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Global bifurcation from intervals for Sturm–Liouville problems which are not linearizable *

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Abstract

In this paper, we study unilateral global bifurcation which bifurcates from the trivial solutions axis or from infinity for nonlinear Sturm–Liouville problems of the form

$$\begin{cases} -\left(pu'\right)' + qu = \lambda au + af\left(x, u, u', \lambda\right) + g\left(x, u, u', \lambda\right) \text{ for } x \in (0,1), \\ b_0 u(0) + c_0 u'(0) = 0, \\ b_1 u(1) + c_1 u'(1) = 0, \end{cases}$$

where $a \in C([0,1], [0,+\infty))$ and $a(x) \not\equiv 0$ on any subinterval of $[0,1]$, $f, g \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$. Suppose that f and g satisfy

$$|f(x, \xi, \eta, \lambda)| \le M_0|\xi| + M_1|\eta|, \ \forall x \in [0,1] \text{ and } \lambda \in \mathbb{R},$$

$$g(x, \xi, \eta, \lambda) = o(|\xi| + |\eta|), \text{ uniformly in } x \in [0,1] \text{ and } \lambda \in \Lambda,$$

as either $|\xi| + |\eta| \to 0$ or $|\xi| + |\eta| \to +\infty$, for some constants M_0, M_1, and any bounded interval Λ.

Keywords: interval bifurcation; Sturm–Liouville problem; unilateral global bifurcation.

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1 Introduction

Consider the following nonlinear Sturm–Liouville problem

$$\begin{cases} -\left(pu'\right)' + qu = \lambda au + h\left(x, u, u', \lambda\right) \text{ for } x \in (0,1), \\ b_0 u(0) + c_0 u'(0) = 0, \\ b_1 u(1) + c_1 u'(1) = 0, \end{cases} \tag{1.1}$$

where p is a positive, continuously differentiable function on $[0,1]$, q is a continuous function on $[0,1]$ and b_i, c_i are real numbers such that $|b_i| + |c_i| \ne 0$, $i = 0, 1$, and a satisfies

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the following condition

(A0) $a \in C([0,1],[0,+\infty))$ and $a(x) \not\equiv 0$ on any subinterval of $[0,1]$.

Moreover, the nonlinear term h has the form $h = af + g$, where f and g are continuous functions on $[0,1] \times \mathbb{R}^3$, satisfying some of the following conditions

(A1) For any $x \in [0,1]$ and $\lambda \in \mathbb{R}$, there are constants M_0^0, M_1^0 such that

$$|f(x,\xi,\eta,\lambda)| \le M_0^0|\xi| + M_1^0|\eta| \ \text{ as } \ |\xi| + |\eta| \to 0;$$

(A2) For any $x \in [0,1]$ and $\lambda \in \mathbb{R}$, there are constants M_0^∞, M_1^∞ such that

$$|f(x,\xi,\eta,\lambda)| \le M_0^\infty|\xi| + M_1^\infty|\eta| \ \text{ as } \ |\xi| + |\eta| \to +\infty;$$

(A3) For any bounded interval $\Lambda \subseteq \mathbb{R}$,

$$g(x,\xi,\eta,\lambda) = o(|\xi| + |\eta|) \text{ near } (\xi,\eta) = (0,0), \text{ uniformly for } (x,\lambda) \in [0,1] \times \Lambda;$$

(A4) For any bounded interval $\Lambda \subseteq \mathbb{R}$,

$$g(x,\xi,\eta,\lambda) = o(|\xi| + |\eta|) \text{ near } (\xi,\eta) = (\infty,\infty), \text{ uniformly for } (x,\lambda) \in [0,1] \times \Lambda.$$

Note that problem (1.1) does not have in general a linearization about $u = 0$ or $u = \infty$. Thus the standard bifurcation theory of [12–15, 19] cannot be applied directly. If a is strictly positive on $[0,1]$, h has the form of $h = f + g$ and (A1), (A3) hold with $M_1^0 = 0$, Berestycki [2] established an important global bifurcation theorem from intervals for (1.1). The authors of [17] obtained similar results as [2] if $p(x) \equiv 1 \equiv a(x)$. Although the conditions may weaker in [17], their results only hold for $k \ge k_0$ with some $k_0 \in \mathbb{N}$. Similar problems have been considered in [3, 10, 11]. These results have been extended by Rynne [16] (with the help of some estimates come from [1]) under the assumption that

$$|h(x,\xi,\eta,\lambda)| \le M_0|\xi| + M_1|\eta|, \ (x,\xi,\eta,\lambda) \in [0,1] \times \mathbb{R}^3,$$

as either $|(\xi,\eta)| \to 0$ or $|(\xi,\eta)| \to +\infty$, for some constants M_0 and M_1. *However, the bifurcation intervals appear to be larger and the assumption $a \in C^1[0,1]$ is too strong. Moreover, it is not clear whether these results of [16] with $M_1 = 0$ degenerates to the corresponding ones of [2].* Recently, Ma and Dai [9] improved Berestycki's result to show a unilateral global bifurcation result for (1.1) with similar conditions as in [2]. We refer to [5, 6, 7, 8, 13, 18] and their references for the theory of unilateral global bifurcation.

The aim of this paper is to improve or extend the corresponding results of [9] and [16] under weaker assumptions. In order to introduce our main results, next, we give some notations.

Let $Lu := -(pu')' + qu$. It is well known (see [4] or [20, p. 269]) that the linear Sturm–Liouville problem

$$\begin{cases} Lu = \lambda a u, \ x \in (0,1), \\ b_0 u(0) + c_0 u'(0) = 0, \\ b_1 u(1) + c_1 u'(1) = 0 \end{cases} \tag{1.2}$$

possesses infinitely many eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_k \to +\infty$, all of which are simple. The eigenfunction φ_k corresponding to λ_k has exactly $k-1$ simple zeros in $(0,1)$. Let

$$E := \left\{u \in C^1[0,1] : b_0 u(0) + c_0 u'(0) = 0, b_1 u(1) + c_1 u'(1) = 0\right\}$$

with the norm $\|u\| = \max_{x\in[0,1]}|u(x)| + \max_{x\in[0,1]}|u'(x)|$. Let S_k^+ denote the set of functions in E which have exactly $k-1$ simple zeros in (0,1) and are positive near $x = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. It is clear that S_k^+ and S_k^- are disjoint and open in E. We also let $\Phi_k^\pm = \mathbb{R}\times S_k^\pm$ and $\Phi_k = \mathbb{R}\times S_k$ under the product topology. Finally, we use $\mathscr{S}$ to denote the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions of (1.1), and $\mathscr{S}_k^\pm$ to denote the subset of $\mathscr{S}$ with $u \in S_k^\pm$ and $\mathscr{S}_k = \mathscr{S}_k^+ \cup \mathscr{S}_k^-$.

The first main result of this paper is the following theorem.

Theorem 1.1. *Let $I_k = [\lambda_k - M_0^0 - c_k^1 M_1^0, \lambda_k + M_0^0 + c_k^2 M_1^0]$ for every $k \in \mathbb{N}^*$ and some constants c_k^1 and c_k^2 which only depend on k. And assume that (A0), (A1) and (A3) hold. Then the component $\mathscr{D}_k^+$ of $\mathscr{S}_k^+ \cup (I_k\times\{0\})$, containing $I_k\times\{0\}$ is unbounded and lies in $\Phi_k^+ \cup (I_k\times\{0\})$ and the component $\mathscr{D}_k^-$ of $\mathscr{S}_k^- \cup (I_k\times\{0\})$, containing $I_k\times\{0\}$ is unbounded and lies in $\Phi_k^- \cup (I_k\times\{0\})$.*

Use $\mathscr{T}$ to denote the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions of (1.1) under conditions (A0), (A2) and (A4). Our second main result is the following theorem.

Theorem 1.2. *Let $I_k = [\lambda_k - M_0^\infty - d_k^1 M_1^\infty, \lambda_k + M_0^\infty + d_k^2 M_1^\infty]$ for every $k \in \mathbb{N}^*$ and some constants d_k^1 and d_k^2 which only depend on k. For every $\nu \in \{+,-\}$, there exists a component $\mathscr{D}_k^\nu$ of $\mathscr{T} \cup (I_k\times\{\infty\})$, containing $I_k\times\{\infty\}$. Moreover, if $\Lambda\subset\mathbb{R}$ is an interval such that $\Lambda\cap(\cup_{k=1}^\infty I_k) = I_k$ and $\mathscr{M}$ is a neighborhood of $I_k\times\{\infty\}$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either*

1^o. $\mathscr{D}_k^\nu - \mathscr{M}$ is bounded in $\mathbb{R}\times E$ in which case $\mathscr{D}_k^\nu - \mathscr{M}$ meets $\mathscr{R} = \{(\lambda,0)\,|\,\lambda\in\mathbb{R}\}$ or

2^o. $\mathscr{D}_k^\nu - \mathscr{M}$ is unbounded.

If 2^o occurs and $\mathscr{D}_k^\nu - \mathscr{M}$ has a bounded projection on $\mathbb{R}$, then $\mathscr{D}_k^\nu - \mathscr{M}$ meets $I_j\times\{\infty\}$ for some $j\neq k$. In addition, there exists a neighborhood $\mathscr{N}\subset\mathscr{M}$ of $I_k\times\{\infty\}$ such that $(\mathscr{D}_k^\nu\cap\mathscr{N}) \subseteq (\Phi_k^\nu \cup (I_k\times\{\infty\}))$.

The rest of this paper is arranged as follows. In Section 2, we give the proof of Theorem 1.1. In Section 3, we present the proof of Theorem 1.2 and give some remarks.

2 Proof of Theorem 1.1

Firstly, by an argument similar to that of [9, Lemma 2.2], we can show the following lemma.

Lemma 2.1. *If (λ,u) is a solution of (1.1) under assumptions (A0), (A1), (A3) and u has a double zero, then $u\equiv 0$.*

Thus if (λ,u) is a nontrivial solution of (1.1) under assumptions (A0), (A1) and (A3), then $u\in\cup_{k=1}^\infty S_k$. We still use the approximation technique introduced in [2] to prove

Theorem 1.1. Hence consider the following approximate problem

$$\begin{cases} -\left(pu'\right)'+qu=\lambda au+af\left(x,u|u|^{\epsilon},u'|u|^{\epsilon},\lambda\right)+g\left(x,u,u',\lambda\right) \text{ for } x\in(0,1),\\ b_0u(0)+c_0u'(0)=0,\\ b_1u(1)+c_1u'(1)=0. \end{cases} \tag{2.1}$$

The next lemma will play a key role in this paper which provides uniform a priori bounds for the solutions of problem (2.1) near the trivial solutions and will also ensure that $(\mathscr{S}_k^\nu\cap(\mathbb{R}\times\{0\}))\subset(I_k\times\{0\})$.

Lemma 2.2. *Let ϵ_n, $0\le\epsilon_n\le1$, be a sequence converging to 0. If there exists a sequence $(\lambda_n,u_n)\in\mathbb{R}\times S_k^\nu$ such that (λ_n,u_n) is a nontrivial solution of problem (2.1) corresponding to $\epsilon=\epsilon_n$, and (λ_n,u_n) converges to $(\lambda,0)$ in $\mathbb{R}\times E$, then $\lambda\in I_k$.*

Proof. Without loss of generality, we may assume that $\|u_n\|\le1$. Let $w_n=u_n/\|u_n\|$, then w_n satisfies the problem

$$\begin{cases} -\left(pw_n'\right)'+qw_n=\lambda aw_n+af_n(x)+g_n(x),\ x\in(0,1),\\ b_0w_n(0)+c_0w_n'(0)=0,\\ b_1w_n(1)+c_1w_n'(1)=0, \end{cases} \tag{2.2}$$

where

$$f_n(x)=\frac{f\left(x,u_n(x)|u_n(x)|^{\varepsilon_n},u_n'(x)|u_n(x)|^{\varepsilon_n},\lambda_n\right)}{\|u_n\|},\quad g_n(x)=\frac{g\left(x,u_n(x),u_n'(x),\lambda_n\right)}{\|u_n\|}.$$

It follows from (A3) that $g_n(x)\to0$ uniformly in $x\in[0,1]$. Furthermore, (A1) implies that

$$\begin{aligned} |f_n(x)| &\le \frac{|u_n(x)|^{\varepsilon_n}\left(M_0^0\left|u_n(x)\right|+M_1^0\left|u_n'(x)\right|\right)}{\|u_n\|}\\ &\le \|u_n\|^{\epsilon_n}\left(M_0^0\left|w_n(x)\right|+M_1^0\left|w_n'(x)\right|\right)\\ &\le M_0^0\left|w_n(x)\right|+M_1^0\left|w_n'(x)\right|\\ &\le M_0^0+M_1^0 \end{aligned}$$

for all $x\in[0,1]$. In view of (2.2), we know that w_n is bounded in C^2. By the Arzelà–Ascoli theorem, we may assume that $w_n\to v$ in C^1 with $\|w\|=1$. Clearly, we have $w\in\overline{S_k^\nu}$.

We claim that $w\in S_k^\nu$. On the contrary, suppose that $w\in\partial S_k^\nu$, then w has at least one double zero $x_*\in[0,1]$. It follows that $w_n\left(x_*\right)\to0$ and $w_n'\left(x_*\right)\to0$ as $n\to+\infty$. Then by the argument of [2, p. 379], we can deduce $w_n\to0$ in C^1, which is a contradiction with $\|w_n\|=1$.

Now, we deduce the boundedness of λ. Let $\varphi_k^\nu\in S_k^\nu$ be an eigenfunction of problem (1.2) corresponding to λ_k and $[\alpha,\beta]\subseteq[0,1]$. Integrating by parts and taking the limit as $n\to+\infty$, we can obtain that

$$\left[p\left(w\left(\varphi_k^\nu\right)'-\varphi_k^\nu w'\right)\right]_\alpha^\beta=\int_\alpha^\beta\left(\lambda-\lambda_k\right)aw\varphi_k^\nu\,dx+\lim_{n\to+\infty}\int_\alpha^\beta af_n(x)\varphi_k^\nu\,dx.$$

It was shown in [2] that there are two intervals (ξ_1,η_1) and (ξ_2,η_2) in $(0,1)$ where w_n and ψ_k^ν do not vanish and have the same sign and such that

$$\left[p\left(w\left(\varphi_k^\nu\right)'-\varphi_k^\nu w'\right)\right]_{\xi_1}^{\eta_1}\ge0,\quad\left[p\left(w\left(\varphi_k^\nu\right)'-\varphi_k^\nu w'\right)\right]_{\xi_2}^{\eta_2}\le0.$$

So we have that

$$\int_{\xi_1}^{\eta_1} (\lambda - \lambda_k)\, aw\varphi_k^\nu \, dx + \lim_{n\to+\infty} \int_{\xi_1}^{\eta_1} af_n(x)\varphi_k^\nu \, dx \geq 0$$

and

$$\int_{\xi_2}^{\eta_2} (\lambda - \lambda_k)\, aw\varphi_k^\nu \, dx + \lim_{n\to+\infty} \int_{\xi_2}^{\eta_2} af_n(x)\varphi_k^\nu \, dx \leq 0.$$

Furthermore, one has that

$$\int_{\xi_1}^{\eta_1} \left(\lambda - \lambda_k + M_0^0\right) aw\varphi_k^\nu \, dx + \int_{\xi_1}^{\eta_1} aM_1^0 \left|w'\varphi_k^\nu\right| \, dx \geq 0 \tag{2.3}$$

and

$$\int_{\xi_2}^{\eta_2} \left(\lambda - \lambda_k - M_0^0\right) aw\varphi_k^\nu \, dx - \int_{\xi_2}^{\eta_2} aM_1^0 \left|w'\varphi_k^\nu\right| \, dx \leq 0. \tag{2.4}$$

We choose $\widetilde{c}_k \geq 1$ and $\overline{c}_k \geq 1$ such that

$$\int_{\xi_1}^{\eta_1} a \left|\varphi_k^\nu\right| \, dx \leq \widetilde{c}_k \int_{\xi_1}^{\eta_1} aw\varphi_k^\nu \, dx, \quad \int_{\xi_2}^{\eta_2} a \left|\varphi_k^\nu\right| \, dx \leq \overline{c}_k \int_{\xi_2}^{\eta_2} aw\varphi_k^\nu \, dx.$$

It follows that

$$\int_{\xi_1}^{\eta_1} a \left|w'\varphi_k^\nu\right| \, dx \leq \int_{\xi_1}^{\eta_1} a(1 - |w|) \left|\varphi_k^\nu\right| \, dx \leq c_k^1 \int_{\xi_1}^{\eta_1} aw\varphi_k^\nu \, dx \tag{2.5}$$

and

$$\int_{\xi_2}^{\eta_2} a \left|w'\varphi_k^\nu\right| \, dx \leq \int_{\xi_2}^{\eta_2} a(1 - |w|) \left|\varphi_k^\nu\right| \, dx \leq c_k^2 \int_{\xi_2}^{\eta_2} aw\varphi_k^\nu \, dx \tag{2.6}$$

where $c_k^1 = \widetilde{c}_k - 1$ and $c_k^2 = \overline{c}_k - 1$. From (2.3)–(2.6), we can see that

$$\lambda \geq \lambda_k - M_0^0 - c_k^1 M_1^0 \text{ and } \lambda \leq \lambda_k + M_0^0 + c_k^2 M_1^0.$$

Therefore, we have that $\lambda \in I_k$. ■

Proof of Theorem 1.1. By Lemma 2.1, 2.2 and an argument similar to that of [10, Theorem 2.1], we can obtain the desired conclusion. ■

3 Proof of Theorem 1.2

We add the points $\{(\lambda, \infty) \big| \lambda \in \mathbb{R}\}$ to the space $\mathbb{R} \times E$.

Proof of Theorem 1.2. If $(\lambda, u) \in \mathscr{T}$ with $\|u\| \neq 0$, dividing (1.1) by $\|u\|^2$ and setting $w = u/\|u\|^2$ yield

$$\begin{cases} -\left(pw'\right)' + qw = \lambda aw + \frac{h(t,u,u',\lambda)}{\|u\|^2} \text{ for } x \in (0,1), \\ b_0 w(0) + c_0 w'(0) = 0, \\ b_1 w(1) + c_1 w'(1) = 0. \end{cases} \tag{3.1}$$

Define

$$\widetilde{h}\left(x, w, w^{\prime}, \lambda\right)=\begin{cases}\|w\|^{2} h\left(x, w /\|w\|^{2}, w^{\prime} /\|w\|^{2}, \lambda\right), & \text{if } w \neq 0, \\ 0, & \text{if } w=0.\end{cases}$$

Then (3.1) can be rewritten as

$$\begin{cases}-\left(p w^{\prime}\right)^{\prime}+q w=\lambda a w+a \widetilde{f}\left(x, w, w^{\prime}, \lambda\right)+\widetilde{g}\left(x, w, w^{\prime}, \lambda\right) & \text{for } x \in(0,1), \\ b_{0} w(0)+c_{0} w^{\prime}(0)=0, \\ b_{1} w(1)+c_{1} w^{\prime}(1)=0.\end{cases} \tag{3.2}$$

It is obvious that $(\lambda, 0)$ is always the solution of (3.2). By an easy calculation, we can show that assumptions (A2) and (A4) imply that $\widetilde{f}$ and $\widetilde{g}$ satisfy (A1) and (A3). Now applying Theorem 1.1 to problem (3.2), we have the component $\mathscr{C}_{k,0}$ of $\mathscr{S}_k \cup (I_k \times \{0\})$, containing $I_k \times \{0\}$ is unbounded and lies in $\Phi_k \cup (I_k \times \{0\})$. Under the inversion $w \to w/\|w\|^2 = u$, $\mathscr{C}_{k,0} \to \mathscr{D}_k$ satisfying (1.1). By an argument similar to that of [9, Theorem 2.3], we can prove the existence of $\mathscr{N}$ such that $(\mathscr{D}_k^{\nu} \cap \mathscr{N}) \subset (\Phi_k^{\nu} \cup (I_k \times \{\infty\}))$ for $\nu = +$ and $-$. ■

Remark 3.1. Note that if $M_1^0 = 0$, Theorem 1.1 degenerates to Theorem 2.1 of [9], and if $M_1^{\infty} = 0$, Theorem 1.2 degenerates to Theorem 2.2 and 2.3 of [9]. In fact, even in these special cases, the bifurcation intervals in this paper are smaller than the corresponding ones of [9].

Remark 3.2. Note that our assumption on a is weaker than any mentioned paper (in introduction) dealing with this kind of problems.

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