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Existence of solutions of nonlinear differential equations with ψ-exponential or ψ-ordinary dichotomous linear part in a Banach space

Atanaska Georgieva, Hristo Kiskinov, Stepan Kostadinov[✉] and **Andrey Zahariev**

Faculty of Mathematics and Informatics, Plovdiv University, 236 Bulgaria Blvd., Plovdiv, BG–4003, Bulgaria

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Abstract. In this article we consider nonlinear differential equations with ψ-exponential and ψ-ordinary dichotomous linear part in a Banach space. By the help of the fixed point principle of Banach sufficient conditions are found for the existence of ψ-bounded solutions of these equations on $\mathbb{R}$ and $\mathbb{R}_+$.

Keywords: ψ-dichotomy for ordinary differential equations, ψ-boundedness.

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1 Introduction

The problem of ψ-boundedness and ψ-stability of the solutions of differential equations in finite dimensional Euclidean spaces has been studied by many auhtors, as e.g. Akinyele [1], Constantin [6]. In these papers, the function ψ is a scalar continuous function (and increasing, differentiable and bounded in [1], nondecreasing and such that $\psi(t) \geq 1$ on $\mathbb{R}_+$ in [6]). In Diamandescu [8–15] and Boi [2–4] ψ is a nonnegative continuous diagonal matrix function.

Inspired by the famous monographs of Coppel [5], Daleckii and Krein [7] and Massera and Schaeffer [17], where the important notion of exponential and ordinary dichotomy is considered in detail, Diamandescu [8–12] and Boi [2–4] introduced and studied the ψ-dichotomy for linear differential equations in finite dimensional Euclidean space.

In our paper [16] we introduced the concept of ψ-dichotomy for arbitrary Banach spaces, where ψ is an arbitrary bounded invertible linear operator.

In this paper nonlinear perturbed differential equations with ψ-dichotomous linear part are considered in an arbitrary Banach space. We will show that some properties of these equations will be influenced by the corresponding ψ-dichotomous homogeneous linear equation. Sufficient conditions for the existence of ψ-bounded solutions of this equations on $\mathbb{R}$ and $\mathbb{R}_+$ in case of ψ-exponential or ψ-ordinary dichotomy are found.

✉ Corresponding author. Email: stkostadinov@uni-plovdiv.bg

2 Preliminaries

Let X be an arbitrary Banach space with norm $|\cdot|$ and identity I. Let $LB(X)$ be the space of all linear bounded operators acting in X with the norm $\|\cdot\|$. By J we shall denote $\mathbb{R}$ or $\mathbb{R}_+ = [0, \infty)$.

We consider the nonlinear differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + F(t,x), \tag{2.1}$$

the corresponding linear homogenous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x \tag{2.2}$$

and the appropriate inhomogeneous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + f(t), \tag{2.3}$$

where $A(\cdot)\colon J \to LB(X)$, $f(\cdot)\colon J \to X$ are strongly measurable and Bochner integrable on the finite subintervals of J and $F(\cdot,\cdot)\colon J \times X \to X$ is a continuous function with respect to t.

By a solution of equation (2.1) (or (2.2) or (2.3)) we will understand a continuous function $x(t)$ that is differentiable (in the sence that it is representable in the form $x(t) = \int_a^t y(\tau)d\tau$ of a Bochner integral of a strongly measurable function y) and satisfies (2.1) (or (2.2) or (2.3)) almost everywhere.

By $V(t)$ we will denote the Cauchy operator of (2.2).

Let $RL(X)$ be the subspace of all invertible operators in $LB(X)$ and $\psi(\cdot)\colon J \to RL(X)$ be continuous for any $t \in J$ operator-function.

Definition 2.1 ([16]). A function $u(\cdot)\colon J \to X$ is said to be ψ-bounded on J if $\psi(t)u(t)$ is bounded on J.

Let $C_\psi(X)$ denote the Banach space of all ψ-bounded and continuous functions with values in X with the norm

$$|||f|||_{C_\psi} = \sup_{t\in J} |\psi(t)f(t)|.$$

Definition 2.2 ([16]). The equation (2.2) is said to have a ψ-exponential dichotomy on J if there exist a pair of mutually complementary projections P_1 and $P_2 = I - P_1$ and positive constants N_1, N_2, ν_1, ν_2 such that

$$||\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)|| \le N_1 e^{-\nu_1(t-s)} \qquad (s \le t;\ s,t \in J) \tag{2.4}$$

$$||\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)|| \le N_2 e^{-\nu_2(s-t)} \qquad (t \le s;\ s,t \in J) \tag{2.5}$$

The equation (2.2) is said to have a ψ-ordinary dichotomy on J if (2.4) and (2.5) hold with $\nu_1 = \nu_2 = 0$.

Remark 2.3. For $\psi(t) = I$ for all $t \in J$ we obtain the notion of exponential and ordinary dichotomy in [5,7,17].

Let us introduce the principal Green function of (2.3) with the projections P_1 and P_2 from the definition for ψ-exponential dichotomy

$$G(t,s) = \begin{cases} V(t)P_1V^{-1}(s) & (t > s;\ t,s \in J) \\ -V(t)P_2V^{-1}(s) & (t < s;\ t,s \in J). \end{cases} \tag{2.6}$$

Clearly G is continuous except at $t = s$ where it has a jump discontinuity.

Definition 2.4. Let $r > 0$ be an arbitrary number. We say that the conditions (H) are fulfilled if there exist positive functions $m(t), k(t)$ such that

H1. $|\psi(t)F(t,x)| \leq m(t) \qquad (|\psi(t)x| \leq r,\ t \in J)$

H2. $|\psi(t)(F(t,x_1) - F(t,x_2))| \leq k(t)|\psi(t)(x_1 - x_2)| \quad (|\psi(t)x_1|, |\psi(t)x_2| \leq r,\ t \in J)$

Definition 2.5. The nonnegative function $m(t)$ is said to be integrally bounded on J if the following inequality holds:

$$B(m(t)) = \sup_{t\in J} \int_t^{t+1} m(s)\,\mathrm{d}s < \infty.$$

Definition 2.6. We say that the function $F(t,x)$ belongs to the class $ED_\psi(a_1, a_2, r)$ if the conditions (H) are fulfilled, the functions $m(t), k(t)$ are integrally bounded on J and $B(m(t)) \leq a_1, B(k(t)) \leq a_2$.

For each integrable on J function $m(t)$ we introduce the notation

$$L(m(t)) = \int_J m(s)\,\mathrm{d}s.$$

Definition 2.7. We say that the function $F(t,x)$ belongs to the class $D_\psi(a_1, a_2, r)$ if the conditions (H) are fulfilled, the functions $m(t), k(t)$ are integrable on J and $L(m(t)) \leq a_1, L(k(t)) \leq a_2$.

3 Main results

Theorem 3.1. *Let the following conditions be fulfilled:*

1. The linear part of (2.1) has ψ-exponential dichotomy on $\mathbb{R}$ with projections P_1 and P_2.

2. The function $F(t,x)$ belongs to the class $ED_\psi(a_1, a_2, r)$.

Then for an arbitrary $r > 0$ for sufficient small values of a_1, a_2 the equation (2.1) has a unique solution $x(t)$, which is defined for $t \in \mathbb{R}$ and for which $|\psi(t)x(t)| \leq r$ $(t \in \mathbb{R})$.

Proof. Let $J = \mathbb{R}$. We consider in the space $C_\psi(X)$ the operator $Q\colon C_\psi(X) \to C_\psi(X)$ defined by the formula

$$Qx(t) = \int_J G(t,\tau)F(\tau, x(\tau))\,\mathrm{d}\tau \tag{3.1}$$

where G is defined by (2.6).

Let $x(t)$ be a solution of equation (2.1) that remains for $t \in J$ in the ball

$$S_{\psi,r} = \{x : |||x|||_{C_\psi} \leq r\}.$$

Then the function $F(t, x(t))$ is ψ-bounded on J and it follows (see [16, Theorem 3.6]) that such solution satisfies the integral equation

$$x(t) = Qx(t). \tag{3.2}$$

The converse is also true: a solution of the integral equation (3.2) which remains for $t \in J$ in the ball $S_{\psi,r}$ satisfies the differential equation (2.1) for $t \in J$.

Now we shall show that the ball $S_{\psi,r}$ is invariant with respect to Q and the operator Q is contracting.

First we shall prove that the operator Q maps the ball $S_{\psi,r}$ into itself. Indeed we have

$$|\psi(t)Qx(t)| \leq \left|\psi(t)\int_J G(t,\tau)F(\tau,x(\tau))\,\mathrm{d}\tau\right|.$$

We have

$$\begin{aligned}
|\psi(t)Qx(t)| &\leq \left|\psi(t)\int_J G(t,\tau)F(\tau,x(\tau))\,\mathrm{d}\tau\right| \\
&\leq \int_J \|\psi(t)G(t,\tau)\psi^{-1}(\tau)\|\,|\psi(\tau)F(\tau,x(\tau))|\,\mathrm{d}\tau \\
&= \int_{t\leq\tau} \|\psi(t)G(t,\tau)\psi^{-1}(\tau)\|\,|\psi(\tau)F(\tau,x(\tau))|\,\mathrm{d}\tau \\
&\quad + \int_{t\geq\tau} \|\psi(t)G(t,\tau)\psi^{-1}(\tau)\|\,|\psi(\tau)F(\tau,x(\tau))|\,\mathrm{d}\tau \\
&\leq N_2\int_{t\leq\tau} e^{-\nu_2(\tau-t)}m(\tau)\,\mathrm{d}\tau + N_1\int_{t\geq\tau} e^{-\nu_1(t-\tau)}m(\tau)\,\mathrm{d}\tau \\
&\leq N_2\int_{s\geq 0} e^{-\nu_2 s}m(t+s)\,\mathrm{d}s + N_1\int_{s\leq 0} e^{\nu_1 s}m(t+s)\,\mathrm{d}s \\
&\leq N_2 a_1\sum_{k=0}^{\infty} e^{-\nu_2 k} + N_1 a_1\sum_{k=0}^{\infty} e^{-\nu_1 k} = \frac{N_2 a_1}{1-e^{-\nu_2}} + \frac{N_1 a_1}{1-e^{-\nu_1}}.
\end{aligned}$$

Hence by $a_1 \leq r\left(\frac{N_2}{1-e^{-\nu_2}} + \frac{N_1}{1-e^{-\nu_1}}\right)^{-1}$ we obtain

$$\left|\psi(t)\int_J G(t,\tau)F(\tau,x(\tau))\,\mathrm{d}\tau\right| \leq r.$$

Thus the operator Q maps the ball $S_{\psi,r}$ into itself.

Now we shall prove that the operator Q is a contraction in the ball $S_{\psi,r}$. Let $x_1, x_2 \in S_{\psi,r}$. We obtain

$$\begin{aligned}
|\psi(t)Qx_1(t) - \psi(t)Qx_2(t)| &\leq \left|\psi(t)\int_J G(t,\tau)(F(\tau,x_1(\tau)) - F(\tau,x_2(\tau))\,\mathrm{d}\tau\right| \\
&\leq \int_J \left\|\psi(t)G(t,\tau)\psi^{-1}(\tau)\right\|\,|\psi(\tau)(F(\tau,x_1(\tau)) - F(\tau,x_2(\tau))|\,\mathrm{d}\tau \\
&\leq \int_J \left\|\psi(t)G(t,\tau)\psi^{-1}(\tau)\right\|\,k(\tau)|\psi(\tau)(x_1(\tau)) - x_2(\tau))|\,\mathrm{d}\tau \\
&\leq \int_J \left\|\psi(t)G(t,\tau)\psi^{-1}(\tau)\right\|\,k(\tau)\,\mathrm{d}\tau\,\sup_{\tau\in J}|\psi(\tau)(x_1(\tau)) - x_2(\tau))| \\
&\leq \left(\frac{N_2 a_2}{1-e^{-\nu_2}} + \frac{N_1 a_2}{1-e^{-\nu_1}}\right)\sup_{\tau\in J}|\psi(\tau)(x_1(\tau)) - x_2(\tau))|.
\end{aligned}$$

Hence

$$|||Qx_1 - Qx_2|||_{C_\psi} \le \left(\frac{N_2 a_2}{1 - e^{-\nu_2}} + \frac{N_1 a_2}{1 - e^{-\nu_1}} \right) |||x_1 - x_2|||_{C_\psi}.$$

Thus by $a_2 < \left(\frac{N_2}{1-e^{-\nu_2}} + \frac{N_1}{1-e^{-\nu_1}} \right)^{-1}$ the operator Q is a contraction in the ball $S_{\psi,r}$.

From Banach's fixed point principle the existence of a unique fixed point of the operator Q follows. □

Corollary 3.2. *If the conditions of Theorem 3.1 are fulfilled and if, moreover, $F(t,0) = 0$ $(t \in \mathbb{R})$ then $x = 0$ is a unique solution of (2.1) in $C_\psi(X)$.*

Proof. Let $F(t,0) = 0$ $(t \in \mathbb{R})$. Then from H2 it follows

$$|\psi(t)F(t, x(t))| \le k(t)|\psi(t)x(t)| \quad (t \in \mathbb{R}).$$

Thus every solution $x(t)$ except $x(t) \equiv 0$ $(t \in \mathbb{R})$ will leave any ball S_{ψ,r_1} $(r_1 < r)$ by $t \to \infty$ or $t \to -\infty$. □

Theorem 3.3. *Let the following conditions be fulfilled:*

1. The linear part of (2.1) has ψ-ordinary dichotomy on $\mathbb{R}$ with projections P_1 and P_2.

2. The function $F(t,x)$ belongs to the class $D_\psi(a_1, a_2, r)$.

Then for each $r > 0$ for sufficient small values of a_1, a_2 the equation (2.1) has a unique solution $x(t)$, which is defined for $t \in \mathbb{R}$ and for which $|\psi(t)x(t)| \le r$ $(t \in \mathbb{R})$.

Proof. Let $J = \mathbb{R}$. In the proof of Theorem 3.1 it was mentioned that each solution $x(t)$ of equation (2.1) that remains for $t \in J$ in the ball $S_{\psi,r}$ satisfies the integral equation

$$x(t) = \int_J G(t,\tau)F(\tau, x(\tau))\mathrm{d}\tau$$

and vice versa.

We consider again in the space $C_\psi(X)$ the operator $Q\colon C_\psi(X) \to C_\psi(X)$ defined in (3.1).

For $|\psi(t)Qx(t)|$ we obtain the following estimate:

$$|\psi(t)Qx(t)| \le \left| \psi(t) \int_J G(t,\tau)F(\tau, x(\tau))\mathrm{d}\tau \right|.$$

With $a_1 \le r \max\{N_1, N_2\}$ we have

$$\begin{aligned}
|\psi(t)Qx(t)| &\le \left| \psi(t) \int_J G(t,\tau)F(\tau, x(\tau))\,\mathrm{d}\tau \right| \\
&\le \int_J \left\| \psi(t)G(t,\tau)\psi^{-1}(\tau) \right\| \, |\psi(\tau)F(\tau, x(\tau))|\,\mathrm{d}\tau \\
&= \int_{t\le\tau} \left\| \psi(t)G(t,\tau)\psi^{-1}(\tau) \right\| \, |\psi(\tau)F(\tau, x(\tau))|\,\mathrm{d}\tau \\
&\quad + \int_{t\ge\tau} \left\| \psi(t)G(t,\tau)\psi^{-1}(\tau) \right\| \, |\psi(\tau)F(\tau, x(\tau))|\,\mathrm{d}\tau \\
&\le N_2 \int_{t\le\tau} m(\tau)\,\mathrm{d}\tau + N_1 \int_{t\ge\tau} m(\tau)\,\mathrm{d}\tau \\
&\le \max\{N_1, N_2\} \int_J m(\tau)\,\mathrm{d}\tau \le \max\{N_1, N_2\}a_1 \le r.
\end{aligned}$$

Thus the operator Q maps the ball $S_{\psi,r}$ into itself.

Now we shall prove that the operator Q is a contraction in the ball $S_{\psi,r}$. Let $x_1, x_2 \in S_{\psi,r}$. We obtain

$$\begin{aligned}|\psi(t)Qx_1(t) - \psi(t)Qx_2(t)| &\le \left|\psi(t)\int_J G(t,\tau)(F(\tau,x_1(\tau)) - F(\tau,x_2(\tau))\,\mathrm{d}\tau\right| \\ &\le \int_J \left\|\psi(t)G(t,\tau)\psi^{-1}(\tau)\right\| \; |\psi(\tau)(F(\tau,x_1(\tau)) - F(\tau,x_2(\tau))|\,\mathrm{d}\tau \\ &\le \int_J \left\|\psi(t)G(t,\tau)\psi^{-1}(\tau)\right\| \; k(\tau)|\psi(\tau)(x_1(\tau)) - x_2(\tau))|\,\mathrm{d}\tau \\ &\le \int_J \left\|\psi(t)G(t,\tau)\psi^{-1}(\tau)\right\| \; k(\tau)\,\mathrm{d}\tau \;\sup_{\tau\in J}|\psi(\tau)(x_1(\tau)) - x_2(\tau))| \\ &\le (\max\{N_1,N_2\}a_2)\sup_{\tau\in J}|\psi(\tau)(x_1(\tau)) - x_2(\tau))|.\end{aligned}$$

Hence

$$|||Qx_1 - Qx_2|||_{C_\psi} \le (a_2 \max\{N_1,N_2\})\,|||x_1 - x_2|||_{C_\psi}.$$

Thus by $a_2 < (\max\{N_1,N_2\})^{-1}$ the operator Q is a contraction in the ball $S_{\psi,r}$.

From Banach's fixed point principle the existence of a unique fixed point of the operator Q follows. □

Theorem 3.4. *Let the following conditions be fulfilled:*

1. The linear part of (2.1) *has ψ-exponential dichotomy on $\mathbb{R}_+$ with projections P_1 and P_2.*

2. The function $F(t,x)$ belongs to the class $ED_\psi(a_1,a_2,r)$.

Then for any $r > 0$ by sufficient small a_1, a_2 there exists $\rho < r$ such that the equation (2.1) *has for each $\xi \in X_1 = P_1X$ with $|\psi(0)\xi| \le \rho$ a unique solution $x(t)$ on $\mathbb{R}_+$ for which $P_1x(0) = \xi$ and $|\psi(t)x(t)| \le r \;\; (t \in \mathbb{R}_+)$.*

Proof. Let $J = \mathbb{R}_+$ and $x(t)$ be a solution of equation (2.1) that remains for $t \in J$ in the ball $S_{\psi,r} = \{x : |||x|||_{C_\psi} \le r\}$. From the results obtained in [16, Theorem 3.6 and Remark 3.8] it follows that such $x(t)$ satisfies the integral equation

$$x(t) = V(t)\xi + \int_J G(t,\tau)F(\tau,x(\tau))\,\mathrm{d}\tau \tag{3.3}$$

where $\xi = P_1x(0)$. The converse is also true: a solution of the integral equation (3.3) satisfies the differential equation (2.1) for $t \in J$.

Let $\xi \in X_1$ and $|\psi(0)\xi| \le \rho < r$. We consider in the space $C_\psi(X)$ the operator $Q\colon C_\psi(X) \to C_\psi(X)$ defined by the formula

$$Qx(t) = V(t)\xi + \int_J G(t,\tau)F(\tau,x(\tau))\,\mathrm{d}\tau \tag{3.4}$$

First we shall prove, that the operator Q maps the ball $S_{\psi,r}$ into itself. Indeed we have

$$|\psi(t)Qx(t)| \le |\psi(t)V(t)\xi| + \left|\psi(t)\int_J G(t,\tau)F(\tau,x(\tau))\,\mathrm{d}\tau\right|.$$

For the first addend with $\rho \le \frac{r}{2N_1}$ we obtain

$$|\psi(t)V(t)\xi| \le N_1e^{-\nu_1 t}|\psi(0)\xi| \le N_1e^{-\nu_1 t}\rho \le \frac{r}{2}.$$

Using the same technique and notations as in the proof of Theorem 3.1 we obtain for the second addend the estimate

$$\left|\psi(t)\int_J G(t,\tau)F(\tau,x(\tau))\,\mathrm{d}\tau\right| \le \frac{N_2a_1}{1-e^{-\nu_2}} + \frac{N_1a_1}{1-e^{-\nu_1}}.$$

Hence by $a_1 \le \frac{r}{2}\left(\frac{N_2}{1-e^{-\nu_2}} + \frac{N_1}{1-e^{-\nu_1}}\right)^{-1}$ we obtain

$$\left|\psi(t)\int_J G(t,\tau)F(\tau,x(\tau))\,\mathrm{d}\tau\right| \le \frac{r}{2}.$$

Thus the operator Q maps the ball $S_{\psi,r}$ into itself.

Now we shall prove that the operator Q is a contraction in the ball $S_{\psi,r}$. Let $x_1, x_2 \in S_{\psi,r}$. We obtain as in the proof of Theorem 3.1 the estimate

$$|||Qx_1 - Qx_2|||_{C_\psi} \le \left(\frac{N_2a_2}{1-e^{-\nu_2}} + \frac{N_1a_2}{1-e^{-\nu_1}}\right) |||x_1 - x_2|||_{C_\psi}$$

By $a_2 < \left(\frac{N_2}{1-e^{-\nu_2}} + \frac{N_1}{1-e^{-\nu_1}}\right)^{-1}$ the operator Q is a contraction in the ball $S_{\psi,r}$.

From Banach's fixed point principle the existence of a unique fixed point of the operator Q follows. □

Theorem 3.5. *Let the following conditions be fulfilled:*

1. The linear part of (2.1) has ψ-ordinary dichotomy on $\mathbb{R}_+$ with projections P_1 and P_2.

2. The function $F(t,x)$ belongs to the class $D_\psi(a_1,a_2,r)$.

Then for any $r > 0$ by sufficiently small a_1, a_2 there exists $\rho < r$ such that the equation (2.1) has for each $\xi \in X_1 = P_1X$ with $|\psi(0)\xi| \le \rho$ a unique solution $x(t)$ on $\mathbb{R}_+$ for which $P_1x(0) = \xi$ and $|\psi(t)x(t)| \le r \ (t \in \mathbb{R}_+)$.

Proof. Let $J = \mathbb{R}_+$, $\xi \in X_1$ and $|\psi(0)\xi| \le \rho < r$. We consider again in the space $C_\psi(X)$ the operator $Q\colon C_\psi(X) \to C_\psi(X)$ defined by the formula (3.4).

First we shall prove, that the operator Q maps the ball $S_{\psi,r}$ into itself. We have

$$|\psi(t)Qx(t)| \le |\psi(t)V(t)\xi| + \left|\psi(t)\int_J G(t,\tau)F(\tau,x(\tau))\,\mathrm{d}\tau\right|.$$

For the first addend with $\rho \le \frac{r}{2N_1}$ we obtain

$$|\psi(t)V(t)\xi| \le N_1|\psi(0)\xi| \le N_1\rho \le \frac{r}{2}.$$

For the second addend with $a_1 \le \frac{r}{2\max\{N_1,N_2\}}$ as in the proof of Theorem 3.3 we have

$$\left|\psi(t)\int_J G(t,\tau)F(\tau,x(\tau))\,\mathrm{d}\tau\right| \le \max\{N_1,N_2\}a_1 \le \frac{r}{2}.$$

Thus the operator Q maps the ball $S_{\psi,r}$ into itself.

Let $x_1, x_2 \in S_{\psi,r}$. As in the proof of Theorem 3.3 we obtain the estimate

$$|||Qx_1 - Qx_2|||_{C_\psi} \le (a_2 \max\{N_1,N_2\}) \, |||x_1 - x_2|||_{C_\psi}.$$

Hence by $a_2 < (\max\{N_1,N_2\})^{-1}$ the operator Q is a contraction in the ball $S_{\psi,r}$.

From the fixed point principle of Banach it follows the existence of a unique fixed point of the operator Q.

In the proof of Theorem 3.4 it was already mentioned that every solution of the differential equation (2.1) which lies in the ball $S_{\psi,r}$ fulfil the equality

$$x(t) = Qx(t)$$

and vice versa. □

Corollary 3.6. *Let the conditions of Theorem 3.5 hold and let $x_1(t)$ and $x_2(t)$ be two solutions whose initial values fulfil $P_1x_1(0) = \xi$ and $P_1x_2(0) = \eta$. Let $N = \max\{N_1, N_2\}$.*

Then for $Na_2 < 1$ the following estimate holds

$$|\psi(t)(x_1(t) - x_2(t))| \leq \frac{N}{1 - Na_2}|\psi(0)(\xi - \eta)| \quad (t \in \mathbb{R}_+).$$

Proof. Applying the presentation (3.3) for the solutions x_1 and x_2 we obtain

$$x_1(t) - x_2(t) = V(t)(\xi - \eta) + \int_0^\infty G(t,\tau)(F(\tau, x_1(\tau)) - F(\tau, x_2(\tau)))\,\mathrm{d}\tau.$$

From here and the conditions of Theorem 3.5 for $u(t) = \psi(t)(x_1(t) - x_2(t))$ we obtain

$$|u(t)| \leq N|\psi(0)(\xi - \eta)| + N\int_0^\infty k(\tau)u(\tau)\,\mathrm{d}\tau.$$

Let us consider the equation

$$u(t) = \alpha + N\int_0^\infty k(\tau)u(\tau)\,\mathrm{d}\tau, \tag{3.5}$$

where $\alpha = N|\psi(0)(\xi - \eta)|$. Let us introduce the functional $\Phi\colon C \to \mathbb{R}_+$, where C is the space of all bounded functions on $\mathbb{R}_+$ with values in $\mathbb{R}_+$ by the formula

$$(\Phi u)(t) = N\int_0^\infty k(\tau)u(\tau)\,\mathrm{d}\tau.$$

For the norm of Φ we obtain the estimate

$$\|\Phi\| \leq N\int_0^\infty k(\tau)\,\mathrm{d}\tau \leq Na_2.$$

For sufficiently small a_2 we have $\|\Phi\| \leq 1$.

Let I_C be the identity of the space C. Then the equation $(I_C - \Phi)u = \alpha$ has a bounded solution $u(t)$, i.e. there exists a constant $c = \sup_{t\in\mathbb{R}_+}|u(t)| < \infty$. We shall estimate the constant c from equation (3.5):

$$c \leq \alpha + Nc\int_0^\infty k(\tau)\,\mathrm{d}\tau \leq \alpha + Nca_2,$$

i.e.

$$c \leq \frac{\alpha}{1 - Na_2}.$$

Finally we obtain

$$|\psi(t)(x_1(t) - x_2(t))| \leq \frac{N|\psi(0)(\xi - \eta)|}{1 - Na_2}.$$

□

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