View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Repository of the Academy's Library

Electronic Journal of Qualitative Theory of Differential Equations
2007, No. **1**, 1 - 19; **http://www.math.u-szeged.hu/ejqtde/**

An Extended Method of Quasilinearization for Nonlinear Impulsive Differential Equations with a Nonlinear Three-Point Boundary Condition

Bashir Ahmad[1] and Ahmed Alsaedi
Department of Mathematics,
Faculty of science, King Abdulaziz University,
P.O. Box. 80257, Jeddah 21589, Saudi Arabia

Abstract

In this paper, we discuss an extended form of generalized quasilinearization technique for first order nonlinear impulsive differential equations with a nonlinear three-point boundary condition. In fact, we obtain monotone sequences of upper and lower solutions converging uniformly and quadratically to the unique solution of the problem.

Key words: Impulsive differential equations, three-point boundary conditions, quasilinearization, quadratic convergence.
AMS(MOS) Subject Classification: 34A37, 34B10.

1 Introduction

The method of quasilinearizaion provides an adequate approach for obtaining approximate solutions of nonlinear problems. The origin of the quasilinearizaion lies in the theory of dynamic programming [1-3]. This method applies to semilinear equations with convex (concave) nonlinearities and generates a monotone scheme whose iterates converge quadratically to the solution of the problem at hand. The assumption of convexity proved to be a stumbling block for the further development of the method. The nineties brought new dimensions to this technique. The most interesting new idea was introduced by Lakshmikantham [4-5] who generalized the method of quasilinearizaion by relaxing the convexity assumption. This development proved to be quite significant and the method was studied and applied to a wide range of initial and boundary value problems for different types of differential equations, see [6-17] and references therein. Some real-world applications of the quasilinearizaion technique can be found in [18-20].
Many evolution processes are subject to short term perturbations which act instantaneously in the form of impulses. Examples include biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models

[1]Corresponding author
bashir_qau@yahoo.com (B. Ahmad), aalsaedi@hotmail.com (A. Alsaedi)

in economics and frequency modulated systems. Thus, impulsive differential equations provide a natural description of observed evolution processes of several real world problems. Moreover, the theory of impulsive differential equations is much richer than the corresponding theory of ordinary differential equations without impulse effects since a simple impulsive differential equation may exhibit several new phenomena such as rhythmical beating, merging of solutions and noncontinuability of solutions. Thus, the theory of impulsive differential equations is quite interesting and has attracted the attention of many scientists, for example, see [21-24]. In particular, Eloe and Hristova [23] discussed the method of quasilinearization for first order nonlinear impulsive differential equations with linear boundary conditions.
Multi-point nonlinear boundary value problems, which refer to a different family of boundary conditions in the study of disconjugacy theory [25], have been addressed by many authors, for instance, see [26-27] and the references therein. In this paper, we develop an extended method of quasilinearization for a class of first order nonlinear impulsive differential equations involving a mixed type of nonlinearity with a nonlinear three-point boundary condition

$$x'(t) = F(t, x(t)) \text{ for } t \in [0, T],\ t \neq \tau_k,\ \tau_k \in (0, T), \tag{1}$$

$$x(\tau_k + 0) = I_k(x(\tau_k)), \quad k = 1, 2, ..., p, \tag{2}$$

$$\gamma_1 x(0) - \gamma_2 x(T) = h(x(\frac{T}{2})), \tag{3}$$

where $F \in C[[0, T] \times \mathbf{R}, \mathbf{R}]$ and $F(t, x(t)) = f(t, x(t)) + g(t, x(t)),\ \gamma_1, \gamma_2$ are constants with $\gamma_1 \geq \gamma_2 > 0,\ \tau_k < \tau_{k+1},\ k = 1, 2, ..., p$ and the nonlinearity $h : \mathbf{R} \longrightarrow \mathbf{R}$ is continuous. Here, it is worthmentioning that the convexity assumption on $f(t, x)$ has been relaxed and instead $f(t, x) + M_1 x^2$ is taken to be convex for some $M_1 > 0$ while a less restrictive condition is demanded on $g(t, x)$, namely, $[g(t, x) + M_2 x^{1+\epsilon}]$ satisfies a nondecreasing condition for some $\epsilon > 0$ and $M_2 > 0$. Moreover, we also relax the concavity assumption ($h''(x) \leq 0$) on the nonlinearity $h(x)$ in the boundary condition (3) by requiring $h''(x) + \psi''(x) \leq 0$ for some continuous function $\psi(x)$ satisfying $\psi'' \leq 0$ on $\mathbf{R}$. We construct two monotone sequences of upper and lower solutions converging uniformly and quadratically to the unique solution of the problem. Some special cases of our main result have also been recorded.

2 Some Basic Results

For $A \subset \mathbf{R},\ B \subset \mathbf{R}$, let $PC(A, B)$ denotes the set of all functions $v : A \to B$ which are piecewise continuous in A with points of discontinuity of first kind at the points $\tau_k \in A$, that is, there exist the limits $\lim_{t \downarrow \tau_k} v(t) = v(\tau_k + 0) < \infty$ and $\lim_{t \uparrow \tau_k} v(t) = v(\tau_k - 0) = v(\tau_k)$. The set $PC^1(A, B)$ consists of all functions $v \in PC(A, B)$ that are continuously differentiable for $t \in A, t \neq \tau_k$.

Definition 1. The function $\alpha(t) \in PC^1([0,T],\mathbf{R})$ is called a lower solution of the BVP (1)-(3) if

$$\alpha'(t) \le F(t,\alpha(t)) \;\text{ for }\; t \in [0,T],\; t \ne \tau_k, \tag{4}$$

$$\alpha(\tau_k + 0) \le I_k(\alpha(\tau_k)), \quad k = 1,2,...,p, \tag{5}$$

$$\gamma_1\alpha(0) - \gamma_2\alpha(T) \le h(\alpha(\frac{T}{2})). \tag{6}$$

The function $\beta(t) \in PC^1([0,T],\mathbf{R})$ is called an upper solution of the BVP (1)-(3) if the inequalities are reversed in (4)-(6).

Let us set the following notations for the sequel.

$$\begin{aligned} S(\alpha,\beta) &= \{x \in PC([0,T],\mathbf{R}):\; \alpha(t) \le x(t) \le \beta(t) \;\text{ for }\; t \in [0,T]\}, \\ \Omega(\alpha,\beta) &= \{(t,x) \in [0,T] \times \mathbf{R} : \alpha(t) \le x(t) \le \beta(t)\}, \\ D_k(\alpha,\beta) &= \{x \in \mathbf{R} : \alpha(\tau_k) \le x \le \beta(\tau_k)\},\; k = 1,2,...,p. \end{aligned}$$

Theorem 1. (Comparison Result)
Let $\alpha,\beta \in PC^1([0,T],\mathbf{R})$ be lower and upper solutions of (1)-(3) respectively. Further, $F(t,x) \in C(\Omega(\alpha,\beta),\mathbf{R})$ is quasimonotone nondecreasing in x for each $t \in [0,T]$ and satisfies $F(t,x) - F(t,y) \le L(x-y),\; L \ge 0$ whenever $y \le x$. Moreover, h is nondecreasing on $\mathbf{R}$ and $I_k : D_k(\alpha,\beta) \to \mathbf{R}$ are nondecreasing in $D_k(\alpha,\beta)$ for each $k = 1,2,...,p$ and satisfies $I_k(x) - I_k(y) \le M(x-y),\; M \ge 0$. Then $\alpha(t) \le \beta(t)$ on $[0,T]$.

Proof. The method of proof is similar to the one used in proving Theorem 2.6.1 (page 87 [21]), so we omit the proof.

Theorem 2. (Existence of solution)
Assume that F is continuous on $\Omega(\alpha,\beta)$ and h is nondecreasing on $\mathbf{R}$. Further, we assume that $I_k : D_k(\alpha,\beta) \to \mathbf{R}$ are nondecreasing in $D_k(\alpha,\beta)$ for each $k = 1,2,...,p$ and α,β are respectively lower and upper solutions of (1)-(3) such that $\alpha(t) \le \beta(t)$ on $[0,T]$. Then there exists a solution $x(t)$ of (1)-(3) such that $x(t) \in S(\alpha,\beta)$.

Proof. There is no loss of generality if we consider the case $p = 1$, that is, $0 < t_1 < T$. Let x_0 be an arbitrary point such that $\alpha(0) \le x_0 \le \beta(0)$. Define $\overline{F}$ and H by

$$\overline{F}(t,x) = \begin{cases} F(t,\beta(t)) + \frac{\beta(t)-x}{1+|x|}, & \text{if } x(t) > \beta(t), \\ F(t,x), & \text{if } \alpha(t) \le x(t) \le \beta(t), \\ F(t,\alpha(t)) + \frac{\alpha(t)-x}{1+|x|}, & \text{if } x(t) < \alpha(t), \end{cases}$$

$$H(x) = \begin{cases} h(\beta(\frac{T}{2})), & \text{if } x > \beta(\frac{T}{2}), \\ h(x), & \text{if } \alpha(\frac{T}{2}) \le x \le \beta(\frac{T}{2}), \\ h(\alpha(\frac{T}{2})), & \text{if } x < \alpha(\frac{T}{2}). \end{cases}$$

Since $\overline{F}(t,x)$ and $H(x)$ are continuous and bounded, therefore, there exists a function $\mu \in ([0,T],[0,\infty))$ such that $\sup\{|F(t,x)| : x \in R\} \le \mu(t)$ for $t \in [0,T]$. Thus, the initial value problem $x'(t) = \overline{F}(t,x),\;\; x(0) = x_0$ has a solution $X(t;x_0)$ for $t \in [0,t_1]$. We define $u(t) = X(t;x_0) - \beta(t)$ and prove that the function $u(t)$ is non-positive on $[0,t_1]$. For the sake of contradiction, assume that $u(t) > 0$, that is, $\sup\{u(t) : t \in [0,t_1]\} > 0$. Therefore, there exists a point $t_0 \in (0,t_1)$ such that $u(t_0) > 0$ and $u'(t_0) \ge 0$. On the other hand, we have

$$\begin{aligned} u'(t_0) &= X'(t_0;x_0) - \beta'(t_0) \\ &\le \overline{F}(t_0,x) - F(t_0,\beta(t_0)) \\ &= F(t_0,\beta(t_0)) + \frac{\beta(t_0) - X(t_0;x_0)}{1+|X(t_0;x_0)|} - F(t_0,\beta(t_0)) \\ &= \frac{-u(t_0)}{1+|X(t_0;x_0)|} < 0, \end{aligned}$$

which is a contradiction. Hence we conclude that $X(t;x_0) \le \beta(t),\;\; t \in [0,t_1)$. Similarly, it can be shown that $X(t;x_0) \ge \alpha(t),\;\; t \in [0,t_1]$.
Now we set $y_0 = I_1(X(t_1;x_0))$ and note that y_0 depends on x_0. From the nondecreasing property of $I_1(x)$, we obtain

$$\alpha(t_1+0) \le I_1(\alpha(t_1)) \le I_1(X(t_1;x_0) \le I_1(\beta(t_1)) \le \beta(t_1+0),$$

that is, $\alpha(t_1+0) \le y_0 \le \beta(t_1+0)$.
Consider the initial value problem $x' = \overline{F}(t,x),\;\; x(t_1) = y_0$ for $t \in [t_1,T]$ which has a solution $Y(t;y_0)$ for $t \in [t_1,T]$. Employing the earlier arguments, It is not hard to prove that $\alpha(t) \le Y(t;y_0) \le \beta(t)$ for $t \in [t_1,T]$. Also, we notice that $Y(t_1,y_0) = y_0 = I_1(X(t_1,x_0))$.
Let us define

$$x(t;x_0) = \begin{cases} X(t;x_0) & \text{for } t \in [0,t_1], \\ Y(t;y_0) & \text{for } t \in (t_1,T]. \end{cases}$$

Obviously the function $x(t;x_0)$ such that $\alpha(t) \le x(t;x_0) \le \beta(t)$ is a solution of the impulsive differential equation (1)-(2) with the initial condition $x(0) = x_0$.
In view of the inequality $\alpha(t) \le \beta(t)$ for $t \in [0,T]$, there are following two possible cases:
Case 1. Let $\alpha(0) = \beta(0)$. Then $x_0 = \alpha(0) = \beta(0)$ and

$$\gamma_1 x(0;x_0) - \gamma_2 x(T;x_0) = \gamma_1 x_0 - \gamma_2 x(T;x_0) \le \gamma_1\alpha(0) - \gamma_2\alpha(T) \le h(\alpha(\frac{T}{2})),$$

$$\gamma_1 x(0;x_0) - \gamma_2 x(T;x_0) \ge \gamma_1\beta(0) - \gamma_2\beta(T) \ge h(\beta(\frac{T}{2})).$$

Thus,

$$h(\beta(\frac{T}{2})) \le \gamma_1 x(0;x_0) - \gamma_2 x(T;x_0) \le h(\alpha(\frac{T}{2})). \tag{10}$$

Using the nondecreasing property of $h(t)$ together with the fact that $\alpha(t) \leq x(t; x_0) \leq \beta(t),\ t \in [0, T]$, we find that $h(\alpha(t)) \leq h(x(t; x_0)) \leq h(\beta(t)),\ t \in [0, T]$. In particular, for $t = \frac{T}{2}$, we have

$$h(\alpha(\frac{T}{2})) \leq h(x(\frac{T}{2}; x_0)) \leq h(\beta(\frac{T}{2})). \tag{11}$$

Combining (10) and (11), we obtain

$$\gamma_1 x(0; x_0) - \gamma_2 x(T; x_0) = h(x(\frac{T}{2}; x_0)).$$

This shows that the function $x(t; x_0)$ is a solution of the BVP (1)-(3).

Case 2. Let $\alpha(0) < \beta(0)$. We will prove that there exists a point $x_0 \in [\alpha(0), \beta(0)]$ such that the solution $x(t; x_0)$ of the impulsive differential equation (1)-(2) with the initial condition $x(0) = x_0$ satisfies the boundary condition (3). For the sake of contradiction, let us assume that $\gamma_1 x(0; x_0) - \gamma_2 x(T; x_0) \neq h(\beta(\frac{T}{2}))$, where $x(t; x_0)$ is the solution of (1)-(2).

Letting $x_0 = \beta(0)$ together with the relation $\alpha(t) \leq x(t; x_0) \leq \beta(t)$, we obtain

$$\begin{aligned} \gamma_1 x(0; x_0) - \gamma_2 x(T; x_0) &= \gamma_1 \beta(0) - \gamma_2 x(T; x_0) \\ &\geq \gamma_1 \beta(0) - \gamma_2 \beta(T) \geq h(\beta(\frac{T}{2})) \geq h(x(\frac{T}{2}; x_0)), \end{aligned}$$

which, according to the above assumption, reduces to

$$\gamma_1 x(0; x_0) - \gamma_2 x(T; x_0) > h(x(\frac{T}{2}; x_0)).$$

Then there exists a number δ satisfying $0 < \delta < \beta(0) - \alpha(0)$, such that for $x_0 : 0 \leq \beta(0) - x_0 < \delta$, the corresponding solution $x(t; x_0)$ of (1)-(2) satisfies the inequality

$$\gamma_1 x(0; x_0) - \gamma_2 x(T; x_0) > h(x(\frac{T}{2}; x_0)). \tag{12}$$

We can indeed assume that for every natural number n there exists a point ν_n satisfying $0 \leq \beta(0) - \nu_n < \frac{1}{n}$ such that the corresponding solution $x^{(n)}(t; \nu_n)$ of (1)-(2) with the initial condition $x(0) = \nu_n$ satisfies the inequality

$$\gamma_1 x^{(n)}(0; \nu_n) - \gamma_2 x^{(n)}(T; \nu_n) < h(x(\frac{T}{2}; \nu_n)).$$

Let $\{\nu_{n_j}\}$ is a subsequence such that $\lim_{j\to\infty} \nu_{n_j} = \beta(0)$ and $\lim_{j\to\infty} x^{(n_j)}(t; \nu_{n_j}) = x(t)$ uniformly on the intervals $[0, t_1]$ and $(t_1, T]$. The function $x(t)$ is a solution of (1)-(2) such that $x(0) = \beta(0),\ \alpha(t) \leq x(t) \leq \beta(t)$ and

$$\gamma_1 x(0) - \gamma_2 x(T) \leq h(x(\frac{T}{2})), \tag{13}$$

which contradicts the inequality (12) and consequently our assumption is false. Now, we define

$$\begin{aligned}\delta^* &= \sup\{\delta \in (0, \beta(0)-\alpha(0)] : \text{ for which there exists a point } x_0 \in (\beta(0)-\delta, \beta(0)] \\ &\quad \text{such that the solution } x(t;x_0) \text{ satisfies the inequality (12) }\}.\end{aligned}$$

Select a sequence of points $x_n \in (\alpha(0), \beta(0)-\delta^*)$ such that $\lim_{n\to\infty} x_n = \beta(0)-\delta^*$. From the choice of δ^* and the assumption, it follows that the corresponding solutions $x^{(n)}(t;x_n)$ satisfy the inequality

$$\gamma_1 x^{(n)}(0;x_n) - \gamma_2 x^{(n)}(T;x_n) < h(x(\frac{T}{2};x_n)).$$

Thus, there exists a subsequence $\{x_{n_j}\}_0^\infty$ of the sequence $\{x_n\}_0^\infty$ such that $\lim_{j\to\infty} x^{(n_j)}(t;x_{n_j}) = x^*(t)$ uniformly on the intervals $[0,t_1]$ and $(t_1,T]$. The function $x^*(t)$ satisfying $\alpha(t) \le x^*(t) \le \beta(t)$, is a solution of(1)-(2) with the initial condition $x(0) = \beta(0)-\delta^*$ and satisfies the inequality $\gamma_1 x^*(0) - \gamma_2 x^*(T) \le h(x(\frac{T}{2}))$. This contradicts the choice of δ^*. Therefore, there exists a point $x_0 \in [\alpha(0), \beta(0)]$ such that the solution $x(t,x_0)$ of (1)-(2) satisfies (3). Thus, the function $x(t,x_0)$ is a solution of (1)-(3). This completes the proof of the theorem.

Theorem 3. Let $g, \eta \in PC([0,T], \mathbf{R})$ and $\gamma_1,\ \gamma_2, \zeta,\ b_k,\ \delta_k (k = 1,2,...,p)$ be constants such that $[\gamma_1/\gamma_2 - (\Pi_{k=1}^p b_k)\exp(\int_0^T g(m)dm)] \ne 0$. Then the following linear BVP

$$x'(t) = g(t)x(t) + \eta(t), \quad \text{for} \quad t \in [0,T],\ t \ne \tau_k,$$

$$x(\tau_k + 0) = b_k x(\tau_k) + \delta_k,, \quad k = 1,2,...,p,$$

$$\gamma_1 x(0) - \gamma_2 x(T) = \zeta,$$

has a unique solution $u(t)$ on the interval $[0,T]$ given by

$$\begin{aligned} u(t) &= u(0)(\prod_{0<\tau_k<t} b_k)\exp(\int_0^t g(m)dm) + \sum_{0<\tau_k<t} \delta_k(\prod_{\tau_k<\tau_j<t} b_j)\exp(\int_{\tau_k}^t g(m)dm) \\ &+ \int_0^t \eta(s)(\prod_{s<\tau_k<t} b_k)\exp(\int_s^t g(m)dm)ds, \end{aligned}$$

where

$$\tau_0 = 0,\ b_0 = 1,\ \prod_{j=k}^n f(j) = 1, k > n,$$

$$\begin{aligned} u(0) &= [\gamma_1/\gamma_2 - (\prod_{k=1}^p b_k)\exp(\int_0^T g(m)dm)]^{-1}\{\sum_{i=1}^p \delta_i(\prod_{j=i+1}^p b_j)\exp(\int_{\tau_i}^T g(m)dm) \\ &+ \int_0^T \eta(s)(\prod_{s<\tau_j<T} b_j)\exp(\int_s^T g(m)dm)ds + \zeta/\gamma_2\}. \end{aligned}$$

We need the following known theorem (Theorem 1.4.1, page 32 [21]) to prove our main result.

Theorem 4. Let the function $m \in PC^1[\mathbf{R}_+, \mathbf{R}]$ be such that

$$m'(t) \le \sigma(t)m(t) + q(t), \quad \text{for} \ \ t \in [0,T], \ t \neq \tau_k,$$

$$m(\tau_k + 0) \le d_k m(\tau_k) + b_k, \quad k = 1, 2, ..., p,$$

where $\sigma, \ q \in C[\mathbf{R}_+, \mathbf{R}], \ d_k \ge 0$ and b_k are constants. Then

$$\begin{aligned} m(t) &= m(0)(\prod_{0<\tau_k<t} d_k) \exp(\int_0^t \sigma(\xi)d\xi) + \sum_{0<\tau_k<t} (\prod_{\tau_k<\tau_j<t} d_j) \exp(\int_{\tau_k}^t \sigma(\xi)d\xi) b_k \\ &+ \int_0^t (\prod_{s<\tau_k<t} d_k) \exp(\int_s^t \sigma(\xi)d\xi) q(s)ds, \quad t \ge 0. \end{aligned}$$

3 Extended Method of Quasilinearization

Theorem 5. Assume that

($\mathbf{A_1}$) The functions $\alpha_0(t), \ \beta_0(t)$ are lower and upper solutions of the BVP (1)-(3) respectively such that $\alpha_0(t) \le \beta_0(t)$ for $t \in [0,T]$.

($\mathbf{A_2}$) $f_x, \ f_{xx}$ exist, are continuous and $(f(t,x)+M_1x^2)_{xx} \ge 0$ for $(t,x) \in \Omega, M_1 > 0$. For some $\epsilon > 0, \ M_2 > 0, \ [g(t,x)+M_2x^{1+\epsilon}]$ satisfies a nondecreasing condition. Further, g_x satisfies Lipschitz condition and

$$\{[g_x(t,x) + (1+\epsilon)M_2x^\epsilon] - [g_x(t,y) + (1+\epsilon)M_2y^\epsilon]\}(x-y) \ge 0, \quad \epsilon > 0.$$

Moreover, $\int_0^T [F_x(s, \beta_0(s)) - 2M_1\alpha_0(s)]ds < 0$.

($\mathbf{A_3}$) For $k = 1, 2, ..., p$, the functions $I_k \in C^2(D_k(\alpha_0, \beta_0), \mathbf{R})$ and there exists functions $G_k, J_k \in C^2(D_k(\alpha_0, \beta_0), \mathbf{R})$ such that $G_k(x) = I_k(x) + J_k(x), \quad G_k''(x) \ge 0, \quad J_k''(x) \ge 0$,

$$\begin{aligned} G_k'(\beta_0(\tau_k)) - J_k'(\alpha_0(\tau_k)) &< 1, \\ G_k'(\alpha_0(\tau_k)) - J_k'(\beta_0(\tau_k)) &\ge 0. \end{aligned}$$

($\mathbf{A_4}$) $h(x), h'(x), h''(x)$ exist, are continuous on $\mathbf{R}$ with $0 \le h'$ and $h''(x)+\psi''(x) \le 0$ for some continuous function $\psi(x)$ satisfying $\psi'' \le 0$ on $\mathbf{R}$.

Then there exist monotone sequences $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$ of lower and upper solutions respectively that converge uniformly and quadratically on the intervals $(\tau_k, \tau_{k+1}]$ for $k = 1, 2, ..., p$ to the unique solution of the BVP (1)-(3) in $S(\alpha_0, \beta 0)$.

Proof. For $(t,x_1),\ (t,x_2)\in\Omega(\alpha_0,\beta_0)$ with $x_1\geq x_2$, it follows from (A_2) that

$$f(t,x_1)\geq f(t,x_2)+(f_x(t,x_2)+2M_1x_2)(x_1-x_2)-M_1(x_1^2-x_2^2), \tag{14}$$

$$g(t,x_1)\geq g(t,x_2)+(g_x(t,x_2)+(1+\epsilon)M_2x_2^{\epsilon})(x_1-x_2)-M_2(x_1^{1+\epsilon}-x_2^{1+\epsilon}). \tag{15}$$

Define

$$\begin{aligned}Q(t,x_1,x_2) &= f(t,x_2)+(f_x(t,x_2)+2M_1x_2)(x_1-x_2)-M_1(x_1^2-x_2^2)\\ &+ g(t,x_2)+[g_x(t,x_2)+(1+\epsilon)M_2x_2^{\epsilon}](x_1-x_2)-M_2(x_1^{1+\epsilon}-x_2^{1+\epsilon}),\end{aligned}$$

and observe that

$$f(t,x_1)+g(t,x_1)\geq Q(t,x_1,x_2),\qquad f(t,x_1)+g(t,x_1)=Q(t,x_1,x_1). \tag{16}$$

Furthermore, for $\alpha_0(t)\leq y\leq x\leq\beta_0(t)$, we have

$$[f(t,x)+g(t,x)]-[f(t,y)+g(t,y)]\leq L(x-y),\ L>0, \tag{17}$$

$$Q(t,x,x_2)-Q(t,y,x_2)\leq N(x-y),\ N>0. \tag{18}$$

From (A_3), we obtain

$$I_k(x_1)\geq I_k(x_2)+G'_k(x_2)(x_1-x_2)+J_k(x_2)-J_k(x_1), \tag{19}$$

$$G_k(x_1)\geq G_k(x_2)+G'_k(x_2)(x_1-x_2), \tag{20}$$

where $x_1,x_2\in D_k(\alpha_0,\beta_0)$ with $x_1\geq x_2$. Since

$$I'_k(x)=G'_k(x)-J'_k(x)=G'_k(\alpha_0\tau_k)-J'_k(\beta_0\tau_k)\geq 0,$$

it follows that the functions $I_k(x)$ are nondecreasing for $k=1,2,...,p$.
Now, we define $H:\mathbf{R}\to\mathbf{R}$ by $H(x)=h(x)+\psi(x)$. Using the mean value theorem and (A_4), we obtain

$$h(x)\leq h(y)+H'(y)(x-y)+\psi(y)-\psi(x)=E(x,y),\ \ h(x)=E(x,x),\ \ x,y\in\mathbf{R}. \tag{21}$$

Hence, by Theorem 2, the BVP (1)-(3) has a solution in $S(\alpha_0,\beta_0)$. We set

$$\begin{aligned}Q(t,x,\alpha_0) &= f(t,\alpha_0)+[f_x(t,\alpha_0)+2M_1\alpha_0](x-\alpha_0)-M_1(x^2-\alpha_0^2)\\ &+ g(t,\alpha_0)+[g_x(t,\alpha_0)+(1+\epsilon)M_2\alpha_0^{\epsilon}](x-\alpha_0)-M_2(x^{1+\epsilon}-\alpha_0^{1+\epsilon}),\\ \widehat{Q}(t,x,\beta_0) &= f(t,\beta_0)+[f_x(t,\alpha_0)+2M_1\alpha_0](x-\beta_0)-M_1(x^2-\beta_0^2)\\ &+ g(t,\beta_0)+[g_x(t,\alpha_0)+(1+\epsilon)M_2\alpha_0^{\epsilon}](x-\beta_0)-M_2(x^{1+\epsilon}-\beta_0^{1+\epsilon}),\\ C_k(x(\tau_k),\alpha_0(\tau_k)) &= I_k(\alpha_0(\tau_k))+B_k^0(x(\tau_k)-\alpha_0(\tau_k)),\\ B_k^0 &= G'_k(\alpha_0(\tau_k))-J'_k(\beta_0(\tau_k)),\\ \widehat{C}_k(x(\tau_k),\beta_0(\tau_k)) &= I_k(\beta_0(\tau_k))+B_k^0(x(\tau_k)-\beta_0(\tau_k)),\\ E(x(\frac{T}{2});\alpha_0,\beta_0) &= h(\alpha_0(\frac{T}{2}))+H'(\beta_0(\frac{T}{2}))(x(\frac{T}{2})-\alpha_0(\frac{T}{2}))+\psi(\alpha_0(\frac{T}{2}))-\psi(x(\frac{T}{2})),\\ e(x(\frac{T}{2});\beta_0) &= h(\beta_0(\frac{T}{2}))+H'(\beta_0(\frac{T}{2}))(x(\frac{T}{2})-\beta_0(\frac{T}{2}))+\psi(\beta_0(\frac{T}{2}))-\psi(x(\frac{T}{2})).\end{aligned}$$

Obviously

$$Q(t,\alpha_0,\alpha_0)=f(t,\alpha_0)+g(t,\alpha_0)=F(t,\alpha_0),\quad \widehat{Q}(t,\beta_0,\beta_0)=f(t,\beta_0)+g(t,\beta_0)=F(t,\beta_0),$$

$$C_k(\alpha_0(\tau_k),\alpha_0(\tau_k))=I_k(\alpha_0(\tau_k)),\quad \widehat{C}_k(\beta_0(\tau_k),\beta_0(\tau_k))=I_k(\beta_0(\tau_k)),$$

$$E(\alpha_0(\frac{T}{2});\alpha_0,\beta_0)=h(\alpha_0(\frac{T}{2})),\quad e(\beta_0(\frac{T}{2});\beta_0)=h(\beta_0(\frac{T}{2})).$$

Now, we consider the following three-point impulsive boundary value problem

$$x'=Q(t,x,\alpha_0),\quad \text{for}\quad t\in[0,T],\ t\neq\tau_k, \tag{22}$$

$$x(\tau_k+0)=C_k(x(\tau_k),\alpha_0(\tau_k)), \tag{23}$$

$$\gamma_1 x(0)-\gamma_2 x(T)=E(x(\frac{T}{2});\alpha_0,\beta_0), \tag{24}$$

and show that α_0 and β_0 are its lower and upper solutions respectively.
From (A_1) and (14)-(15), we obtain

$$\alpha_0'\leq f(t,\alpha_0)+g(t,\alpha_0)=Q(t,\alpha_0,\alpha_0),$$

$$\alpha_0(\tau_k+0)\leq I_k(\alpha_0(\tau_k))=C_k(\alpha_0(\tau_k),\alpha_0(\tau_k)),$$

$$\gamma_1\alpha_0(0)-\gamma_2\alpha_0(T)\leq h(\alpha_0(\frac{T}{2}))=E(\alpha_0(\frac{T}{2});\alpha_0,\beta_0),$$

which implies that α_0 is a lower solution of the BVP (22)-(24) and

$$\begin{aligned}
\beta_0'(t) &\geq F(t,\beta_0)=f(t,\beta_0)+g(t,\beta_0)\\
&\geq f(t,\alpha_0)+[f_x(t,\alpha_0)+2M_1\alpha_0](\beta_0-\alpha_0)-M_1(\beta_0^2-\alpha_0^2)\\
&+ g(t,\alpha_0)+[g_x(t,\alpha_0)+(1+\epsilon)M_2\alpha_0^{\epsilon}](\beta_0-\alpha_0)-M_2(\beta_0^{1+\epsilon}-\alpha_0^{1+\epsilon})\\
&= Q(t,\beta_0,\alpha_0).
\end{aligned}$$

Using (A_1), (19) and the nondecreasing property of J_k', we get

$$\begin{aligned}
\beta_0(\tau_k+0) &\geq I_k(\beta_0(\tau_k))\\
&\geq I_k(\alpha_0(\tau_k))+G_k'(\alpha_0(\tau_k))(\beta_0(\tau_k)-\alpha_0(\tau_k))+J_k(\alpha_0(\tau_k))-J_k(\beta_0(\tau_k))\\
&= I_k(\alpha_0(\tau_k))+G_k'(\alpha_0(\tau_k))(\beta_0(\tau_k)-\alpha_0(\tau_k))-J_k'(\eta_0)(\beta_0(\tau_k)-\alpha_0(\tau_k))\\
&\geq I_k(\alpha_0(\tau_k))+(G_k'(\alpha_0(\tau_k)-J_k'(\beta_0(\tau_k))(\beta_0(\tau_k)-\alpha_0(\tau_k))\\
&= I_k(\alpha_0(\tau_k))+B_k^0(\beta_0(\tau_k)-\alpha_0(\tau_k))=C_k(\beta_0(\tau_k),\alpha_0(\tau_k)),
\end{aligned}$$

where $\alpha_0(\tau_k)\leq\eta_0\leq\beta_0(\tau_k)$. In view of (A_1) and (A_4), for $\alpha_0(\frac{T}{2})\leq c_0\leq\beta_0(\frac{T}{2})$, we find that

$$\begin{aligned}
& h(\beta_0(\frac{T}{2}))-E(\beta_0(\frac{T}{2});\alpha_0,\beta_0)\\
=\ & h(\beta_0(\frac{T}{2}))-h(\alpha_0(\frac{T}{2}))-H'(\beta_0(\frac{T}{2}))(\beta_0(\frac{T}{2})-\alpha_0(\frac{T}{2}))
\end{aligned}$$

$$
\begin{aligned}
&- \quad \psi(\alpha_0(\frac{T}{2})) + \psi(\beta_0(\frac{T}{2})) \\
&= \quad H(\beta_0(\frac{T}{2})) - H(\alpha_0(\frac{T}{2})) - H'(\beta_0(\frac{T}{2}))(\beta_0(\frac{T}{2}) - \alpha_0(\frac{T}{2})) \\
&= \quad H'(c_0)(\beta_0(\frac{T}{2}) - \alpha_0(\frac{T}{2})) - H'(\beta_0(\frac{T}{2}))(\beta_0(\frac{T}{2}) - \alpha_0(\frac{T}{2})) \\
&= \quad (H'(c_0) - H'(\beta_0(\frac{T}{2})))(\beta_0(\frac{T}{2}) - \alpha_0(\frac{T}{2})) \geq 0.
\end{aligned}
$$

Thus, $\gamma_1\beta_0(0) - \gamma_2\beta_0(T) \geq \beta_0(\frac{T}{2}) \geq E(\beta_0(\frac{T}{2}); \alpha_0, \beta_0)$. Hence $\beta_0(t)$ is an upper solution of the BVP (22)-(24). Then, by Theorem 2, there exists a unique solution $\alpha_1(t) \in S(\alpha_0, \beta_0)$ of the BVP (22)-(24) such that $\alpha_0(t) \leq \alpha_1(t) \leq \beta_0(t), \quad t \in [0, T]$.

Next, consider the problem

$$x' = \widehat{Q}(t, x, \beta_0), \quad \text{for} \quad t \in [0, T], \ t \neq \tau_k, \tag{25}$$

$$x(\tau_k + 0) = \widehat{C}_k(x(\tau_k), \beta_0(\tau_k)), \tag{26}$$

$$\gamma_1 x(0) - \gamma_2 x(T) = e(x(\frac{T}{2}); \beta_0). \tag{27}$$

From (A_1), it follows that

$$
\begin{gathered}
\beta_0'(t) \geq f(t, \beta_0) + g(t, \beta_0) = \widehat{Q}(t, \beta_0, \beta_0), \\
\beta_0(\tau_k + 0) \geq I_k(\beta_0(\tau_k)) = \widehat{C}_k(\beta_0(\tau_k), \beta_0(\tau_k)), \\
\gamma_1\beta_0(0) - \gamma_2\beta_0(T) \geq h(\beta_0(\frac{T}{2})) = e(\beta_0(\frac{T}{2}); \beta_0),
\end{gathered}
$$

which implies that β_0 is an upper solution (25)-(27).
Using (A_1) and (14)-(15) again, we obtain

$$
\begin{aligned}
\alpha_0'(t) &\leq f(t, \alpha_0) + g(t, \alpha_0) \\
&\leq f(t, \beta_0) + g(t, \beta_0) - (f_x(t, \alpha_0) + 2M_1\alpha_0)(\beta_0 - \alpha_0) + M_1(\beta_0^2 - \alpha_0^2) \\
&- (g_x(t, \alpha_0) + (1 + \epsilon)M_2\alpha_0^{\epsilon})(\beta_0 - \alpha_0) + M_2(\beta_0^{1+\epsilon} - \alpha_0^{1+\epsilon}) \\
&= f(t, \beta_0) + (f_x(t, \alpha_0) + 2M_1\alpha_0)(\alpha_0 - \beta_0) - M_1(\alpha_0^2 - \beta_0^2) \\
&+ g(t, \beta_0) + (g_x(t, \alpha_0) + (1 + \epsilon)M_2\alpha_0^{\epsilon})(\alpha_0 - \beta_0) - M_2(\alpha_0^{1+\epsilon} - \beta_0^{1+\epsilon}) \\
&= \widehat{Q}(t, \alpha_0, \beta_0).
\end{aligned}
$$

In a similar manner, it can be shown that

$$\alpha_0(\tau_k + 0) \leq \widehat{C}_k(\alpha_0(\tau_k), \beta_0(\tau_k)).$$

$$\gamma_1\alpha_0(0) - \gamma_2\alpha_0(T) \leq e(\alpha_0(\frac{T}{2}), \beta_0).$$

Hence α_0 is a lower solution of the BVP (25)-(27). Again, by Theorem 2, there exists a unique solution $\beta_1(t) \in S(\alpha_0, \beta_0)$ of the BVP (25)-(27) such that $\alpha_0(t) \leq$

$\beta_1(t) \le \beta_0(t), \quad t \in [0,T]$. Now, we show that $\alpha_1(t) \le \beta_1(t)$, for $t \in [0,T]$. For that, we will prove that $\alpha_1(t)$ and $\beta_1(t)$ are lower and upper solution of the BVP (1)-(3) respectively. Using the fact that α_1 is a solution of (22)-(24) together with (14)-(15), we obtain

$$\begin{aligned}
\alpha_1'(t) &= Q(t,\alpha_1,\alpha_0) \\
&= f(t,\alpha_0) + [f_x(t,\alpha_0) + 2M_1\alpha_0](\alpha_1 - \alpha_0) - M_1(\alpha_1^2 - \alpha_0^2) \\
&+ g(t,\alpha_0) + [g_x(t,\alpha_0) + (1+\epsilon)M_2\alpha_0^{\epsilon}](\alpha_1 - \alpha_0) - M_2(\alpha_1^{1+\epsilon} - \alpha_0^{1+\epsilon}) \\
&\le f(t,\alpha_1) + g(t,\alpha_1) - [f_x(t,\alpha_0) + 2M_1\alpha_0](\alpha_1 - \alpha_0) + M_1(\alpha_1^2 - \alpha_0^2) \\
&- [g_x(t,\alpha_0) + (1+\epsilon)M_2\alpha_0^{\epsilon}](\alpha_1 - \alpha_0) + M_2(\alpha_1^{1+\epsilon} - \alpha_0^{1+\epsilon}) \\
&+ [f_x(t,\alpha_0) + 2M_1\alpha_0](\alpha_1 - \alpha_0) - M_1(\alpha_1^2 - \alpha_0^2) \\
&+ [g_x(t,\alpha_0) + (1+\epsilon)M_2\alpha_0^{\epsilon}](\alpha_1 - \alpha_0) - M_2(\alpha_1^{1+\epsilon} - \alpha_0^{1+\epsilon}) \\
&= f(t,\alpha_1) + g(t,\alpha_1) = F(t,\alpha_1(t)), \quad t \in [0,T], \quad t \ne \tau_k.
\end{aligned}$$

In view of (19) and the nonincreasing property of J_k', we have

$$\begin{aligned}
\alpha_1(\tau_k + 0) &= I_k(\alpha_0(\tau_k)) + B_k^0[\alpha_1(\tau_k) - \alpha_0(\tau_k)] \\
&\le I_k(\alpha_1(\tau_k)) - G_k'(\alpha_0(\tau_k))(\alpha_1(\tau_k) - \alpha_0(\tau_k)) - J_k(\alpha_0(\tau_k)) + J_k(\alpha_1(\tau_k)) \\
&+ B_k^0[\alpha_1(\tau_k) - \alpha_0(\tau_k)] \\
&= I_k(\alpha_1(\tau_k)) + [G_k'(\alpha_0(\tau_k)) - J_k'(\eta_1) - B_k^0](\alpha_0(\tau_k) - \alpha_1(\tau_k)) \\
&\le I_k(\alpha_1(\tau_k)) + [G_k'(\alpha_0(\tau_k)) - J_k'(\beta_0(\tau_k)) - G_k'(\alpha_0(\tau_k)) \\
&+ J_k'(\beta_0(\tau_k))](\alpha_0(\tau_k) - \alpha_1(\tau_k)) \\
&= I_k(\alpha_1(\tau_k)),
\end{aligned}$$

where $\alpha_0 \le \eta_1 \le \alpha_1 \le \beta_0$. Utilizing the nonincreasing property of H' together with (21) yields

$$\begin{aligned}
&M\alpha_1(0) - N\alpha_1(T) \\
&= h(\alpha_0(\tfrac{T}{2})) + H'(\beta_0(\tfrac{T}{2}))(\alpha_1(\tfrac{T}{2}) - \alpha_0(\tfrac{T}{2})) + \psi(\alpha_0(\tfrac{T}{2})) - \psi(\alpha_1(\tfrac{T}{2})) \\
&\le h(\alpha_1(\tfrac{T}{2})) + H'(\alpha_1(\tfrac{T}{2}))(\alpha_0(\tfrac{T}{2}) - \alpha_1(\tfrac{T}{2})) + \psi(\alpha_1(\tfrac{T}{2})) - \psi(\alpha_0(\tfrac{T}{2})) \\
&+ H'(\alpha_1(\tfrac{T}{2}))(\alpha_1(\tfrac{T}{2}) - \alpha_0(\tfrac{T}{2})) + \psi(\alpha_0(\tfrac{T}{2})) - \psi(\alpha_1(\tfrac{T}{2})) \\
&= h(\alpha_1(\tfrac{T}{2})).
\end{aligned}$$

Thus, $\alpha_1(t)$ is a lower solution of the BVP (1)-(3). Similarly, we can show that $\beta_1(t)$ is an upper solution of (1)-(3). Thus, by Theorem 1, $\alpha_1(t) \le \beta_1(t)$ and consequently, we get

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t), \quad t \in [0,T].$$

Continuing this process, by induction, one can construct monotone sequences $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty, \quad \alpha_n, \beta_n \in S(\alpha_{n-1}, \beta_{n-1})$ such that

$$\alpha_0(t) \le \alpha_1(t) \le ... \le \alpha_n(t) \le \beta_n(t) \le ... \le \beta_1(t) \le \beta_0(t), \quad t \in [0,T],$$

where $\alpha_{n+1}(t)$ is the unique solution of the BVP

$$x' = Q(t, x, \alpha_n), \quad \text{for} \quad t \in [0, T], \ t \neq \tau_k, \tag{28}$$

$$x(\tau_k + 0) = C_k(x(\tau_k), \alpha_n(\tau_k)), \tag{29}$$

$$\gamma_1 x(0) - \gamma_2 x(T) = E(x(\frac{T}{2}); \alpha_n, \beta_n), \tag{30}$$

and $\beta_{n+1}(t)$ is the unique solution of

$$x' = \widehat{Q}(t, x, \beta_n), \quad \text{for} \quad t \in [0, T], \ t \neq \tau_k, \tag{31}$$

$$x(\tau_k + 0) = \widehat{C}_k(x(\tau_k), \beta_n(\tau_k)), \tag{32}$$

$$\gamma_1 x(0) - \gamma_2 x(T) = e(x(\frac{T}{2}); \beta_n). \tag{33}$$

Since the sequences $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$ are uniformly bounded and equi-continuous on $(\tau_k, \tau_{k+1}], \ k = 0, 1, ..., p$, it follows that they are uniformly convergent [23] with

$$\lim_{n\to\infty} \alpha_n(t) = x(t), \qquad \lim_{n\to\infty} \beta_n(t) = y(t).$$

Hence we conclude that

$$\alpha_0(t) \leq x(t) \leq y(t) \leq \beta_0(t).$$

Taking the limit $n \to \infty$, we find that

$$Q(t, \alpha_{n+1}, \alpha_n) \to f(t, x(t)) + g(t, x(t)), \quad C_k(\alpha_{n+1}(\tau_k), \alpha_n(\tau_k)) \to I_k(x(t_k)),$$
$$E(\alpha_{n+1}(\frac{T}{2}); \alpha_n, \beta_n) \to h(x(\frac{T}{2})).$$

Now applying Theorem 3 to the BVP (28)-(30) together with Lebesgue dominated convergence theorem, it follows that $x(t)$ is the solution of the BVP (1)-(3) in $S(\alpha_0, \beta_0)$. Similarly, applying Theorem 3 to the BVP (31)-(32), it can be shown that $y(t)$ is the solution of the BVP (1)-(3) in $S(\alpha_0, \beta_0)$. Therefore, by the uniqueness of the solution, $x(t) = y(t)$.

Now, we prove that the convergence of each of the two sequences is quadratic. For that, we set $a_{n+1}(t) = x(t) - \alpha_{n+1}(t), \quad b_{n+1}(t) = \beta_{n+1}(t) - x(t), \ t \in [0, T]$ and note that $a_{n+1}(t) \geq 0$ and $b_{n+1}(t) \geq 0$. We will only prove the quadratic convergence of the sequence $\{a_n(t)\}_0^\infty$ as that of the sequence $\{b_n(t)\}_0^\infty$ is similar one.

Setting $P(t, x) = f(t, x) + g(t, x) + M_1 x^2 + M_2 x^{1+\epsilon}, \ t \in [0, T], \ t \neq \tau_k$ and using the mean value theorem repeatedly, we obtain

$$\begin{aligned} a'_{n+1}(t) &= x'(t) - \alpha'_{n+1}(t) \\ &= f(t, x) + g(t, x) - Q(t, \alpha_{n+1}, \alpha_n) \\ &= P(t, x) - P(t, \alpha_n) - P_x(t, \alpha_n)(a_n(t) - a_{n+1}(t)) - M_1(x^2 - \alpha_{n+1}^2) \end{aligned}$$

$$
\begin{aligned}
&- \quad M_2(x^{1+\epsilon} - \alpha_{n+1}^{1+\epsilon}) \\
&= \quad P_x(t,c_1)a_n(t) - P_x(t,\alpha_n)a_n(t) + P_x(t,\alpha_n)a_{n+1}(t) \\
&- \quad M_1(x^2 - \alpha_{n+1}^2) - M_2(x^{1+\epsilon} - \alpha_{n+1}^{1+\epsilon}) \\
&= \quad [P_x(t,c_1) - P_x(t,\alpha_n)]a_n(t) \\
&+ \quad [P_x(t,\alpha_n) - M_1(x+\alpha_{n+1}) - M_2\sigma(x,\alpha_{n+1})]a_{n+1}(t) \\
&= \quad [f_x(t,c_1) + g_x(t,c_1) + 2M_1c_1 + (1+\epsilon)M_2c_1^\epsilon \\
&- \quad f_x(t,\alpha_n) - g_x(t,\alpha_n) - 2M_1\alpha_n - (1+\epsilon)M_2\alpha_n^\epsilon]a_n(t) \\
&+ \quad [P_x(t,\alpha_n) - M_1(x+\alpha_{n+1}) - M_2\sigma(x,\alpha_{n+1})]a_{n+1}(t) \\
&= \quad [f_{xx}(t,c_2)(c_1-\alpha_n) + g_x(t,c_1) - g_x(t,\alpha_n) + (1+\epsilon)M_2(c_1^\epsilon - \alpha_n^\epsilon) \\
&+ \quad 2M_1(c_1-\alpha_n)]a_n(t) + [P_x(t,\alpha_n) - M_1(x+\alpha_{n+1}) - M_2\sigma(x,\alpha_{n+1})]a_{n+1}(t) \\
&\leq \quad Q_n(t)a_{n+1}(t) + \rho_n, \qquad (34)
\end{aligned}
$$

where L_1 is Lipschitz constant (g_x satisfies the Lipschitz condition), $\alpha_n \leq c_1 \leq c_2 \leq x \leq \beta_n$ and

$$
\begin{aligned}
Q_n &= P_x(t,\alpha_n) - M_1(x+\alpha_{n+1}) - M_2\omega(x,\alpha_{n+1}), \\
\rho_n &= [f_{xx}(t,c_2) + L_1 + (1+\epsilon)M_2\omega(c_1,\alpha_n) + 2M_1]a_n^2(t), \\
\omega(x,\alpha_{n+1}) &= (x^\epsilon + x^{\epsilon-1}\alpha_{n+1} + x^{\epsilon-2}\alpha_{n+1}^2 + ... + x^1\alpha_{n+1}^{\epsilon-1} + \alpha_n^\epsilon) > 0.
\end{aligned}
$$

Similarly it can be shown that

$$
a_{n+1}(\tau_k+0) \leq B_k^n a_{n+1}(\tau_k) + \sigma_k, \qquad (35)
$$

where $\sigma_k = [G_k''(\omega_k) + \frac{3}{2}J_k''(\chi_k)]a_n^2(\tau_k) + \frac{1}{2}J_k''(\chi_k)b_n^2(\tau_k)$, $\alpha_n(\tau_k) \leq \omega_k \leq x(\tau_k)$ and $\alpha_n(\tau_k) \leq \chi_k \leq \beta_n(\tau_k)$, $k = 1,2,...,p$.
Applying Theorem 1.4.1 (page 32 [21]) on (34)-(35), it follows that the function $a_{n+1}(t)$ satisfies the estimate

$$
\begin{aligned}
a_{n+1}(t) \quad &\leq \quad a_{n+1}(0)\left(\prod_{0<\tau_k<t} B_k^n\right)\exp\left(\int_0^t Q_n(\tau)d\tau\right) \\
&+ \quad \sum_{0<\tau_k<t}\left(\prod_{\tau_k<\tau_j<t} B_j^n \exp\left(\int_{\tau_k}^t Q_n(\tau)d\tau\right)\right)\sigma_k \\
&+ \quad \int_0^t \prod_{s<\tau_k<t} B_k^n \exp\left(\int_s^t Q_n(\tau)d\tau\right)\rho_n(s)ds, \quad t \geq t_0. \qquad (36)
\end{aligned}
$$

In view of (21), we have

$$
\begin{aligned}
&\quad \gamma_1 a_{n+1}(0) - \gamma_2 a_{n+1}(T) \\
&= \quad [\gamma_1 x(0) - \gamma_2 x(T)] - [\gamma_1\alpha_{n+1}(0) - \gamma_2\alpha_{n+1}(T)] \\
&= \quad h(x(\frac{T}{2})) - h(\alpha_n(\frac{T}{2})) - H'(\beta_n(\frac{T}{2}))(\alpha_{n+1}(\frac{T}{2}) - \alpha_n(\frac{T}{2}))
\end{aligned}
$$

$$
\begin{aligned}
&- \quad \psi(\alpha_n(\frac{T}{2})) + \psi(\alpha_{n+1}(\frac{T}{2}))\\
&\leq \quad h(\alpha_n(\frac{T}{2})) + H'(\alpha_n(\frac{T}{2}))(x(\frac{T}{2}) - \alpha_n(\frac{T}{2})) + \psi(\alpha_n(\frac{T}{2})) - \psi(x(\frac{T}{2}))\\
&- \quad h(\alpha_n(\frac{T}{2})) - H'(\beta_n(\frac{T}{2}))(\alpha_{n+1}(\frac{T}{2}) - \alpha_n(\frac{T}{2}))\\
&- \quad \psi(\alpha_n(\frac{T}{2})) + \psi(\alpha_{n+1}(\frac{T}{2}))\\
&= \quad H'(\alpha_n(\frac{T}{2}))a_n(\frac{T}{2}) - H'(\beta_n(\frac{T}{2}))a_n(\frac{T}{2}) + H'(\beta_n(\frac{T}{2}))a_{n+1}(\frac{T}{2})\\
&- \quad \psi'(c_3)a_{n+1}(\frac{T}{2})\\
&\leq \quad -H''(c_4)(\beta_n(\frac{T}{2}) - \alpha_n(\frac{T}{2}))a_n(\frac{T}{2}) + H'(\beta_n(\frac{T}{2}))a_{n+1}(\frac{T}{2})\\
&- \quad \psi'(\beta_n(\frac{T}{2}))a_{n+1}(\frac{T}{2})\\
&= \quad -H''(c_4)(b_n(\frac{T}{2}) + a_n(\frac{T}{2}))a_n(\frac{T}{2}) + h'(\beta_n(\frac{T}{2}))a_{n+1}(\frac{T}{2})\\
&\leq \quad -H''(c_4)(\frac{3}{2}a_n^2(\frac{T}{2}) + \frac{1}{2}b_n^2(\frac{T}{2})) + h'(\beta_n(\frac{T}{2}))a_{n+1}(\frac{T}{2})
\end{aligned}
$$

where $\alpha_{n+1}(\frac{T}{2}) \leq c_3 \leq x(\frac{T}{2}) \leq \beta_n(\frac{T}{2})$ and $x(\frac{T}{2}) \leq c_4 \leq \beta_n(\frac{T}{2})$. Thus, we have

$$
a_{n+1}(0) \leq \frac{\gamma_2}{\gamma_1}a_{n+1}(T) + \frac{1}{\gamma_1}[-H''(c_4)(\frac{3}{2}a_n^2(\frac{T}{2}) + \frac{1}{2}b_n^2(\frac{T}{2})) + h'(\beta_n(\frac{T}{2}))a_{n+1}(\frac{T}{2})]. \quad (37)
$$

Combining (36) and (37) yields

$$
\begin{aligned}
a_{n+1}(0) \quad &\leq \quad \frac{\gamma_2}{\gamma_1}a_{n+1}(T) + \frac{1}{\gamma_1}(-H''(c_4)b_n(\frac{T}{2})a_n(\frac{T}{2}) + h'(\beta_n(\frac{T}{2}))a_{n+1}(\frac{T}{2}))\\
&\leq \quad \frac{\gamma_2}{\gamma_1}[a_{n+1}(0)\left(\prod_{0<\tau_k<T} B_k^n\right)\exp\left(\int_0^T Q_n(\tau)d\tau\right)\\
&+ \quad \sum_{0<\tau_k<T}\left(\prod_{\tau_k<\tau_j<T} B_j^n \exp\left(\int_{\tau_k}^T Q_n(\tau)d\tau\right)\right)\sigma_k\\
&+ \quad \int_0^T \prod_{s<\tau_k<T} B_k^n \exp\left(\int_s^T Q_n(\tau)d\tau\right)\rho_n(s)ds]\\
&- \quad \frac{1}{\gamma_1}H''(c_4)(\frac{3}{2}a_n^2(\frac{T}{2}) + \frac{1}{2}b_n^2(\frac{T}{2}))\\
&+ \quad \frac{1}{\gamma_1}h'(\beta_n(\frac{T}{2}))[a_{n+1}(0)\left(\prod_{0<\tau_k<\frac{T}{2}} B_k^n\right)\exp\left(\int_0^{\frac{T}{2}} Q_n(\tau)d\tau\right)\\
&+ \quad \sum_{0<\tau_k<\frac{T}{2}}\left(\prod_{\tau_k<\tau_j<\frac{T}{2}} B_j^n \exp\left(\int_{\tau_k}^{\frac{T}{2}} Q_n(\tau)d\tau\right)\right)\sigma_k
\end{aligned}
$$

$$+ \int_0^{\frac{T}{2}} \prod_{s<\tau_k<\frac{T}{2}} B_k^n \exp\left(\int_s^{\frac{T}{2}} Q_n(\tau)d\tau\right) \rho_n(s)ds].$$

Solving for $a_{n+1}(0)$, we get

$$\begin{aligned}
a_{n+1}(0) \leq & [1 - \frac{\gamma_2}{\gamma_1}\left(\prod_{0<\tau_k<T} B_k^n\right) \exp\left(\int_0^T Q_n(\tau)d\tau\right) \\
- & \frac{1}{\gamma_1}h'(\beta_n(\frac{T}{2}))\left(\prod_{0<\tau_k<\frac{T}{2}} B_k^n\right) \exp\left(\int_0^{\frac{T}{2}} Q_n(\tau)d\tau\right)]^{-1} \\
\times & \{\frac{\gamma_2}{\gamma_1}[\sum_{0<\tau_k<T}\left(\prod_{\tau_k<\tau_j<T} B_j^n \exp\left(\int_{\tau_k}^T Q_n(\tau)d\tau\right)\right)\sigma_k \\
+ & \int_0^T \prod_{s<\tau_k<T} B_k^n \exp\left(\int_s^T Q_n(\tau)d\tau\right)\rho_n(s)ds] \\
+ & \frac{1}{\gamma_1}h'(\beta_n(\frac{T}{2}))[\sum_{0<\tau_k<\frac{T}{2}}\left(\prod_{\tau_k<\tau_j<\frac{T}{2}} B_j^n \exp\left(\int_{\tau_k}^{\frac{T}{2}} Q_n(\tau)d\tau\right)\right)\sigma_k \\
+ & \int_0^{\frac{T}{2}} \prod_{s<\tau_k<\frac{T}{2}} B_k^n \exp\left(\int_s^{\frac{T}{2}} Q_n(\tau)d\tau\right)\rho_n(s)ds] \\
- & \frac{1}{\gamma_1}H''(c_4)(\frac{3}{2}a_n^2(\frac{T}{2}) + \frac{1}{2}b_n^2(\frac{T}{2}))\}. \qquad (38)
\end{aligned}$$

Substituting (38) into (36) yields

$$\begin{aligned}
a_{n+1}(t) \leq & \left(\prod_{0<\tau_k<t} B_k^n\right) \exp\left(\int_0^t Q_n(\tau)d\tau\right)\Phi^{-1} \\
\times & \{\frac{\gamma_2}{\gamma_1}[\sum_{0<\tau_k<T}\left(\prod_{\tau_k<\tau_j<T} B_j^n \exp\left(\int_{\tau_k}^T Q_n(\tau)d\tau\right)\right)(\lambda a_n^2(\tau_k) + \frac{1}{2}\delta_2 b_n^2(\tau_k)) \\
+ & \int_0^T \prod_{s<\tau_k<T} B_k^n \exp\left(\int_s^T Q_n(\tau)d\tau\right)[(\delta_3+\delta_4)a_n^2(s)]ds \\
+ & \frac{1}{\gamma_1}\delta_5(\frac{T}{2}))[\sum_{0<\tau_k<\frac{T}{2}}\left(\prod_{\tau_k<\tau_j<\frac{T}{2}} B_j^n \exp\left(\int_{\tau_k}^{\frac{T}{2}} Q_n(\tau)d\tau\right)\right)(\lambda a_n^2(\tau_k) + \frac{1}{2}\delta_2 b_n^2(\tau_k)) \\
+ & \int_0^{\frac{T}{2}} \prod_{s<\tau_k<\frac{T}{2}} B_k^n \exp\left(\int_s^{\frac{T}{2}} Q_n(\tau)d\tau\right)[(\delta_3+\delta_4)a_n^2(s)]ds \\
+ & \frac{1}{\gamma_1}\delta_6(\frac{3}{2}a_n^2(\frac{T}{2}) + \frac{1}{2}b_n^2(\frac{T}{2}))\} \\
+ & \sum_{0<\tau_k<t}\left(\prod_{\tau_k<\tau_j<t} B_j^n \exp\left(\int_{\tau_k}^t Q_n(\tau)d\tau\right)\right)(\lambda a_n^2(\tau_k) + \frac{1}{2}\delta_2 b_n^2(\tau_k))
\end{aligned}$$

$$+ \int_0^t \prod_{s<\tau_k<t} B_k^n \exp\left(\int_s^t Q_n(\tau)d\tau\right) [(\delta_3+\delta_4)a_n^2(s)]ds$$

where $|G_k''| \le \delta_1$, $|J_k''| \le \delta_2$, $|f_{xx}| \le \delta_3$, $|L_1 + (1+\epsilon)M_2\omega(c_1,\alpha_n) + 2M_1| \le \delta_4$, $|h'| \le \delta_5$, $|H''| \le \delta_6$, $\lambda = \delta_1 + \frac{3}{2}\delta_2$ and

$$\Phi = 1 - \frac{\gamma_2}{\gamma_1} \prod_{0<\tau_k<T} B_k^n \exp\left(\int_0^T Q_n(\tau)d\tau\right) - \frac{1}{\gamma_1} h'(\beta_n(\frac{T}{2})) \prod_{0<\tau_k<\frac{T}{2}} B_k^n \exp\left(\int_0^{\frac{T}{2}} Q_n(\tau)d\tau\right).$$

Taking the maximum on $[0,T]$, it follows that there exist positive constants η_1 and η_2 such that

$$\|a_{n+1}(t)\| \le \eta_1 \|a_n\|^2 + \eta_2 \|b_n\|^2.$$

On the same pattern, it can be proved that

$$\|b_{n+1}(t)\| \le \zeta_1 \|b_n\|^2 + \zeta_2 \|a_n\|^2,$$

where ζ_1 and ζ_2 are positive constants. This establishes the quadratic convergence of the sequences.

Example. The impulsive BVP

$$x'(t) = \frac{1}{15}\ln((x(t))^3+1) + \frac{1}{300}((x(t))+t)^5 \quad \text{for} \quad t \in [0,1],\ t \ne \frac{1}{3},$$

$$x(\frac{1}{3}+0) = x(\frac{1}{3}),$$

$$\gamma_1 x(0) - \gamma_2 x(1) = x(\frac{1}{2}), \quad \gamma_2 \le \frac{1}{6}\gamma_1 - \frac{3}{4},$$

admits the minimal solution $\alpha_0(t) = 0,\ \ t \in [0,1]$ and the maximal solution $\beta_0(t)$ given by

$$\beta_0(t) = \begin{cases} t + \frac{1}{3}, & \text{if } t \in [0, \frac{1}{3}], \\ t+1, & \text{if } t \in (\frac{1}{3}, 1]. \end{cases}$$

Clearly $\alpha_0(t)$ and $\beta_0(t)$ are not the solutions of the BVP and $\alpha_0(t) \le \beta_0(t),\ t \in [0,1]$.

4 Concluding Remarks

This paper addresses a quasilinearization method for a nonlinear impulsive first order ordinary differential equation dealing with a nonlinear function $F(t, x(t))$ which is a sum of two functions of different nature together with a nonlinear three-point boundary condition in contrast to a problem containing a single function and a linear boundary condition considered in [23]. The condition on $g(t, x(t))$ in assumption (A_3) of Theorem 3 is motivated by the well known fact that $\chi(t) = t^p$ is convex for $p > 1$. The following results can be recorded as a special case of this problem:

(**i**) If we take $g(t, x(t)) = 0,\ h(x(\frac{T}{2})) = c$ (constant), we obtain the generalized quasilinearization for first order impulsive differential equations with linear boundary conditions [23].

(**ii**) By taking $h(x(\frac{T}{2})) = u_0, \gamma_1 = 1,\ \gamma_2 = 0$, we can record the results of usual initial value problems with impulse. In reference [28], the authors have developed an extension of generalized quasilinearization for initial value problems without impulse. Thus our problem generalizes the results of [28] in the sense that impulsive effects have been taken into account along with a three-point nonlinear boundary condition.

(**iii**) The extension of generalized quasilinearization technique for periodic boundary value problems involving impulsive differential equations follows if we take $\gamma_1 = 1,\ \gamma_2 = 1$ and $h(x(\frac{T}{2})) = 0$.

Acknowledgement. The authors thank the reviewer for his/her valuable suggestions and comments for the improvement of the original manuscript.

References

[1] R. Bellman, Methods of Nonlinear Analysis, Vol. 2, Academic Press, New York, 1973.

[2] R. Bellman and R. Kalaba, Quasilinearization and Nonlinear Boundary Value Problems, Amer. Elsevier, New York, 1965.

[3] E.S. Lee, Quasilinearization and Invariant Embedding, Academic Press, New York, 1968.

[4] V. Lakshmikantham, An extension of the method of quasilinearization, J. Optim. Theory Appl. 82(1994), 315-321.

[5] V. Lakshmikantham, Further improvement of generalized quasilinearization, Nonlinear Anal. 27(1996), 223-227.

[6] J.J. Nieto, Generalized quasilinearization method for a second order differential equation with Dirichlet boundary conditions, Proc. Amer. Math. Soc. 125(1997), 2599-2604.

[7] V. Lakshmikantham and A.S.Vatsala, Generalized Quasilinearization for Nonlinear Problems, Kluwer Academic Publishers, Dordrecht, 1998.

[8] A. Cabada, J.J. Nieto and Rafael Pita-da-veige, A note on rapid convergence of approximate solutions for an ordinary Dirichlet problem, Dynam. Contin. Discrete Impuls. Systems 4(1998), 23-30.

[9] A. Cabada and J.J. Nieto, Quasilinearization and rate of convergence for higher order nonlinear periodic boundary value problems, J. Optim. Theory Appl. 108(2001), 97-107.

[10] B. Ahmad, J.J. Nieto and N. Shahzad, The Bellman-Kalaba-Lakshamikantham quasilinearization method for Neumann problems, J. Math. Anal. Appl. 257(2001), 356-363.

[11] P. Eloe and Y. Gao, The method of quasilinearization and a three-point boundary value problem, J. Korean Math. Soc. 39(2002), 319-330.

[12] V. Lakshmikantham and A.S. Vatsala, Generalized quasilinearization versus Newton's method, Appl. Math. Comput. 164(2005), 523-530.

[13] A. Buica, Quasilinearization method for nonlinear elliptic boundary value problems, J. Optim. Theory Appl. 124(2005), 323-338.

[14] B. Ahmad, R.A. Khan and S. Sivasundaram, Generalized quasilinearization method for a first order differential equation with integral boundary condition, Dynam. Contin. Discrete Impuls. Systems Ser. A Math. Anal. 12(2005), 289-296.

[15] B. Ahmad, A. Al-Saedi and S. Sivasundaram, Approximation of the solution of nonlinear second order integro-differential equations, Dynamic Systems Appl. 14 (2005), 253-263.

[16] F.M. Atici and S.G. Topal, The generalized quasilinearization method and three-point boundary value problems on time scales, Appl. Math. Lett. 18(2005), 577-585.

[17] B. Ahmad, A quasilinearization method for a class of integro-differential equations with mixed nonlinearities, Nonlinear Anal. Real World Appl. 7(2006), 997-1004.

[18] S. Nikolov, S. Stoytchev, A. Torres and J.J. Nieto, Biomathematical modeling and analysis of blood flow in an intracranial aneurysms, Neurological Research 25(2003), 497-504.

[19] Yermachenko and F. Sadyrbaev, Quasilinearization and multiple solutions of the Emden-Fowler type equation, Math. Model. Anal. 10(2005), 41-50.

[20] V.B. Mandelzweig and F. Tabakin, Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs, Computer Physics Comm. 141(2001), 268-281.

[21] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore (1989).

[22] V. Doddaballapur and P.W. Eloe, Monotone and quadratic convergence of approximate solutions of ordinary differential equations with impulse, Comm. Appl. Anal. 2(1998), 373-382.

[23] P.W. Eloe and S.G. Hristova, Method of the quasilinearization for nonlinear impulsive differential equations with linear boundary conditions, EJQTDE, 10(2002), 1-14.

[24] Bashir Ahmad and S. Sivasundaram, Rapid unilateral monotone scheme for impulsive hybrid nonlinear boundary value problems, Int. J. Hybrid Systems 4(2004), 345-354.

[25] W. Coppel, Disconjugacy, Lecture Notes in Mathematics, Vol.220, Springer-Verlag, NewYork/Berlin (1971).

[26] I.T. Kiguradze and A.G. Lomtatidze, On certain boundary value problems for second-order linear ordinary differential equations with singularities, J. Math. Anal. Appl. 101(1984), 325-347.

[27] C.P. Gupta, A second order m-point boundary value problem at resonance, Nonlinear Anal. 24(1995), 1483-1489.

[28] J.O. Adeyeye and U.U. Livinus, An extension of the generalized quasilinearization method of Lakshmikantham, Nonlinear Studies 10(2003), 195-199.

(Received September 19, 2006)