\|\nabla\Delta\phi^m\|_{L^2(\Omega)}^2 + \|\nabla\Delta\varphi^m\|_{L^2(\Omega)}^2) + g_1\Box\frac{\Delta\phi^m}{\gamma} + \|\nabla\varphi_t^m\|_{L^2(\Omega)}^2 \\
&+\gamma^{-2}M(\gamma^{n-2}(||\nabla\phi^m||_{L^2(\Omega)}^2 + ||\nabla\varphi^m||_{L^2(\Omega)}^2)(\|\Delta\phi^m\|_{L^2(\Omega)}^2 + \|\Delta\varphi^m\|_{L^2(\Omega)}^2) \\
&+\left(\frac{1}{\gamma^2(t)} - \int_0^t g_2(s)\gamma^{-2}(s)ds\right)||\Delta\varphi^m||_{L^2(\Omega)}^2 + g_2\Box\frac{\Delta\varphi^m}{\gamma}.
\end{aligned}
$$

From (1.16)-(1.17) and (1.19)-(1.20), we have

$$
\begin{aligned}
&\gamma^{-4}(\|\nabla\Delta\phi^m\|^2_{L^2(\Omega)}+\|\nabla\Delta\varphi^m\|^2_{L^2(\Omega)})\\
&+\gamma^{-2}M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)}+||\nabla\varphi^m||^2_{L^2(\Omega)})(\|\Delta\phi^m\|^2_{L^2(\Omega)}+\|\Delta\varphi^m\|^2_{L^2(\Omega)})\\
&\geq \frac{m_1}{||\gamma||^2_\infty}(||\Delta\phi^m||^2_{L^2(\Omega)}+||\Delta\varphi^m||^2_{L^2(\Omega)}).
\end{aligned}
$$

Using similar arguments as (2.7), the hypothesis for the function h and observing the above inequality we obtain

$$
\pounds_2^m(t)+\int_0^t(\|\nabla\phi_s^m\|^2_{L^2(\Omega)}+\|\nabla\varphi_s^m\|^2_{L^2(\Omega)})ds\leq C,\quad \forall m\in\mathbb{N},\quad \forall t\in[0,T]. \tag{2.8}
$$

A priori estimate III

It easy to see from (2.2)-(2.3) and of the growth hypothesis for the function h together with the Sobolev's imbedding that

$$
||\phi^m_{tt}(0)||^2_{L^2(\Omega)}+||\varphi^m_{tt}(0)||^2_{L^2(\Omega)}\leq C,\quad \forall m\in\mathbb{N}. \tag{2.9}
$$

Differentiating the equations (2.2)-(2.3) with respect to the time, we obtain

$$
\begin{aligned}
&\int_\Omega\phi^m_{ttt}w_jdy+\alpha\int_\Omega\phi^m_{tt}w_jdy+\gamma^{-4}\int_\Omega\Delta^2\phi^m_tw_jdy\\
&-\int_\Omega\frac{d}{dt}(\gamma^{-2}M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)}+||\nabla\varphi^m||^2_{L^2(\Omega)}))\Delta\phi^m)w_jdy\\
&-\frac{4\gamma'}{\gamma^5}\int_\Omega\Delta^2\phi^mw_jdy-\frac{g_0(0)}{\gamma^2(0)}\int_\Omega\Delta\phi^m_0w_jdy\\
&+\int_\Omega\int_0^t g_1'(t-s)\gamma^{-2}(s)\nabla\phi^m(s)\cdot\nabla w_jdsdy\\
&+\int_\Omega\frac{d}{dt}(A(t)\phi^m)w_jdy+\int_\Omega\frac{d}{dt}(a_1\cdot\nabla\phi^m_t)w_jdy\\
&+\int_\Omega h'(\phi^m-\varphi^m)(\phi^m_t-\varphi^m_t)w_jdy+\int_\Omega\frac{d}{dt}(a_2\cdot\nabla\phi^m)w_jdy=0
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&\int_\Omega\varphi^m_{ttt}w_jdy+\alpha\int_\Omega\varphi^m_{tt}w_jdy+\gamma^{-4}\int_\Omega\Delta^2\varphi^m_tw_jdy\\
&-\int_\Omega\frac{d}{dt}(\gamma^{-2}M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)}+||\nabla\varphi^m||^2_{L^2(\Omega)}))\Delta\varphi^m)w_jdy\\
&-\frac{4\gamma'}{\gamma^5}\int_\Omega\Delta^2\varphi^mw_jdy-\frac{g_2(0)}{\gamma^2(0)}\int_\Omega\Delta\varphi^m_0w_jdy\\
&+\int_\Omega\int_0^t g'(t-s)\gamma^{-2}(s)\nabla\varphi^m(s)\cdot\nabla w_jdsdy\\
&+\int_\Omega\frac{d}{dt}(A(t)\varphi^m)w_jdy+\int_\Omega\frac{d}{dt}(a_1\cdot\nabla\varphi^m_t)w_jdy\\
&-\int_\Omega h'(\phi^m-\varphi^m)(\phi^m_t-\varphi^m_t)w_jdy+\int_\Omega\frac{d}{dt}(a_2\cdot\nabla\varphi^m)w_jdy=0.
\end{aligned}
\tag{2.11}
$$

Multiplying the equations (2.10)-(2.11) by $g''_{jm}(t)$ and $f''_{jm}(t)$, respectively, and summing up the product result in $j = 1, 2, \ldots, m$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\phi^m_{tt}|^2 dy + \alpha \int_\Omega |\phi^m_{tt}|^2 dy + \frac{\gamma^{-4}}{2}\frac{d}{dt}\int_\Omega |\Delta\phi^m_t|^2 dy \\
&-\int_\Omega \frac{d}{dt}(\gamma^{-2} M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)}))\Delta\phi^m)\phi^m_{tt} dy \\
&-\frac{4\gamma'}{\gamma^5}\int_\Omega \Delta^2\phi^m\phi^m_{tt} dy - \frac{g_1(0)}{\gamma^2(0)}\int_\Omega \Delta\phi^m_0\phi^m_{tt} dy \\
&+\int_\Omega\int_0^t g'_1(t-s)\gamma^{-2}(s)\nabla\phi^m(s)\cdot\nabla\phi^m_{tt} ds dy \\
&+\int_\Omega \frac{d}{dt}(A(t)\phi^m)\phi^m_{tt} dy + \int_\Omega \frac{d}{dt}(a_1\cdot\nabla\phi^m_t)\phi^m_{tt} dy \\
&+\int_\Omega h'(\phi^m - \varphi^m)(\phi^m_t - \varphi^m_t)\phi^m_{tt} dy + \int_\Omega \frac{d}{dt}(a_2\cdot\nabla\phi^m)\phi^m_{tt} dy = 0 \qquad (2.12)\\
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi^m_{tt}|^2 dy + \alpha \int_\Omega |\varphi^m_{tt}|^2 dy + \frac{\gamma^{-4}}{2}\frac{d}{dt}\int_\Omega |\Delta\varphi^m_t|^2 dy \\
&-\int_\Omega \frac{d}{dt}(\gamma^{-2} M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)}))\Delta\varphi^m)\varphi^m_{tt} dy \\
&-\frac{4\gamma'}{\gamma^5}\int_\Omega \Delta^2\varphi^m\varphi^m_{tt} dy - \frac{g_2(0)}{\gamma^2(0)}\int_\Omega \Delta\varphi^m_0\varphi^m_{tt} dy \\
&+\int_\Omega\int_0^t g'(t-s)\gamma^{-2}(s)\nabla\varphi^m(s)\cdot\nabla\varphi^m_{tt} ds dy \\
&+\int_\Omega \frac{d}{dt}(A(t)\varphi^m)\varphi^m_{tt} dy + \int_\Omega \frac{d}{dt}(a_1\cdot\nabla\varphi^m_t)\varphi^m_{tt} dy \\
&-\int_\Omega h'(\phi^m - \varphi^m)(\phi^m_t - \varphi^m_t)\varphi^m_{tt} dy + \int_\Omega \frac{d}{dt}(a_2\cdot\nabla\varphi^m)\varphi^m_{tt} dy = 0. \qquad (2.13)
\end{aligned}$$

Let us take $p_n = \frac{2n}{n-2}$. From the growth condition of the function h and from the Sobolev imbedding we obtain

$$\begin{aligned}
&\int_\Omega h'(\phi^m - \varphi^m)\phi^m_t\phi^m_{tt} dy \le C\int_\Omega (1 + 2|\phi^m - \varphi^m|^{\rho-1}|\phi^m_t||\phi^m_{tt}| dy \\
&\le C\left[\int_\Omega (1 + 2|\phi^m - \varphi^m|^{\rho-1})^n dy\right]^{\frac{1}{n}}\left[\int_\Omega |\phi^m_t|^{p_n} dy\right]^{\frac{1}{p_n}}\left[\int_\Omega |\phi^m_{tt}|^2 dy\right]^{\frac{1}{2}} \\
&\le C\left[\int_\Omega (1 + |\nabla(\phi^m - \varphi^m)|^2) dy\right]^{\frac{\rho-1}{2}}\left[\int_\Omega |\nabla\phi^m_t|^2 dy\right]^{\frac{1}{2}}\left[\int_\Omega |\phi^m_{tt}|^2 dy\right]^{\frac{1}{2}}.
\end{aligned}$$

Taking into account the estimate (2.7), we conclude that

$$\begin{aligned}
\int_\Omega h'(\phi^m - \varphi^m)\phi^m_t\phi^m_{tt} dy &\le C\left[\int_\Omega |\nabla\phi^m_t|^2 dy\right]^{\frac{1}{2}}\left[\int_\Omega |\phi^m_{tt}|^2 dy\right]^{\frac{1}{2}} \\
&\le C\left\{\int_\Omega |\nabla\phi^m_t|^2 dy + \int_\Omega |\phi^m_{tt}|^2 dy\right\}. \qquad (2.14)
\end{aligned}$$

Similarly we get

$$\int_\Omega h'(\phi^m-\varphi^m)\varphi_t^m\phi_{tt}^m dy \le C\left\{\int_\Omega |\nabla\varphi_t^m|^2dy+\int_\Omega |\phi_{tt}^m|^2dy\right\}, \tag{2.15}$$

$$\int_\Omega h'(\phi^m-\varphi^m)\phi_t^m\varphi_{tt}^m dy \le C\left\{\int_\Omega |\nabla\phi_t^m|^2dy+\int_\Omega |\varphi_{tt}^m|^2dy\right\}, \tag{2.16}$$

$$\int_\Omega h'(\phi^m-\varphi^m)\varphi_t^m\varphi_{tt}^m dy \le C\left\{\int_\Omega |\nabla\varphi_t^m|^2dy+\int_\Omega |\varphi_{tt}^m|^2dy\right\}. \tag{2.17}$$

Summing the equations (2.12) and (2.13), substituting the inequalities (2.14)-(2.17) and using similar arguments as (2.7)-(2.8) we obtain, after some calculations and taking into account the lemma 2.1 and hypothesis on M

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\pounds_3^m(t)+\alpha(||\phi_{tt}^m(t)||^2_{L^2(\Omega)}+||\varphi_{tt}^m(t)||^2_{L^2(\Omega)})\\
&\le C(|\gamma'|+|\gamma''|)(\|\Delta\phi^m\|^2_{L^2(\Omega)}+\|\nabla\phi_t^m\|^2_{L^2(\Omega)})\\
&+C(|\gamma'|+|\gamma''|)(\|\Delta\varphi^m\|^2_{L^2(\Omega)}+\|\nabla\varphi_t^m\|^2_{L^2(\Omega)})\\
&+C(||\nabla\phi_t^m||^2_{L^2(\Omega)}+||\nabla\varphi_t^m||^2_{L^2(\Omega)})+C(|\gamma'|+|\gamma''|)\pounds_3^m(t),
\end{aligned}$$

where

$$\begin{aligned}
\pounds_3^m(t) &= ||\phi_{tt}^m||^2_{L^2(\Omega)}+||\varphi_{tt}^m||^2_{L^2(\Omega)}+g_1\Box\frac{\nabla\phi_t}{\gamma}+g_2\Box\frac{\nabla\varphi_t}{\gamma}\\
&+\gamma^{-2}M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)}+||\nabla\varphi^m||^2_{L^2(\Omega)}))(||\nabla\phi_t^m||^2_{L^2(\Omega)}+||\nabla\varphi_t^m||^2_{L^2(\Omega)})\\
&-(\int_0^t g_1(s)\gamma^{-2}(s)ds)||\nabla\phi_t||^2_{L^2(\Omega)}-(\int_0^t g_2(s)\gamma^{-2}(s)ds)||\nabla\varphi_t||^2_{L^2(\Omega)}.
\end{aligned}$$

Using Gronwall's lemma and relations (1.17), (2.7)-(2.8) we get

$$\pounds_3^m(t)+\alpha\int_0^t(||\phi_{ss}^m(s)||^2_{L^2(\Omega)}+||\varphi_{ss}^m(s)||^2_{L^2(\Omega)})ds\le C, \quad \forall t\in[0,T], \quad \forall m\in\mathbb{N}. \tag{2.18}$$

A priori estimate IV

Multiplying the equations (2.2)-(2.3) by $g_{jm}(t)$ and $f_{jm}(t)$, respectively, summing up the product result in $j=1,2,\dots,m$ and using the hypothesis on h we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\pounds_4^m(t)-||\phi_t^m||^2_{L^2(\Omega)}-||\varphi_t^m||^2_{L^2(\Omega)}+\gamma^{-4}(||\Delta\phi||^2_{L^2(\Omega)}+||\Delta\varphi||^2_{L^2(\Omega)})\\
&+\gamma^{-2}M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)}+||\nabla\varphi||^2_{L^2(\Omega)}))(||\nabla\phi^m||^2_2+||\nabla\varphi||^2_{L^2(\Omega)})\\
&+\int_\Omega\int_0^t g_1(s)\gamma^{-2}(s)\Delta\phi^m(s)\phi^m(t)dsdy\\
&+\int_\Omega\int_0^t g_2(s)\gamma^{-2}(s)\Delta\varphi^m(s)\varphi^m(t)dsdy\\
&\le C(|\gamma'|+|\gamma''|)(||\phi^m||^2_{L^2(\Omega)}+||\varphi^m||^2_{L^2(\Omega)}+||\Delta\phi^m||^2_{L^2(\Omega)}\\
&+||\Delta\varphi^m||^2_{L^2(\Omega)}+||\phi_t^m||^2_{L^2(\Omega)}+||\varphi_t^m||^2_{L^2(\Omega)}),
\end{aligned} \tag{2.19}$$

where

$$\pounds_4^m(t) = 2\int_\Omega (\phi^m \phi_t^m + \varphi^m \varphi_t^m) dy + \alpha(||\phi^m||^2_{L^2(\Omega)} + ||\varphi^m||^2_{L^2(\Omega)}).$$

Choosing $k > \frac{2}{\alpha}$, we obtain

$$\begin{aligned} k\pounds_1^m(t) + \pounds_4^m(t) &\geq (k - \frac{2}{\alpha})(||\phi_t^m||^2_{L^2(\Omega)} + ||\varphi_t^m||^2_{L^2(\Omega)} + ||\phi^m||^2_{L^2(\Omega)} + ||\varphi^m||^2_{L^2(\Omega)}) \\ &+ (k - \frac{2}{\alpha})\gamma^{-4}(||\Delta\phi^m||^2_{L^2(\Omega)} + ||\Delta\varphi^m||^2_{L^2(\Omega)}) \\ &+ \gamma^{-n}\widehat{M}(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)}))) > 0. \end{aligned} \tag{2.20}$$

Now, multiplying the equations (2.2) and (2.3) by $kg_{jm}(t)$ and $kf_{jm}(t)$, respectively, summing up the product result and combining with (2.19), we get, taking into account (2.5)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(k\pounds_1^m(t) + \pounds_4^m(t)) + (k\alpha - 1)(||\phi_t^m||^2_{L^2(\Omega)} + ||\varphi_t^m||^2_{L^2(\Omega)}) \\ &+ (\gamma^{-4}(||\Delta\phi^m||^2_{L^2(\Omega)} + ||\Delta\varphi^m||^2_{L^2(\Omega)}) \leq C(|\gamma'| + |\gamma''|)(k\pounds_1^m(t) + \pounds_4^m(t)). \end{aligned} \tag{2.21}$$

From (2.21), using the Gronwall's Lemma we obtain the following estimate, taking into account (2.5) and (1.17)

$$\begin{aligned} &k\pounds_1^m(t) + \pounds_4^m(t) + \int_0^t (||\phi_s^m(s)||^2_{L^2(\Omega)} + ||\varphi_s^m(s)||^2_{L^2(\Omega)} + ||\Delta\phi^m(s)||^2_{L^2(\Omega)} \\ &+ ||\Delta\varphi^m(s)||^2_{L^2(\Omega)}) ds \leq C(||\phi_1||^2_{L^2(\Omega)} + ||\varphi_1||^2_{L^2(\Omega)} + ||\Delta\phi_0||^2_{L^2(\Omega)} + ||\Delta\varphi_0||^2_{L^2(\Omega)}). \end{aligned} \tag{2.22}$$

From estimates (2.7), (2.8), (2.18) and (2.22) it's follows that (ϕ^m, φ^m) converge strong to $(\phi, \varphi) \in L^2(0, \infty : H_0^2(\Omega))$. Moreover, since $M \in C^1[0, \infty[$ and $\nabla\phi^m$, $\nabla\varphi^m$ are bounded in $L^\infty(0, \infty : L^2(\Omega)) \cap L^2(0, \infty : L^2(\Omega))$, we have for any $t > 0$

$$\begin{aligned} &\int_0^t |M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)})) - M(\gamma^{n-2}(||\nabla\phi||^2_{L^2(\Omega)} + ||\nabla\varphi||^2_{L^2(\Omega)}))| \\ &\leq C\int_0^t (||\phi^m - \phi||^2_{H^1(\Omega)} + ||\varphi^m - \varphi||^2_{H^1(\Omega)}, \end{aligned}$$

where C is a positive constant independent of m and t, so that

$$M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)}))(\Delta\phi^m, w_j) \to M(\gamma^{n-2}(||\nabla\phi||^2_{L^2(\Omega)} + ||\nabla\varphi||^2_{L^2(\Omega)}))(\Delta\phi, w_j)$$

and

$$M(\gamma^{n-2}(||\nabla\phi^m||^2_{L^2(\Omega)} + ||\nabla\varphi^m||^2_{L^2(\Omega)}))(\Delta\varphi^m, w_j) \to M(\gamma^{n-2}(||\nabla\phi||^2_{L^2(\Omega)} + ||\nabla\varphi||^2_{L^2(\Omega)}))(\Delta\varphi, w_j).$$

Using similar arguments as above we conclude that

$$h(\phi^m - \varphi^m) \rightharpoonup h(\phi - \varphi) \quad \text{in} \quad L^2(0, T; L^2(\Omega)).$$

Letting $m \to \infty$ in the equations (2.2)-(2.3) we conclude that (ϕ, φ) satisfies (1.10)-(1.11) in $L^\infty(0,\infty : L^2(\Omega))$. Therefore we have that

$$\phi, \varphi \in L^\infty(0,\infty : H_0^2(\Omega) \cap H^4(\Omega)),$$

$$\phi_t, \varphi_t \in L^\infty(0,\infty : H_0^1(\Omega)),$$

$$\phi_{tt}, \varphi_{tt} \in L^\infty(0,\infty : L^2(\Omega)).$$

To prove the uniqueness of solutions of problem (1.10), (1.11), (1.12) and (1.13) we use the method of the energy introduced by Lions [16], coupled with Gronwall's inequality and the hypotheses introduced in the paper about the functions g_i, h,M and the obtained estimates. ■

To show the existence in non cylindrical domain, we return to our original problem in the non cylindrical domain by using the change of variable given in (1.8) by $(y,t) = \tau(x,t)$, $(x,t) \in \hat{Q}$. Let (ϕ, φ) be the solution obtained from Theorem 2.1 and (u, v) defined by (1.9), then (u, v) belong to the class

$$u, v \in L^\infty(0,\infty : H_0^2(\Omega_t) \cap H^4(\Omega_t)), \tag{2.23}$$

$$u_t, v_t \in L^\infty(0,\infty : H_0^1(\Omega_t)), \tag{2.24}$$

$$u_{tt}, v_{tt} \in L^\infty(0,\infty : L^2(\Omega_t)). \tag{2.25}$$

Denoting by

$$u(x,t) = \phi(y,t) = (\phi \circ \tau)(x,t), \quad v(x,t) = \varphi(y,t) = (\varphi \circ \tau)(x,t)$$

then from (1.9) it is easy to see that (u, v) satisfies the equations (1.1)-(1.2) in $L^\infty(0,\infty : L^2(\Omega_t))$. If (u_1, v_1), (u_2, v_2) are two solutions obtained through the diffeomorphism τ given by (1.7), then (ϕ_1, φ_1), (ϕ_2, φ_2) are the solution to (1.10)-(1.11). By uniqueness result of Theorem 2.1, we have $(\phi_1, \varphi_1) = (\phi_2, \varphi_2)$, so $(u_1, v_1) = (u_2, v_2)$. Therefore, we have the following result.

Theorem 2.2 *Let us take $(u_0, v_0) \in (H_0^2(\Omega_0) \cap H^4(\Omega_0))^2$, $(u_1, v_1) \in (H_0^2(\Omega_0))^2$ and let us suppose that assumptions (1.15)-(1.17), (1.19)-(1.20) and (2.1) hold. Then there exists a unique solution (u, v) of the problem (1.1)-(1.4) satisfying (2.23)-(2.25) and the equations (1.1)-(1.2) in $L^\infty(0,\infty : L^2(\Omega_t))$.*

■

3. EXPONENTIAL DECAY

In this section we show that the solution of system (1.1)-(1.4) decays exponentially. To this end we will assume that the memory g_i satisfies:

$$g_i'(t) \leq -C_1 g_i(t) \tag{3.1}$$

$$\left(1 - \int_0^{\infty} g_i(s)ds\right) = \beta_i > 0, \quad \forall i = 1, 2 \tag{3.2}$$

for all $t \geq 0$, with positive constant C_1. Additionally, we assume that the function $\gamma(\cdot)$ satisfies the conditions

$$\gamma' \leq 0, \quad t \geq 0, \quad n > 2, \tag{3.3}$$

$$0 < \max_{0 \leq t < \infty} |\gamma'(t)| \leq \frac{1}{d}, \tag{3.4}$$

where $d = diam(\Omega)$. The condition (3.4) (see also (1.5)) imply that our domain is "time like" in the sense that

$$|\underline{\nu}| < |\overline{\nu}|$$

where $\underline{\nu}$ and $\overline{\nu}$ denote the t-component and x-component of the outer unit normal of $\hat{\sum}$.

Remark: It is important to observe that to prove the main Theorem of this section, that is, Theorem 3.1 as well the Lemmas 3.4 and 3.5 we use the following substantial hypothesis:

$$M(s) \geq m_0 > 0, \quad \forall s \in [0, \infty[. \tag{3.5}$$

This because we worked directly in our domain with moving boundary, where the geometry of our domain influence directly in the problem, what generated several technical difficulties in limiting some terms in Lemma 3.5 and consequent to prove Theorem 3.1.

To facilitate our calculations we introduce the following notation

$$(g \Box \nabla u)(t) = \int_{\Omega_t} \int_0^t g(t-s) |\nabla u(t) - \nabla u(s)|^2 ds dx.$$

First of all we will prove the following three lemmas that will be used in the sequel.

Lemma 3.1 *Let $F(\cdot, \cdot)$ be the smooth function defined in $\Omega_t \times [0, \infty[$. Then*

$$\frac{d}{dt} \int_{\Omega_t} F(x,t)dx = \int_{\Omega_t} \frac{d}{dt} F(x,t)dx + \frac{\gamma'}{\gamma} \int_{\Gamma_t} F(x,t)(x \cdot \overline{\nu}) d\Gamma_t, \tag{3.6}$$

where $\overline{\nu}$ is the x-component of the unit normal exterior ν.

Proof. We have by a change variable $x = \gamma(t)y$, $y \in \Omega$

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} F(x,t)dx &= \frac{d}{dt}\int_{\Omega} F(\gamma(t)y,t)\gamma^n(t)dy \\ &= \int_{\Omega}(\frac{\partial F}{\partial t}(\gamma(t)y,t))\gamma^n(t)dy \\ &+\sum_{i=1}^{n}\int_{\Omega}\frac{\gamma'}{\gamma}x_i(\frac{\partial F}{\partial t}(\gamma(t)y,t))\gamma^n(t)dy \\ &+n\int_{\Omega}\gamma'(t)\gamma^{n-1}(t)F(\gamma(t)y,t)dy.\end{aligned}$$

If we return at the variable x, we get

$$\frac{d}{dt}\int_{\Omega_t} F(x,t)dx = \int_{\Omega_t}\frac{\partial F}{\partial t}(x,t)dx + \frac{\gamma'}{\gamma}\int_{\Omega_t} x\cdot\nabla F(x,t)dx + n\frac{\gamma'}{\gamma}\int_{\Omega_t} F(x,t)dx.$$

Integrating by parts in the last equality we obtain the formula (3.6). ■

Lemma 3.2 *Let $v \in H^2(\Omega)\cap H_0^1(\Omega)$. Then for all $i = 1,\dots,n$ we have*

$$\frac{\partial v}{\partial y_i} = \eta_i\frac{\partial v}{\partial \eta} \tag{3.7}$$

Proof. We consider $r \in C^2(\overline{\Omega},\mathbb{R}^n)$ such that

$$r = \nu \quad \text{on} \quad \Gamma. \tag{3.8}$$

(It is possible to choose such a field $r(\cdot)$ if we consider that the boundary Γ is sufficiently smooth). Let $\theta \in \mathcal{D}(\Gamma)$ and $\varphi \in H^m(\Omega)$ with $m > max(\frac{n}{2},2)$ such that $\varphi|_\Gamma = \theta$. Since $\mathcal{D}(\Gamma) \subset H^{m-\frac{1}{2}}(\Gamma)$, such function φ exists and we have

$$\int_\Omega \frac{\partial^2}{\partial y_i\partial y_j}(vr_j\varphi)dy = \int_\Gamma \eta_j\frac{\partial}{\partial y_j}(vr_j\varphi)d\Gamma = \int_\Gamma \theta\eta_i\frac{\partial v}{\partial \eta}d\Gamma \quad (i,j = 1,\dots,n).$$

Note that Ω is regular, we also obtain

$$\begin{aligned}\int_\Omega \frac{\partial^2}{\partial y_j\partial y_i}(vr_j\varphi)dy &= \int_\Gamma \eta_j\frac{\partial}{\partial y_i}(vr_j\varphi)d\Gamma = \int_\Gamma \theta\eta_j^2\frac{\partial v}{\partial y_i}d\Gamma \\ &= \int_\Gamma \theta\frac{\partial v}{\partial y_i}d\Gamma.\end{aligned}$$

It follows that

$$\int_\Gamma \theta\frac{\partial v}{\partial y_i}d\Gamma = \int_\Gamma \theta(\eta_i\frac{\partial v}{\partial y_i})d\Gamma \quad \forall\theta \in \mathcal{D}(\Gamma)$$

which implies (3.7). ■

From the Lemma 3.2 it is easy to see that

$$\nabla u \cdot \overline{\nu} = \frac{\partial u}{\partial \overline{\nu}} \quad \text{on} \quad \Gamma_t, \tag{3.9}$$

and for $u \in H_0^2(\Omega_t) \cap H^4(\Omega_t)$ (see Komornik [8] page 26) we have

$$|\nabla u| = 0, \quad \frac{\partial^2 u}{\partial \overline{\nu}^2} = \Delta u - \frac{\partial u}{\partial \overline{\nu}} div\overline{\nu} = \Delta u \quad \text{on} \quad \Gamma_t. \tag{3.10}$$

Lemma 3.3 *For any function $g \in C^1(\mathbb{R}_+)$ and $u \in C^1(0, \infty : H_0^2(\Omega_t) \cap H^4(\Omega_t))$ we have that*

$$\begin{aligned}\int_{\Omega_t}\int_0^t g(t-s)\nabla u(s)\cdot\nabla u_t dsdx &= -\frac{1}{2}g(t)\int_{\Omega_t}|\nabla u(t)|^2dx + \frac{1}{2}g'\square\nabla u\\ &\quad -\frac{1}{2}\frac{d}{dt}\left[g\square\nabla u - \left(\int_0^t g(s)ds\right)\int_{\Omega_t}|\nabla u|^2\right].\end{aligned}$$

Proof. Differentiating the term $g\square\nabla u$ and applying the lemma 3.1 we obtain

$$\begin{aligned}&\frac{d}{dt}g\square\nabla u = \int_{\Omega_t}\frac{d}{dt}\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2dsdx\\ &+\frac{\gamma'}{\gamma}\int_{\Gamma_t}\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2(x\cdot\overline{\nu})dsd\Gamma_t.\end{aligned}$$

Using (3.10) we have

$$\begin{aligned}&\frac{d}{dt}g\square\nabla u = \int_{\Omega_t}\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2dsdx\\ &-2\int_{\Omega_t}\int_0^t g(t-s)\nabla u_t(t)\cdot\nabla u(s)dsdx + \left(\int_0^t g(t-s)ds\right)\int_{\Omega_t}\frac{d}{dt}|\nabla u(t)|^2dx\end{aligned}$$

from where it follows that

$$\begin{aligned}&2\int_{\Omega_t}\int_0^t g(t-s)\nabla u_t(t)\cdot\nabla u(s)dsdx = -\frac{d}{dt}\{g\square\nabla u - \int_0^t g(t-s)ds\int_{\Omega_t}|\nabla u(t)|^2dx\}\\ &+\int_{\Omega_t}\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2dsdx - g(t)\int_{\Omega_t}|\nabla u(t)|^2dx.\end{aligned}$$

The proof is now complete. ■

Let us introduce the functional of energy

$$\begin{aligned}E(t) &= \|u_t\|^2_{L^2(\Omega_t)} + \|\Delta u\|^2_{L^2(\Omega_t)} + \left(1-\int_0^t g_1(s)ds\right)\|\nabla u\|^2_{L^2(\Omega_t)} + g_1\square\nabla u\\ &\quad +\|v_t\|^2_{L^2(\Omega_t)} + \|\Delta v\|^2_{L^2(\Omega_t)} + \left(1-\int_0^t g_2(s)ds\right)\|\nabla v\|^2_{L^2(\Omega_t)} + g_2\square\nabla v\\ &\quad +\widehat{M}(||\nabla u||^2_{L^2(\Omega_t)} + ||\nabla v||^2_{L^2(\Omega_t)}) + 2\int_{\Omega_t}H(u-v)dx.\end{aligned}$$

Lemma 3.4 *Let us take* $(u_0, v_0) \in (H_0^2(\Omega_0) \cap H^4(\Omega_0))^2$, $(u_1, v_1) \in (H_0^2(\Omega_0))^2$ *and let us suppose that assumptions (1.15), (1.16), (1.17), (1.19), (1.20), (2.1) and (3.5) hold. Then any strong solution of system (1.1)-(1.4) satisfies*

$$\begin{aligned}
&\frac{d}{dt}E(t) + 2\alpha(\|u_t\|^2_{L^2(\Omega_t)} + \|v_t\|^2_{L^2(\Omega_t)}) - \int_{\Gamma_t} \frac{\gamma'}{\gamma}(\overline{\nu} \cdot x)(|u_t|^2 + |\Delta u|^2) d\Gamma_t \\
&- \int_{\Gamma_t} \frac{\gamma'}{\gamma}(\overline{\nu} \cdot x)(|v_t|^2 + |\Delta v|^2) d\Gamma_t - 2\int_{\Gamma_t} \frac{\gamma'}{\gamma}(\overline{\nu} \cdot x) H(u-v) d\Gamma_t \\
&= - \int_{\Omega_t} g_1(t)|\nabla u|^2 dx + g_1' \Box \nabla u - \int_{\Omega_t} g_2(t)|\nabla v|^2 dx + g_2' \Box \nabla v.
\end{aligned}$$

Proof. Multiplying the equation (1.1) by u_t, performing an integration by parts over Ω_t and using the lemmas 3.1, 3.2 and 3.3 we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2(\Omega_t)} + M(||\nabla u||^2_{L^2(\Omega_)} + ||\nabla v||^2_{L^2(\Omega_t)})\frac{d}{dt}||\nabla u||^2_{L^2(\Omega_t)} \\
&+\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2_{L^2(\Omega)} - \int_{\Gamma_t} \frac{\gamma'}{2\gamma}(\overline{\nu} \cdot x)(|u_t|^2 + |\Delta u|^2) d\Gamma_t \\
&+\alpha\|u_t\|^2_{L^2(\Omega_t)} + \frac{1}{2}g_1(t)\int_{\Omega_t} |\nabla u|^2 dx - \frac{1}{2}g_1' \Box \nabla u \\
&+\frac{1}{2}\frac{d}{dt}\left[g_1 \Box \nabla u - \left(\int_0^t g_1(s) ds\right)\int_{\Omega_t} |\nabla u|^2 dx\right] \\
&+ \int_{\Omega_t} h(u-v) u_t dx = 0.
\end{aligned}$$

Similarly, using equation (1.2) instead (1.1) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_t\|^2_{L^2(\Omega_t)} + M(||\nabla u||^2_{L^2(\Omega_)} + ||\nabla v||^2_{L^2(\Omega_t)})\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2_{L^2(\Omega_t)} \\
&+\frac{1}{2}\frac{d}{dt}\|\Delta v\|^2_{L^2(\Omega)} - \int_{\Gamma_t} \frac{\gamma'}{2\gamma}(\overline{\nu} \cdot x)(|v_t|^2 + |\Delta v|^2) d\Gamma_t \\
&+\alpha\|v_t\|^2_{L^2(\Omega_t)} + \frac{1}{2}g_2(t)\int_{\Omega_t} |\nabla v|^2 dx - \frac{1}{2}g_2' \Box \nabla v \\
&+\frac{1}{2}\frac{d}{dt}\left[g_2 \Box \nabla v - \left(\int_0^t g_2(s) ds\right)\int_{\Omega_t} |\nabla v|^2 dx\right] \\
&- \int_{\Omega_t} h(u-v) v_t dx = 0.
\end{aligned}$$

Summing these two last equalities, taking into account the lemma 3.1 and hypothesis on h we obtain the conclusion of lemma. ■

Let us consider the follow functional

$$\psi(t) = 2\int_{\Omega_t} u_t u dx + \alpha\|u\|^2_{L^2(\Omega_t)} + 2\int_{\Omega_t} v_t v dx + \alpha\|v\|^2_{L^2(\Omega_t)}.$$

Lemma 3.5 *Let us take $(u_0, v_0) \in (H_0^2(\Omega_0) \cap H^4(\Omega_0))^2$, $(u_1, v_1) \in (H_0^2(\Omega_0))^2$ and let us suppose that assumptions (1.15), (1.16), (1.17), (1.19), (1.20), (2.1) and (3.5) hold. Then any strong solution of system (1.1)-(1.4) satisfies*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\psi(t) \leq\ & \|u_t\|^2_{L^2(\Omega_t)} + \|v_t\|^2_{L^2(\Omega_t)} - \|\Delta u\|^2_{L^2(\Omega_t)} - \|\Delta v\|^2_{L^2(\Omega_t)} \\ & -\|\nabla u\|^2_{L^2(\Omega_t)} + \|\nabla u\|_{L^2(\Omega_t)} \left(\int_0^t g_1(s)ds\right)^{\frac{1}{2}} (g_1 \Box \nabla u)^{\frac{1}{2}} \\ & -\|\nabla v\|^2_{L^2(\Omega_t)} + \|\nabla v\|_{L^2(\Omega_t)} \left(\int_0^t g_2(s)ds\right)^{\frac{1}{2}} (g_2 \Box \nabla v)^{\frac{1}{2}}.\end{aligned}$$

Proof. Multiplying the equations (1.1) by u and (1.2) by v, integrating by parts over Ω_t and summing up the results we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\psi(t) =\ & \|u_t\|^2_{L^2(\Omega_t)} + \|v_t\|^2_{L^2(\Omega_t)} - \|\Delta u\|^2_{L^2(\Omega_t)} - \|\Delta v\|^2_{L^2(\Omega_t)} \\ & -M(||\nabla u||^2_{L^2(\Omega_)} + ||\nabla v||^2_{L^2(\Omega_t)})\|\nabla u\|^2_{L^2(\Omega_t)} \\ & +\int_{\Omega_t}\int_0^t g_1(t-s)\nabla u(s)\cdot\nabla u(t)dsdx \\ & -M(||\nabla u||^2_{L^2(\Omega_)} + ||\nabla v||^2_{L^2(\Omega_t)})\|\nabla v\|^2_{L^2(\Omega_t)} \\ & +\int_{\Omega_t}\int_0^t g_2(t-s)\nabla v(s)\cdot\nabla v(t)dsdx \\ & -\int_{\Omega_t} h(u-v)(u-v)dx.\end{aligned}$$

Noting that

$$\begin{aligned}\int_{\Omega_t}\int_0^t g_1(t-s)\nabla u(s)\cdot\nabla u(t)dsdx =\ & \int_{\Omega_t}\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))\cdot\nabla u dsdx \\ & +\int_{\Omega_t}\left(\int_0^t g_1(s)ds\right)|\nabla u|^2dx, \\ \int_{\Omega_t}\int_0^t g_2(t-s)\nabla v(s)\cdot\nabla v(t)dsdx =\ & \int_{\Omega_t}\int_0^t g_2(t-s)(\nabla v(s)-\nabla v(t))\cdot\nabla v dsdx \\ & +\int_{\Omega_t}\left(\int_0^t g_2(s)ds\right)|\nabla v|^2dx,\end{aligned}$$

and taking into account that

$$\begin{aligned}&\left|\int_{\Omega_t}\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))\cdot\nabla u dx\right| \leq \|\nabla u\|_{L^2(\Omega_t)}\left(\int_0^t g_1(s)ds\right)^{\frac{1}{2}}(g_1\Box\nabla u)^{\frac{1}{2}}, \\ &\left|\int_{\Omega_t}\int_0^t g_2(t-s)(\nabla v(s)-\nabla v(t))\cdot\nabla v dx\right| \leq \|\nabla v\|_{L^2(\Omega_t)}\left(\int_0^t g_2(s)ds\right)^{\frac{1}{2}}(g_2\Box\nabla v)^{\frac{1}{2}}, \\ &-M(||\nabla u||^2_{L^2(\Omega_)} + ||\nabla v||^2_{L^2(\Omega_t)})(||\nabla u||^2_{L^2(\Omega_)} + ||\nabla v||^2_{L^2(\Omega_t)}) \\ &\leq -\widehat{M}(||\nabla u||^2_{L^2(\Omega_)} + ||\nabla v||^2_{L^2(\Omega_t)}), \\ &-\int_{\Omega_t} h(u-v)(u-v)dx \leq -\int_{\Omega_t} H(u-v)dx\end{aligned}$$

follows the conclusion of lemma. ■

Let us introduce the functional

$$\mathcal{L}(t) = NE(t) + \psi(t), \tag{3.11}$$

with $N > 0$. It is not difficult to see that $\mathcal{L}(t)$ verifies

$$k_0 E(t) \leq \mathcal{L}(t) \leq k_1 E(t), \tag{3.12}$$

for k_0 and k_1 positive constants. Now we are in a position to show the main result of this paper.

Theorem 3.1 *Let us take $(u_0, v_0) \in (H_0^2(\Omega_0) \cap H^4(\Omega_0))^2$, $(u_1, v_1) \in (H_0^2(\Omega_0))^2$ and let us suppose that assumptions (1.15), (1.16), (1.17), (1.19), (1.20), (2.1), (3.1), (3.2), (3.4) and (3.5) hold. Then any strong solution of system (1.1)-(1.4) satisfies*

$$E(t) \leq Ce^{-\zeta t} E(0), \quad \forall t \geq 0$$

where C and ζ are positive constants.

Proof. Using the lemmas (3.4) and (3.5) we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \quad \leq \quad & -2N\alpha\|u_t\|^2_{L^2(\Omega_t)} - C_1 N g_1 \Box \nabla u + \|u_t\|^2_{L^2(\Omega_t)} \\
& -\|\Delta u\|^2_{L^2(\Omega_t)} - \widehat{M}(||\nabla u||^2_{L^2(\Omega_)} + ||\nabla v||^2_{L^2(\Omega_t)}) \\
& + \left(\int_0^t g_1(s)ds\right) \|\nabla u\|^2_{L^2(\Omega_t)} \\
& + \|\nabla u\|^2_{L^2(\Omega_t)} \left(\int_0^t g_1(s)ds\right)^{\frac{1}{2}} (g_1 \Box \nabla u)^{\frac{1}{2}} \\
& -2N\alpha\|v_t\|^2_{L^2(\Omega_t)} - C_1 N g_2 \Box \nabla v + \|v_t\|^2_{L^2(\Omega_t)} \\
& -\|\Delta v\|^2_{L^2(\Omega_t)} + \left(\int_0^t g_2(s)ds\right) \|\nabla v\|^2_{L^2(\Omega_t)} \\
& + \|\nabla v\|^2_{L^2(\Omega_t)} \left(\int_0^t g_2(s)ds\right)^{\frac{1}{2}} (g_2 \Box \nabla v)^{\frac{1}{2}} \\
& - \int_{\Omega_t} H(u - v)dx.
\end{aligned}$$

Using Young inequality we obtain for $\epsilon > 0$

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \quad \leq \quad & -2N\alpha\|u_t\|^2_{L^2(\Omega_t)} - C_1 N g_1 \Box \nabla u + \|u_t\|^2_{L^2(\Omega_t)} \\
& - \|\Delta u\|^2_{L^2(\Omega_t)} - \widehat{M}(||\nabla u||^2_{L^2(\Omega_)} + ||\nabla v||^2_{L^2(\Omega_t)}) \\
& + \left(\int_0^t g_1(s)ds\right) \|\nabla u\|^2_{L^2(\Omega_t)} \\
& + \frac{\epsilon}{2}\|\nabla u\|^2_{L^2(\Omega_t)} + \frac{\|g_1\|_{L^1(0,\infty)}}{2\epsilon} g_1 \Box \nabla u \\
& -2N\alpha\|v_t\|^2_{L^2(\Omega_t)} - C_1 N g_2 \Box \nabla v + \|v_t\|^2_{L^2(\Omega_t)} \\
& - \|\Delta v\|^2_{L^2(\Omega_t)} + \left(\int_0^t g_2(s)ds\right) \|\nabla v\|^2_{L^2(\Omega_t)} \\
& + \frac{\epsilon}{2}\|\nabla v\|^2_{L^2(\Omega_t)} + \frac{\|g_2\|_{L^1(0,\infty)}}{2\epsilon} g_2 \Box \nabla v \\
& - \int_{\Omega_t} H(u - v)dx.
\end{aligned}$$

Choosing N large enough and ϵ small we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -\lambda_0 E(t) \tag{3.13}$$

where λ_0 is a positive constant independent of t. From (3.12) and (3.13) follows that

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\frac{\lambda_0}{k_1}t}, \quad \forall t \geq 0.$$

From equivalence relation (3.12) our conclusion follows. The proof now is completed. ■

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