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Positive solutions for systems of second-order integral boundary value problems

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Abstract

We investigate the existence and nonexistence of positive solutions of a system of second-order nonlinear ordinary differential equations, subject to integral boundary conditions.
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1 Introduction

Boundary value problems with positive solutions describe many phenomena in the applied sciences such as the nonlinear diffusion generated by nonlinear sources, thermal ignition of gases and concentration in chemical or biological problems. Problems with integral boundary conditions arise in thermal conduction problems, semiconductor problems and hydrodynamic problems. In the last decades, many authors studied various problems with integral boundary conditions. In the paper [17], by using the fixed point index theory for compact maps, the authors investigated the existence and multiplicity of positive solutions for general systems of perturbed Hammerstein integral equations. We also mention the recent papers [2], [3], [5],

[16], [18], [19], [22], [23], where the authors studied the existence of positive solutions for some systems of boundary value problems with integral conditions.

In this paper, we consider the system of nonlinear second-order ordinary differential equations

$$(S) \qquad \begin{cases} (a(t)u'(t))' - b(t)u(t) + \lambda p(t) f(t, u(t), v(t)) = 0, & 0 < t < 1, \\ (c(t)v'(t))' - d(t)v(t) + \mu q(t) g(t, u(t), v(t)) = 0, & 0 < t < 1, \end{cases}$$

with the integral boundary conditions

$$(BC) \qquad \begin{cases} \alpha u(0) - \beta a(0)u'(0) = \int_0^1 u(s)\, dH_1(s), & \gamma u(1) + \delta a(1)u'(1) = \int_0^1 u(s)\, dH_2(s), \\ \widetilde{\alpha} v(0) - \widetilde{\beta} c(0)v'(0) = \int_0^1 v(s)\, dK_1(s), & \widetilde{\gamma} v(1) + \widetilde{\delta} c(1)v'(1) = \int_0^1 v(s)\, dK_2(s), \end{cases}$$

where the above integrals are Riemann–Stieltjes integrals.

We shall give sufficient conditions on λ, μ, f and g such that positive solutions of problem $(S)-(BC)$ exist. By a positive solution of problem $(S)-(BC)$ we mean a pair of functions $(u,v) \in C^2([0,T]) \times C^2([0,T])$ satisfying (S) and (BC) with $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0,T]$ and $(u,v) \neq (0,0)$. We also present a nonexistence result for the positive solutions of the above problem. The case in which the functions H_1, H_2, K_1, K_2 are scale functions, that is the boundary conditions (BC) become multi-point boundary conditions

$$(BC_1) \qquad \begin{cases} \alpha u(0) - \beta a(0)u'(0) = \sum_{i=1}^{m} a_i u(\xi_i), & \gamma u(1) + \delta a(1)u'(1) = \sum_{i=1}^{n} b_i u(\eta_i), \\ \widetilde{\alpha} v(0) - \widetilde{\beta} c(0)v'(0) = \sum_{i=1}^{r} c_i v(\zeta_i), & \widetilde{\gamma} v(1) + \widetilde{\delta} c(1)v'(1) = \sum_{i=1}^{l} d_i v(\rho_i), \end{cases}$$

where m, n, r, $l \in \mathbb{N}$, has been studied in [10] (with $T = 1$). The system (S) with $a(t) = 1$, $c(t) = 1$, $b(t) = 0$, $d(t) = 0$ for all $t \in [0,T]$, $f(t,u,v) = \widetilde{f}(u,v)$, $g(t,u,v) = \widetilde{g}(u,v)$ (denoted by (S_1)) and (BC_1) was investigated in [11]. Some particular cases of the problem from [11] have been studied in [6] (where in (BC_1), $a_i = 0$ for all $i = 1, \dots, m$, $c_i = 0$ for all $i = 1, \dots, r$, $\gamma = \widetilde{\gamma} = 1$ and $\delta = \widetilde{\delta} = 0$ (denoted by (BC_2)), in [20] (where in (S_1), $\widetilde{f}(u,v) = \widetilde{\widetilde{f}}(v)$, $\widetilde{g}(u,v) = \widetilde{\widetilde{g}}(u)$ – denoted by (S_2), and in (BC_2) we have $n = l$, $b_i = d_i$, $\eta_i = \rho_i$ for all $i = 1, \dots, n$, $\alpha = \widetilde{\alpha}$ and $\beta = \widetilde{\beta}$ – denoted by (BC_3)), in [14] (the problem $(S_2)-(BC_3)$ with $\alpha = \widetilde{\alpha} = 1$, $\beta = \widetilde{\beta} = 0$, $T = 1$), in [12] and [15] (the system (S_2) with $T = 1$ and the boundary conditions $u(0) = 0$, $u(1) = \alpha u(\eta)$, $v(0) = 0$, $v(1) = \alpha v(\eta)$, $\eta \in (0,1)$ and $0 < \alpha < 1/\eta$, or $u(0) = \beta u(\eta)$, $u(1) = \alpha u(\eta)$, $v(0) = \beta v(\eta)$, $v(1) = \alpha v(\eta)$). In [13], the authors investigated the system (S_2) with $T = 1$ and the boundary conditions $\alpha u(0) - \beta u'(0) = 0$, $\gamma u(1) + \delta u'(1) = 0$, $\alpha v(0) - \beta v'(0) = 0$, $\gamma v(1) + \delta v'(1) = 0$, where α, β, γ, $\delta \geq 0$, $\alpha + \beta + \gamma + \delta > 0$. The existence of positive solutions for a second-order singular system of multi-point boundary value problems was studied in [8]. The multiplicity of positive solutions of some systems of multi-point boundary value problems has been investigated in [7], [9].

In Section 2, we shall present some auxiliary results which investigate a boundary value problem for second-order equations. In Section 3, we shall prove two existence theorems for the positive solutions with respect to a cone for our problem $(S)-(BC)$, which are based on the Guo–Krasnosel'skii fixed point theorem (see [4]) which we present now.

Theorem 1.1 *Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ be a completely continuous operator such that, either*

i) $\|\mathcal{A}u\| \leq \|u\|,\ \ u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \geq \|u\|,\ \ u \in C \cap \partial\Omega_2$, *or*

ii) $\|\mathcal{A}u\| \geq \|u\|,\ \ u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \leq \|u\|,\ \ u \in C \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

The nonexistence of positive solutions of $(S)-(BC)$ is also studied in this section. Finally, an example is presented in Section 4 to illustrate our main results.

2 Auxiliary results

In this section, we shall present some auxiliary results, related to the following second-order differential equation with integral boundary conditions

$$(a(t)u'(t))' - b(t)u(t) + y(t) = 0,\ \ t \in (0,1), \tag{1}$$

$$\alpha u(0) - \beta a(0)u'(0) = \int_0^1 u(s)\,dH_1(s),\ \ \gamma u(1) + \delta a(1)u'(1) = \int_0^1 u(s)\,dH_2(s). \tag{2}$$

For $a \in C^1([0,1],(0,\infty))$, $b \in C([0,1],[0,\infty))$, $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$, $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$, we denote by ψ and ϕ the solutions of the following linear problems

$$\begin{cases} (a(t)\psi'(t))' - b(t)\psi(t) = 0,\ \ 0 < t < 1, \\ \psi(0) = \beta,\ \ a(0)\psi'(0) = \alpha, \end{cases} \tag{3}$$

and

$$\begin{cases} (a(t)\phi'(t))' - b(t)\phi(t) = 0,\ \ 0 < t < 1, \\ \phi(1) = \delta,\ \ a(1)\phi'(1) = -\gamma, \end{cases} \tag{4}$$

respectively.

We denote by θ_1 the function $\theta_1(t) = a(t)(\phi(t)\psi'(t) - \phi'(t)\psi(t))$ for $t \in [0,1]$. By using the equations (3) and (4), we deduce that $\theta_1'(t) = 0$, that is $\theta_1(t) = \text{const.}$, for all $t \in [0,1]$. We denote this constant by τ_1. Then $\theta_1(t) = \tau_1$ for all $t \in [0,1]$, and so $\tau_1 = \theta_1(0) = a(0)(\phi(0)\psi'(0) - \phi'(0)\psi(0)) = \alpha\phi(0) - \beta a(0)\phi'(0)$ and $\tau_1 = \theta_1(1) = a(1)(\phi(1)\psi'(1) - \phi'(1)\psi(1)) = \delta a(1)\psi'(1) + \gamma\psi(1)$.

Lemma 2.1 *We assume that $a \in C^1([0,1],(0,\infty))$, $b \in C([0,1],[0,\infty))$, α, β, γ, $\delta \in \mathbb{R}$, $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$, and H_1, $H_2 : [0,1] \to \mathbb{R}$ are functions of bounded variation. If $\tau_1 \neq 0$, $\Delta_1 = \left(\tau_1 - \int_0^1 \psi(s)\, dH_2(s)\right)\left(\tau_1 - \int_0^1 \phi(s)\, dH_1(s)\right) - \left(\int_0^1 \psi(s)\, dH_1(s)\right)\left(\int_0^1 \phi(s)\, dH_2(s)\right) \neq 0$, and $y \in C([0,1])$, then the (unique) solution of (1)-(2) is given by $u(t) = \int_0^1 G_1(t,s)y(s)\, ds$, where the Green's function G_1 is defined by*

$$\begin{aligned} G_1(t,s) =& g_1(t,s) \\ &+ \frac{1}{\Delta_1}\left[\psi(t)\left(\int_0^1 \phi(s)\, dH_2(s)\right) + \phi(t)\left(\tau_1 - \int_0^1 \psi(s)\, dH_2(s)\right)\right]\int_0^1 g_1(\tau,s)\, dH_1(\tau) \\ &+ \frac{1}{\Delta_1}\left[\psi(t)\left(\tau_1 - \int_0^1 \phi(s)\, dH_1(s)\right) + \phi(t)\left(\int_0^1 \psi(s)\, dH_1(s)\right)\right]\int_0^1 g_1(\tau,s)\, dH_2(\tau), \end{aligned} \tag{5}$$

for all $(t,s) \in [0,1] \times [0,1]$, where

$$g_1(t,s) = \frac{1}{\tau_1}\begin{cases} \phi(t)\psi(s), & 0 \le s \le t \le 1, \\ \phi(s)\psi(t), & 0 \le t \le s \le 1, \end{cases} \tag{6}$$

and ψ, ϕ are the functions defined by (3) and (4), respectively.

Proof. Because $\tau_1 \neq 0$, the functions ψ and ϕ are two linearly independent solutions of the equation $(a(t)u'(t))' - b(t)u(t) = 0$. Then the general solution of equation (1) is $u(t) = A\psi(t) + B\phi(t) + u_0(t)$, with A, $B \in \mathbb{R}$ and u_0 is a particular solution of (1). We shall determine u_0 by the method of variation of constants, namely we look for two functions $C(t)$ and $D(t)$ such that $u_0(t) = C(t)\psi(t) + D(t)\phi(t)$ is a solution of equation (1). The derivatives of $C(t)$ and $D(t)$ satisfy the system

$$\begin{cases} C'(t)\psi(t) + D'(t)\phi(t) = 0, \\ C'(t)a(t)\psi'(t) + D'(t)a(t)\phi'(t) = -y(t), \quad t \in (0,1). \end{cases}$$

The above system has the determinant $d_0 = -\tau_1 \neq 0$, and the solution of the above system is $C'(t) = -\frac{1}{\tau_1}\phi(t)y(t)$ and $D'(t) = \frac{1}{\tau_1}\psi(t)y(t)$. Then we choose $C(t) = \frac{1}{\tau_1}\int_t^1 \phi(s)y(s)\, ds$ and $D(t) = \frac{1}{\tau_1}\int_0^t \psi(s)y(s)\, ds$. We deduce that the general solution of equation (1) is

$$u(t) = \frac{\psi(t)}{\tau_1}\int_t^1 \phi(s)y(s)\, ds + \frac{\phi(t)}{\tau_1}\int_0^t \psi(s)y(s)\, ds + A\psi(t) + B\phi(t).$$

Then we obtain

$$u(t) = \int_0^1 g_1(t,s)y(s)\,ds + A\psi(t) + B\phi(t),$$

where g_1 is defined in (6).

By using the conditions (2), we conclude

$$\begin{cases} \alpha\left(\frac{1}{\tau_1}\int_0^1 \psi(0)\phi(s)y(s)\,ds + A\psi(0) + B\phi(0)\right) - \beta a(0)\left(\frac{1}{\tau_1}\int_0^1 \phi(s)\psi'(0)y(s)\,ds\right. \\ \left. + A\psi'(0) + B\phi'(0)\right) = \int_0^1 \left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau + A\psi(s) + B\phi(s)\right) dH_1(s), \\ \gamma\left(\frac{1}{\tau_1}\int_0^1 \phi(1)\psi(s)y(s)\,ds + A\psi(1) + B\phi(1)\right) + \delta a(1)\left(\frac{1}{\tau_1}\int_0^1 \phi'(1)\psi(s)y(s)\,ds\right. \\ \left. + A\psi'(1) + B\phi'(1)\right) = \int_0^1 \left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau + A\psi(s) + B\phi(s)\right) dH_2(s), \end{cases}$$

or

$$\begin{cases} A\left(-\alpha\psi(0) + \beta a(0)\psi'(0) + \int_0^1 \psi(s)\,dH_1(s)\right) + B\left(-\alpha\phi(0) + \beta a(0)\phi'(0) + \int_0^1 \phi(s)\,dH_1(s)\right) \\ = \frac{\alpha}{\tau_1}\int_0^1 \phi(s)\psi(0)y(s)\,ds - \frac{\beta a(0)}{\tau_1}\int_0^1 \phi(s)\psi'(0)y(s)\,ds - \int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right) dH_1(s), \\ A\left(\gamma\psi(1) + \delta a(1)\psi'(1) - \int_0^1 \psi(s)\,dH_2(s)\right) + B\left(\gamma\phi(1) + \delta a(1)\phi'(1) - \int_0^1 \phi(s)\,dH_2(s)\right) \\ = -\frac{\gamma}{\tau_1}\int_0^1 \phi(1)\psi(s)y(s)\,ds - \frac{\delta a(1)}{\tau_1}\int_0^1 \phi'(1)\psi(s)y(s)\,ds + \int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right) dH_2(s). \end{cases}$$

Therefore, we obtain

$$\begin{cases} A\left(\int_0^1 \psi(s)\,dH_1(s)\right) + B\left(-\tau_1 + \int_0^1 \phi(s)\,dH_1(s)\right) = -\int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right) dH_1(s), \\ A\left(\tau_1 - \int_0^1 \psi(s)\,dH_2(s)\right) + B\left(-\int_0^1 \phi(s)\,dH_2(s)\right) = \int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right) dH_2(s). \end{cases} \tag{7}$$

The above system with the unknown A and B has the determinant

$$\Delta_1 = \left(\tau_1 - \int_0^1 \psi(s)\,dH_2(s)\right)\left(\tau_1 - \int_0^1 \phi(s)\,dH_1(s)\right) - \left(\int_0^1 \psi(s)\,dH_1(s)\right)\left(\int_0^1 \phi(s)\,dH_2(s)\right).$$

By using the assumptions of this lemma, we have $\Delta_1 \neq 0$. Hence, the system (7) has a unique solution, namely

$$\begin{aligned} A =& \frac{1}{\Delta_1}\left[\left(\int_0^1 \phi(s)\,dH_2(s)\right)\left(\int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right) dH_1(s)\right)\right. \\ &\left. + \left(\tau_1 - \int_0^1 \phi(s)\,dH_1(s)\right)\left(\int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right) dH_2(s)\right)\right], \\ B =& \frac{1}{\Delta_1}\left[\left(\int_0^1 \psi(s)\,dH_1(s)\right)\left(\int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right) dH_2(s)\right)\right. \\ &\left. + \left(\tau_1 - \int_0^1 \psi(s)\,dH_2(s)\right)\left(\int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right) dH_1(s)\right)\right]. \end{aligned}$$

Then, the solution of problem (1)–(2) is

$$
\begin{aligned}
u(t) = \int_0^1 g_1(t,s)y(s)\,ds &+ \frac{\psi(t)}{\Delta_1}\left[\left(\int_0^1 \phi(s)\,dH_2(s)\right)\left(\int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right)dH_1(s)\right)\right.\\
&+ \left.\left(\tau_1 - \int_0^1 \phi(s)\,dH_1(s)\right)\left(\int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right)dH_2(s)\right)\right]\\
&+ \frac{\phi(t)}{\Delta_1}\left[\left(\int_0^1 \psi(s)\,dH_1(s)\right)\left(\int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right)dH_2(s)\right)\right.\\
&+ \left.\left(\tau_1 - \int_0^1 \psi(s)\,dH_2(s)\right)\left(\int_0^1\left(\int_0^1 g_1(s,\tau)y(\tau)\,d\tau\right)dH_1(s)\right)\right].
\end{aligned}
$$

Therefore, we deduce

$$
\begin{aligned}
u(t) &= \int_0^1 g_1(t,s)y(s)\,ds + \frac{1}{\Delta_1}\left[\psi(t)\left(\int_0^1 \phi(s)\,dH_2(s)\right) + \phi(t)\left(\tau_1 - \int_0^1 \psi(s)\,dH_2(s)\right)\right]\\
&\quad\times \int_0^1\left(\int_0^1 g_1(s,\tau)\,dH_1(s)\right)y(\tau)\,d\tau + \frac{1}{\Delta_1}\left[\psi(t)\left(\tau_1 - \int_0^1 \phi(s)\,dH_1(s)\right)\right.\\
&\quad+ \left.\phi(t)\int_0^1 \psi(s)\,dH_1(s)\right]\int_0^1\left(\int_0^1 g_1(s,\tau)\,dH_2(s)\right)y(\tau)\,d\tau\\
&= \int_0^1 g_1(t,s)y(s)\,ds + \frac{1}{\Delta_1}\left[\psi(t)\left(\int_0^1 \phi(s)\,dH_2(s)\right) + \phi(t)\left(\tau_1 - \int_0^1 \psi(s)\,dH_2(s)\right)\right]\\
&\quad\times \int_0^1\left(\int_0^1 g_1(\tau,s)\,dH_1(\tau)\right)y(s)\,ds + \frac{1}{\Delta_1}\left[\psi(t)\left(\tau_1 - \int_0^1 \phi(s)\,dH_1(s)\right)\right.\\
&\quad+\left.\phi(t)\int_0^1 \psi(s)\,dH_1(s)\right]\int_0^1\left(\int_0^1 g_1(\tau,s)\,dH_2(\tau)\right)y(s)\,ds.
\end{aligned}
$$

So, the solution u of (1)–(2) is $u(t) = \int_0^1 G_1(t,s)y(s)\,ds,\ \ t \in [0,1]$, where G_1 is given in (5). □

Now, we introduce the assumptions

$(A1)$ $a \in C^1([0,T],(0,\infty))$, $b \in C([0,T],[0,\infty))$.

$(A2)$ $\alpha,\ \beta,\ \gamma,\ \delta \in [0,\infty)$ with $\alpha+\beta>0$ and $\gamma+\delta>0$.

$(A3)$ If $b(t)\equiv 0$, then $\alpha+\gamma>0$.

$(A4)$ $H_1,\ H_2 : [0,1] \to \mathbb{R}$ are nondecreasing functions.

$(A5)$ $\tau_1 - \int_0^1 \phi(s)\,dH_1(s) > 0$, $\tau_1 - \int_0^1 \psi(s)\,dH_2(s) > 0$ and $\Delta_1 > 0$.

Lemma 2.2 *([1]) Let* $(A1)$ *and* $(A2)$ *hold. Then*

a) the function ψ *is nondecreasing on* $[0,1]$, $\psi(t)\ge 0$ *for all* $t\in[0,1]$ *and* $\psi(t)>0$ *on* $(0,1]$*;*

b) the function ϕ *is nonincreasing on* $[0,1]$, $\phi(t)\ge 0$ *for all* $t\in[0,1]$ *and* $\phi(t)>0$ *on* $[0,1)$.

Lemma 2.3 *([1]) Let* $(A1)$ *and* $(A2)$ *hold.*

a) If $b(t)$ *is not identically zero, then* $\tau_1>0$.

b) If $b(t)$ *is identically zero, then* $\tau_1>0$ *if and only if* $\alpha+\gamma>0$.

Lemma 2.4 *Let (A1)–(A3) hold. Then the function g_1 given by (6) has the properties*
a) g_1 is a continuous function on $[0,1] \times [0,1]$.
b) $g_1(t,s) \geq 0$ for all $t,\ s \in [0,1]$, and $g_1(t,s) > 0$ for all $t,\ s \in (0,1)$.
c) For any $\sigma \in (0,1/2)$, we have $\min_{t\in[\sigma,1-\sigma]} g_1(t,s) \geq \nu_1 g_1(s,s)$ for all $s \in [0,1]$, where $\nu_1 = \min\left\{\frac{\phi(1-\sigma)}{\phi(0)}, \frac{\psi(\sigma)}{\psi(1)}\right\}$.

For the proof of Lemma 2.4 a)–b) see [1], and for the proof of Lemma 2.4 c) see [21].

Lemma 2.5 *Let (A1)–(A5) hold. Then the Green's function G_1 of the problem (1)-(2) is continuous on $[0,1] \times [0,1]$ and satisfies $G_1(t,s) \geq 0$ for all $(t,s) \in [0,1] \times [0,1]$. Moreover, if $y \in C([0,1])$ satisfies $y(t) \geq 0$ for all $t \in [0,1]$, then the unique solution u of problem (1)–(2) satisfies $u(t) \geq 0$ for all $t \in [0,1]$.*

Proof. By using the assumptions of this lemma, we deduce $G_1(t,s) \geq 0$ for all $(t,s) \in [0,1] \times [0,1]$, and so $u(t) \geq 0$ for all $t \in [0,1]$. □

Lemma 2.6 *Assume that (A1)–(A5) hold. Then the Green's function G_1 of the problem (1)–(2) satisfies the inequalities*
a) $G_1(t,s) \leq J_1(s),\ \ \forall\,(t,s) \in [0,1] \times [0,1]$, where

$$\begin{aligned} J_1(s) =& g_1(s,s) \\ &+ \frac{1}{\Delta_1}\left[\psi(T)\left(\int_0^1 \phi(s)\,dH_2(s)\right) + \phi(0)\left(\tau_1 - \int_0^1 \psi(s)\,dH_2(s)\right)\right]\int_0^1 g_1(\tau,s)\,dH_1(\tau) \\ &+ \frac{1}{\Delta_1}\left[\psi(T)\left(\tau_1 - \int_0^1 \phi(s)\,dH_1(s)\right) + \phi(0)\left(\int_0^1 \psi(s)\,dH_1(s)\right)\right]\int_0^1 g_1(\tau,s)\,dH_2(\tau). \end{aligned}$$

b) For every $\sigma \in (0,1/2)$, we have

$$\min_{t\in[\sigma,1-\sigma]} G_1(t,s) \geq \nu_1 J_1(s) \geq \nu_1 G_1(t',s),\ \ \forall\, t',\ s \in [0,1],$$

where ν_1 is given in Lemma 2.4.

Proof. The first inequality *a*) is evident. For the second inequality *b*), for $\sigma \in (0,1/2)$ and

$t \in [\sigma, 1-\sigma]$, $s \in [0,1]$, we conclude

$$G_1(t,s) \geq \nu_1 g_1(s,s) + \frac{1}{\Delta_1}\left[\psi(\sigma)\left(\int_0^1 \phi(s)\,dH_2(s)\right) + \phi(1-\sigma)\left(\tau_1 - \int_0^1 \psi(s)\,dH_2(s)\right)\right]$$
$$\times \int_0^1 g_1(\tau,s)\,dH_1(\tau) + \frac{1}{\Delta_1}\left[\psi(\sigma)\left(\tau_1 - \int_0^1 \phi(s)\,dH_1(s)\right) + \phi(1-\sigma)\left(\int_0^1 \psi(s)\,dH_1(s)\right)\right]$$
$$\times \int_0^1 g_1(\tau,s)\,dH_2(\tau)$$
$$= \nu_1 g_1(s,s) + \frac{1}{\Delta_1}\left[\frac{\psi(\sigma)}{\psi(1)}\psi(1)\left(\int_0^1 \phi(s)\,dH_2(s)\right) + \frac{\phi(1-\sigma)}{\phi(0)}\phi(0)\right.$$
$$\left. \times \left(\tau_1 - \int_0^1 \psi(s)\,dH_2(s)\right)\right]\int_0^1 g_1(\tau,s)\,dH_1(\tau) + \frac{1}{\Delta_1}\left[\frac{\psi(\sigma)}{\psi(1)}\psi(1)\left(\tau_1 - \int_0^1 \phi(s)\,dH_1(s)\right)\right.$$
$$\left. + \frac{\phi(1-\sigma)}{\phi(0)}\phi(0)\left(\int_0^1 \psi(s)\,dH_1(s)\right)\right]\int_0^1 g_1(\tau,s)\,dH_2(\tau) \geq \nu_1 J_1(s).$$

□

Lemma 2.7 *Assume that (A1)–(A5) hold and let $\sigma \in (0,1/2)$. If $y \in C([0,1])$, $y(t) \geq 0$ for all $t \in [0,1]$, then the solution $u(t)$, $t \in [0,1]$ of problem (1)–(2) satisfies the inequality* $\min_{t \in [\sigma,1-\sigma]} u(t) \geq \nu_1 \max_{t' \in [0,1]} u(t')$.

Proof. For $\sigma \in (0,1/2)$, $t \in [\sigma, 1-\sigma]$, $t' \in [0,1]$, we have

$$u(t) = \int_0^1 G_1(t,s)y(s)\,ds \geq \nu_1 \int_0^1 J_1(s)y(s)\,ds \geq \nu_1 \int_0^1 G_1(t',s)y(s)\,ds = \nu_1 u(t'),$$

so $\min_{t \in [\sigma,1-\sigma]} u(t) \geq \nu_1 \max_{t' \in [0,1]} u(t')$. □

We can also formulate similar results as Lemmas 2.1–2.7 above for the boundary value problem

$$(c(t)v'(t))' - d(t)v(t) + h(t) = 0, \;\; 0 < t < 1, \tag{8}$$

$$\widetilde{\alpha}v(0) - \widetilde{\beta}c(0)v'(0) = \int_0^1 v(s)\,dK_1(s), \;\; \widetilde{\gamma}v(1) + \widetilde{\delta}c(1)v'(1) = \int_0^1 v(s)\,dK_2(s), \tag{9}$$

under similar assumptions as (*A*1)–(*A*5) and $h \in C([0,1])$. We denote by $\widetilde{\psi}$, $\widetilde{\phi}$, θ_2, τ_2, Δ_2, g_2, G_2, ν_2 and J_2 the corresponding constants and functions for the problem (8)–(9) defined in a similar manner as ψ, ϕ, θ_1, τ_1, Δ_1, g_1, G_1, ν_1 and J_1, respectively.

3 Main results

In this section, we shall give sufficient conditions on λ, μ, f and g such that positive solutions with respect to a cone for our problem $(S)-(BC)$ exist. We shall also investigate the nonexistence of positive solutions of $(S)-(BC)$.

We present the assumptions that we shall use in the sequel.

$(I1)$ The functions $a,\ c \in C^1([0,1],(0,\infty))$ and $b,\ d \in C([0,1],[0,\infty))$.

$(I2)$ $\alpha,\ \beta,\ \gamma,\ \delta,\ \widetilde{\alpha},\ \widetilde{\beta},\ \widetilde{\gamma},\ \widetilde{\delta} \in [0,\infty)$ with $\alpha+\beta>0$, $\gamma+\delta>0$, $\widetilde{\alpha}+\widetilde{\beta}>0$, $\widetilde{\gamma}+\widetilde{\delta}>0$; if $b\equiv 0$ then $\alpha+\gamma>0$; if $d\equiv 0$ then $\widetilde{\alpha}+\widetilde{\gamma}>0$.

$(I3)$ $H_1,\ H_2,\ K_1,\ K_2 : [0,1]\to\mathbb{R}$ are nondecreasing functions.

$(I4)$ $\tau_1-\int_0^1\phi(s)\,dH_1(s)>0$, $\tau_1-\int_0^1\psi(s)\,dH_2(s)>0$, $\tau_2-\int_0^1\widetilde{\phi}(s)\,dK_1(s)\,ds>0$, $\tau_2-\int_0^1\widetilde{\psi}(s)\,dK_2(s)>0$, $\Delta_1>0$, $\Delta_2>0$, where $\tau_1,\ \tau_2,\ \Delta_1,\ \Delta_2$ are defined in Section 2.

$(I5)$ The functions $p,\ q\in C([0,1],[0,\infty))$ and there exist $t_1,\ t_2\in(0,1)$ such that $p(t_1)>0$, $q(t_2)>0$.

$(I6)$ The functions $f,\ g\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$.

From assumption $(I5)$, there exists $\sigma\in(0,1/2)$ such that $t_1,\ t_2\in(\sigma,1-\sigma)$. We shall work in this section with this number σ. This implies that

$$\int_\sigma^{1-\sigma} g_1(t,s)p(s)\,ds>0,\ \int_\sigma^{1-\sigma} g_2(t,s)q(s)\,ds>0,\ \int_\sigma^{1-\sigma} J_1(s)p(s)\,ds>0,\ \int_\sigma^{1-\sigma} J_2(s)q(s)\,ds>0,$$

for all $t\in[0,1]$, where $g_1,\ g_2,\ J_1$ and J_2 are defined in Section 2 (Lemma 2.1 and Lemma 2.6).

For σ defined above, we introduce the following extreme limits

$$\begin{aligned}
f_0^s &= \limsup_{u+v\to 0^+}\max_{t\in[0,1]}\frac{f(t,u,v)}{u+v}, & g_0^s &= \limsup_{u+v\to 0^+}\max_{t\in[0,1]}\frac{g(t,u,v)}{u+v},\\
f_0^i &= \liminf_{u+v\to 0^+}\min_{t\in[\sigma,1-\sigma]}\frac{f(t,u,v)}{u+v}, & g_0^i &= \liminf_{u+v\to 0^+}\min_{t\in[\sigma,1-\sigma]}\frac{g(t,u,v)}{u+v},\\
f_\infty^s &= \limsup_{u+v\to \infty}\max_{t\in[0,1]}\frac{f(t,u,v)}{u+v}, & g_\infty^s &= \limsup_{u+v\to \infty}\max_{t\in[0,1]}\frac{g(t,u,v)}{u+v},\\
f_\infty^i &= \liminf_{u+v\to \infty}\min_{t\in[\sigma,1-\sigma]}\frac{f(t,u,v)}{u+v}, & g_\infty^i &= \liminf_{u+v\to \infty}\min_{t\in[\sigma,1-\sigma]}\frac{g(t,u,v)}{u+v}.
\end{aligned}$$

By using the Green's functions G_1 and G_2 from Section 2 (Lemma 2.1), our problem $(S)-(BC)$ can be written equivalently as the following nonlinear system of integral equations

$$\begin{cases} u(t)=\lambda\displaystyle\int_0^1 G_1(t,s)p(s)f(s,u(s),v(s))\,ds, & 0\le t\le 1,\\ v(t)=\mu\displaystyle\int_0^1 G_2(t,s)q(s)g(s,u(s),v(s))\,ds, & 0\le t\le 1.\end{cases}$$

We consider the Banach space $X=C([0,1])$ with supremum norm $\|\cdot\|$, and the Banach space $Y=X\times X$ with the norm $\|(u,v)\|_Y=\|u\|+\|v\|$. We define the cone $P\subset Y$ by

$$P=\{(u,v)\in Y;\ \ u(t)\ge 0,\ \ v(t)\ge 0,\ \ \forall\, t\in[0,1]\ \text{ and }\ \inf_{t\in[\sigma,1-\sigma]}(u(t)+v(t))\ge \nu\|(u,v)\|_Y\},$$

where $\nu = \min\{\nu_1, \nu_2\}$ and ν_1, ν_2 are the constants defined in Section 2 (Lemma 2.4) with respect to the above σ.

For λ, $\mu > 0$, we introduce the operators Q_1, $Q_2 : Y \to X$ and $\mathcal{Q} : Y \to Y$ defined by

$$Q_1(u,v)(t) = \lambda \int_0^1 G_1(t,s)p(s)f(s,u(s),v(s))\,ds, \quad 0 \le t \le 1,$$
$$Q_2(u,v)(t) = \mu \int_0^1 G_2(t,s)q(s)g(s,u(s),v(s))\,ds, \quad 0 \le t \le 1,$$

and $\mathcal{Q}(u,v) = (Q_1(u,v), Q_2(u,v))$, $(u,v) \in Y$. The solutions of our problem $(S)-(BC)$ are the fixed points of the operator $\mathcal{Q}$. By using similar arguments as those used in the proof of Lemma 3.1 from [10], we obtain the following lemma.

Lemma 3.1 *If $(I1)$–$(I6)$ hold, then $\mathcal{Q} : P \to P$ is a completely continuous operator.*

Let us introduce the notations

$$A = \int_\sigma^{1-\sigma} J_1(s)p(s)\,ds, \quad B = \int_0^1 J_1(s)p(s)\,ds,$$
$$C = \int_\sigma^{1-\sigma} J_2(s)q(s)\,ds, \quad D = \int_0^1 J_2(s)q(s)\,ds.$$

First, for f_0^s, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$ and numbers α_1, $\alpha_2 \ge 0$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ such that $\alpha_1+\alpha_2 = 1$ and $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$, we define the numbers L_1, L_2, L_3, L_4, L_2', L_4' by

$$L_1 = \frac{\alpha_1}{\nu\nu_1 f_\infty^i A}, \quad L_2 = \frac{\widetilde{\alpha}_1}{f_0^s B}, \quad L_3 = \frac{\alpha_2}{\nu\nu_2 g_\infty^i C}, \quad L_4 = \frac{\widetilde{\alpha}_2}{g_0^s D}, \quad L_2' = \frac{1}{f_0^s B}, \quad L_4' = \frac{1}{g_0^s D}.$$

Theorem 3.1 *Assume that $(I1)$–$(I6)$ hold, α_1, $\alpha_2 \ge 0$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ such that $\alpha_1 + \alpha_2 = 1$, $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.*

1) If f_0^s, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$, $L_1 < L_2$ and $L_3 < L_4$, then for each $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

2) If $f_0^s = 0$, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$ and $L_3 < L_4'$, then for each $\lambda \in (L_1, \infty)$ and $\mu \in (L_3, L_4')$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

3) If $g_0^s = 0$, f_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$ and $L_1 < L_2'$, then for each $\lambda \in (L_1, L_2')$ and $\mu \in (L_3, \infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

4) If $f_0^s = g_0^s = 0$, f_∞^i, $g_\infty^i \in (0,\infty)$, then for each $\lambda \in (L_1, \infty)$ and $\mu \in (L_3, \infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

5) If $\{f_0^s,\ g_0^s,\ f_\infty^i \in (0,\infty),\ g_\infty^i = \infty\}$ or $\{f_0^s,\ g_0^s,\ g_\infty^i \in (0,\infty),\ f_\infty^i = \infty\}$ or $\{f_0^s,\ g_0^s \in (0,\infty),\ f_\infty^i = g_\infty^i = \infty\}$, then for each $\lambda \in (0, L_2)$ and $\mu \in (0, L_4)$ there exists a positive solution $(u(t), v(t)),\ t \in [0,1]$ for $(S)-(BC)$.

6) If $\{f_0^s = 0,\ g_0^s,\ f_\infty^i \in (0,\infty),\ g_\infty^i = \infty\}$ or $\{f_0^s = 0,\ f_\infty^i = \infty,\ g_0^s,\ g_\infty^i \in (0,\infty)\}$ or $\{f_0^s = 0,\ g_0^s \in (0,\infty),\ f_\infty^i = g_\infty^i = \infty\}$ then for each $\lambda \in (0,\infty)$ and $\mu \in (0, L_4')$ there exists a positive solution $(u(t), v(t)),\ t \in [0,1]$ for $(S)-(BC)$.

7) If $\{f_0^s,\ f_\infty^i \in (0,\infty),\ g_0^s = 0,\ g_\infty^i = \infty\}$ or $\{f_0^s,\ g_\infty^i \in (0,\infty),\ g_0^s = 0,\ f_\infty^i = \infty\}$ or $\{f_0^s \in (0,\infty),\ g_0^s = 0,\ f_\infty^i = g_\infty^i = \infty\}$ then for each $\lambda \in (0, L_2')$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t)),\ t \in [0,1]$ for $(S)-(BC)$.

8) If $\{f_0^s = g_0^s = 0,\ f_\infty^i \in (0,\infty),\ g_\infty^i = \infty\}$ or $\{f_0^s = g_0^s = 0,\ f_\infty^i = \infty,\ g_\infty^i \in (0,\infty)\}$ or $\{f_0^s = g_0^s = 0,\ f_\infty^i = g_\infty^i = \infty\}$ then for each $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t)),\ t \in [0,1]$ for $(S)-(BC)$.

Proof. We consider the above cone $P \subset Y$ and the operators Q_1, Q_2 and $\mathcal{Q}$. Because the proofs of the above cases are similar, in what follows we shall prove one of them, namely the case 3). So, we suppose $g_0^s = 0$, $f_0^s,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$ and $L_1 < L_2'$. Let $\lambda \in (L_1, L_2')$ and $\mu \in (L_3, \infty)$, that is $\lambda \in \left(\frac{\alpha_1}{\nu\nu_1 f_\infty^i A}, \frac{1}{f_0^s B}\right)$, $\mu \in \left(\frac{\alpha_2}{\nu\nu_2 g_\infty^i C}, \infty\right)$. We choose $\widetilde{\alpha}_1' \in (\lambda f_0^s B, 1)$. Let $\widetilde{\alpha}_2' = 1 - \widetilde{\alpha}_1'$ and let $\varepsilon > 0$ be a positive number such that $\varepsilon < \min\{f_\infty^i, g_\infty^i\}$ and

$$\frac{\alpha_1}{\nu\nu_1(f_\infty^i - \varepsilon)A} \le \lambda, \quad \frac{\alpha_2}{\nu\nu_2(g_\infty^i - \varepsilon)C} \le \mu, \quad \frac{\widetilde{\alpha}_1'}{(f_0^s + \varepsilon)B} \ge \lambda, \quad \frac{\widetilde{\alpha}_2'}{\varepsilon D} \ge \mu.$$

By using $(I6)$ and the definitions of f_0^s and g_0^s, we deduce that there exists $R_1 > 0$ such that for all $t \in [0,1]$, $u,\ v \in \mathbb{R}_+$, with $0 \le u + v \le R_1$, we have $f(t,u,v) \le (f_0^s + \varepsilon)(u+v)$ and $g(t,u,v) \le \varepsilon(u+v)$. We define the set $\Omega_1 = \{(u,v) \in Y,\ \|(u,v)\|_Y < R_1\}$. Now let $(u,v) \in P \cap \partial\Omega_1$, that is $(u,v) \in P$ with $\|(u,v)\|_Y = R_1$ or equivalently $\|u\| + \|v\| = R_1$. Then $u(t) + v(t) \le R_1$ for all $t \in [0,1]$, and by Lemma 2.6, we obtain

$$\begin{aligned} Q_1(u,v)(t) &\le \lambda \int_0^1 J_1(s)p(s)f(s,u(s),v(s))\,ds \le \lambda \int_0^1 J_1(s)p(s)(f_0^s + \varepsilon)(u(s) + v(s))\,ds \\ &\le \lambda(f_0^s + \varepsilon) \int_0^1 J_1(s)p(s)(\|u\| + \|v\|)\,ds = \lambda(f_0^s + \varepsilon)B\|(u,v)\|_Y \le \widetilde{\alpha}_1'\|(u,v)\|_Y, \ \ \forall\, t \in [0,1]. \end{aligned}$$

Therefore, $\|Q_1(u,v)\| \le \widetilde{\alpha}_1'\|(u,v)\|_Y$. In a similar manner, we conclude

$$\begin{aligned} Q_2(u,v)(t) &\le \mu \int_0^1 J_2(s)q(s)g(s,u(s),v(s))\,ds \le \mu \int_0^1 J_2(s)q(s)\varepsilon(u(s) + v(s))\,ds \\ &\le \mu\varepsilon \int_0^1 J_2(s)q(s)(\|u\| + \|v\|)\,ds = \mu\varepsilon D\|(u,v)\|_Y \le \widetilde{\alpha}_2'\|(u,v)\|_Y, \ \ \forall\, t \in [0,1]. \end{aligned}$$

Therefore, $\|Q_2(u,v)\| \leq \widetilde{\alpha}_2'\|(u,v)\|_Y$.

Then, for $(u,v) \in P \cap \partial\Omega_1$, we deduce

$$\|\mathcal{Q}(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| \leq \widetilde{\alpha}_1'\|(u,v)\|_Y + \widetilde{\alpha}_2'\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By the definitions of f_∞^i and g_∞^i, there exists $\bar{R}_2 > 0$ such that $f(t,u,v) \geq (f_\infty^i - \varepsilon)(u+v)$ and $g(t,u,v) \geq (g_\infty^i - \varepsilon)(u+v)$ for all $u,\ v \geq 0$ with $u+v \geq \bar{R}_2$ and $t \in [\sigma, 1-\sigma]$. We consider $R_2 = \max\{2R_1, \bar{R}_2/\nu\}$, and we define $\Omega_2 = \{(u,v) \in Y,\ \|(u,v)\|_Y < R_2\}$. Then for $(u,v) \in P$ with $\|(u,v)\|_Y = R_2$, we obtain

$$u(t) + v(t) \geq \inf_{t\in[\sigma,1-\sigma]}(u(t)+v(t)) \geq \nu\|(u,v)\|_Y = \nu R_2 \geq \bar{R}_2,$$

for all $t \in [\sigma, 1-\sigma]$.

Then, by Lemma 2.6, we conclude

$$\begin{aligned} Q_1(u,v)(\sigma) &\geq \lambda\nu_1 \int_0^1 J_1(s)p(s)f(s,u(s),v(s))\,ds \geq \lambda\nu_1 \int_\sigma^{1-\sigma} J_1(s)p(s)f(s,u(s),v(s))\,ds \\ &\geq \lambda\nu_1 \int_\sigma^{1-\sigma} J_1(s)p(s)(f_\infty^i - \varepsilon)(u(s)+v(s))ds \geq \lambda\nu_1(f_\infty^i - \varepsilon)A\nu\|(u,v)\|_Y \geq \alpha_1\|(u,v)\|_Y. \end{aligned}$$

So, $\|Q_1(u,v)\| \geq Q_1(u,v)(\sigma) \geq \alpha_1\|(u,v)\|_Y$.

In a similar manner, we deduce

$$\begin{aligned} Q_2(u,v)(\sigma) &\geq \mu\nu_2 \int_0^1 J_2(s)q(s)g(s,u(s),v(s))\,ds \geq \mu\nu_2 \int_\sigma^{1-\sigma} J_2(s)q(s)g(s,u(s),v(s))\,ds \\ &\geq \mu\nu_2 \int_\sigma^{1-\sigma} J_2(s)q(s)(g_\infty^i - \varepsilon)(u(s)+v(s))\,ds \geq \mu\nu_2(g_\infty^i - \varepsilon)C\nu\|(u,v)\|_Y \geq \alpha_2\|(u,v)\|_Y. \end{aligned}$$

So, $\|Q_2(u,v)\| \geq Q_2(u,v)(\sigma) \geq \alpha_2\|(u,v)\|_Y$.

Hence, for $(u,v) \in P \cap \partial\Omega_2$, we obtain

$$\|\mathcal{Q}(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| \geq (\alpha_1+\alpha_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By using Lemma 3.1 and Theorem 1.1 i), we conclude that $\mathcal{Q}$ has a fixed point $(u,v) \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $R_1 \leq \|u\| + \|v\| \leq R_2$. □

Remark 3.1 *We mention that in Theorem 3.1 we have the possibility to choose $\alpha_1 = 0$ or $\alpha_2 = 0$. Therefore, each of the first four cases contains three subcases. For example, in the second case $f_0^s = 0$, $g_0^s, f_\infty^i, g_\infty^i \in (0,\infty)$, we have the following situations:*

a) if $\alpha_1, \alpha_2 \in (0,1)$, $\alpha_1+\alpha_2 = 1$ and $L_3 < L_4'$, then $\lambda \in (L_1,\infty)$ and $\mu \in (L_3, L_4')$;

b) if $\alpha_1 = 1$, $\alpha_2 = 0$, then $\lambda \in (L_1',\infty)$ and $\mu \in (0, L_4')$, where $L_1' = \frac{1}{\nu\nu_1 f_\infty^i A}$;

c) if $\alpha_1 = 0$, $\alpha_2 = 1$ and $L_3' < L_4'$, then $\lambda \in (0,\infty)$ and $\mu \in (L_3', L_4')$, where $L_3' = \frac{1}{\nu\nu_2 g_\infty^i C}$.

In what follows, for f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$ and numbers α_1, $\alpha_2 \geq 0$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ such that $\alpha_1 + \alpha_2 = 1$ and $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$, we define the numbers $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$, $\widetilde{L}_4$, $\widetilde{L}_2'$ and $\widetilde{L}_4'$ by

$$\widetilde{L}_1 = \frac{\alpha_1}{\nu\nu_1 f_0^i A}, \ \ \widetilde{L}_2 = \frac{\widetilde{\alpha}_1}{f_\infty^s B}, \ \ \widetilde{L}_3 = \frac{\alpha_2}{\nu\nu_2 g_0^i C}, \ \ \widetilde{L}_4 = \frac{\widetilde{\alpha}_2}{g_\infty^s D}, \ \ \widetilde{L}_2' = \frac{1}{f_\infty^s B}, \ \ \widetilde{L}_4' = \frac{1}{g_\infty^s D}.$$

Theorem 3.2 *Assume that* $(I1)-(I6)$ *hold,* α_1, $\alpha_2 \geq 0$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ *such that* $\alpha_1 + \alpha_2 = 1$, $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.

1) If f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$, $\widetilde{L}_1 < \widetilde{L}_2$ *and* $\widetilde{L}_3 < \widetilde{L}_4$, *then for each* $\lambda \in (\widetilde{L}_1, \widetilde{L}_2)$ *and* $\mu \in (\widetilde{L}_3, \widetilde{L}_4)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

2) If f_0^i, g_0^i, $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$ *and* $\widetilde{L}_1 < \widetilde{L}_2'$, *then for each* $\lambda \in (\widetilde{L}_1, \widetilde{L}_2')$ *and* $\mu \in (\widetilde{L}_3, \infty)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

3) If f_0^i, g_0^i, $g_\infty^s \in (0,\infty)$, $f_\infty^s = 0$ *and* $\widetilde{L}_3 < \widetilde{L}_4'$, *then for each* $\lambda \in (\widetilde{L}_1, \infty)$ *and* $\mu \in (\widetilde{L}_3, \widetilde{L}_4')$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

4) If f_0^i, $g_0^i \in (0,\infty)$, $f_\infty^s = g_\infty^s = 0$, *then for each* $\lambda \in (\widetilde{L}_1, \infty)$ *and* $\mu \in (\widetilde{L}_3, \infty)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

5) If $\{f_0^i = \infty,\ g_0^i,\ f_\infty^s,\ g_\infty^s \in (0,\infty)\}$ *or* $\{f_0^i,\ f_\infty^s,\ g_\infty^s \in (0,\infty),\ g_0^i = \infty\}$ *or* $\{f_0^i = g_0^i = \infty,\ f_\infty^s,\ g_\infty^s \in (0,\infty)\}$, *then for each* $\lambda \in (0, \widetilde{L}_2)$ *and* $\mu \in (0, \widetilde{L}_4)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

6) If $\{f_0^i = \infty,\ g_0^i,\ f_\infty^s \in (0,\infty),\ g_\infty^s = 0\}$ *or* $\{f_0^i,\ f_\infty^s \in (0,\infty),\ g_0^i = \infty,\ g_\infty^s = 0\}$ *or* $\{f_0^i = g_0^i = \infty,\ f_\infty^s \in (0,\infty),\ g_\infty^s = 0\}$, *then for each* $\lambda \in (0, \widetilde{L}_2')$ *and* $\mu \in (0,\infty)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

7) If $\{f_0^i = \infty,\ g_0^i,\ g_\infty^s \in (0,\infty),\ f_\infty^s = 0\}$ *or* $\{f_0^i,\ g_\infty^s \in (0,\infty),\ g_0^i = \infty,\ f_\infty^s = 0\}$ *or* $\{f_0^i = g_0^i = \infty,\ f_\infty^s = 0,\ g_\infty^s \in (0,\infty)\}$, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0, \widetilde{L}_4')$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

8) If $\{f_0^i = \infty,\ g_0^i \in (0,\infty),\ f_\infty^s = g_\infty^s = 0\}$ *or* $\{f_0^i \in (0,\infty),\ g_0^i = \infty,\ f_\infty^s = g_\infty^s = 0\}$ *or* $\{f_0^i = g_0^i = \infty,\ f_\infty^s = g_\infty^s = 0\}$, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0,\infty)$ *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$ *for* $(S)-(BC)$.

Proof. We consider the above cone $P \subset Y$ and the operators Q_1, Q_2 and $\mathcal{Q}$. Because the proofs of the above cases are similar, in what follows we shall prove one of them, namely the third case of 7). So, we suppose $f_0^i = g_0^i = \infty$, $f_\infty^s = 0$, $g_\infty^s \in (0,\infty)$. Let $\lambda \in (0,\infty)$ and

$\mu \in (0, \widetilde{L}_4')$, that is $\mu \in \left(0, \frac{1}{g_\infty^s D}\right)$. We choose $\alpha_1' \in (0,1)$ and $\widetilde{\alpha}_2' \in (\mu g_\infty^s D, 1)$. Let $\alpha_2' = 1 - \alpha_1'$, $\widetilde{\alpha}_1' = 1 - \widetilde{\alpha}_2'$ and let $\varepsilon > 0$ be a positive number such that

$$\frac{\varepsilon \alpha_1'}{\nu\nu_1 A} \le \lambda, \quad \frac{\varepsilon \alpha_2'}{\nu\nu_2 C} \le \mu, \quad \frac{\widetilde{\alpha}_1'}{\varepsilon B} \ge \lambda, \quad \frac{\widetilde{\alpha}_2'}{(g_\infty^s + \varepsilon) D} \ge \mu.$$

By using (I6) and the definitions of f_0^i and g_0^i, we deduce that there exists $R_3 > 0$ such that $f(t,u,v) \ge \frac{1}{\varepsilon}(u+v)$, $g(t,u,v) \ge \frac{1}{\varepsilon}(u+v)$ for all u, $v \ge 0$ with $0 \le u+v \le R_3$ and $t \in [\sigma, 1-\sigma]$. We denote by $\Omega_3 = \{(u,v) \in Y,\ \|(u,v)\|_Y < R_3\}$. Let $(u,v) \in P$ with $\|(u,v)\|_Y = R_3$, that is $\|u\| + \|v\| = R_3$. Because $u(t) + v(t) \le \|u\| + \|v\| = R_3$ for all $t \in [0,1]$, then by using Lemma 2.6, we obtain

$$Q_1(u,v)(\sigma) \ge \lambda\nu_1 \int_\sigma^{1-\sigma} J_1(s)p(s)f(s,u(s),v(s))\,ds \ge \lambda\nu_1 \int_\sigma^{1-\sigma} J_1(s)p(s)\frac{1}{\varepsilon}(u(s)+v(s))\,ds$$
$$\ge \lambda\nu\nu_1 \tfrac{1}{\varepsilon} \int_\sigma^{1-\sigma} J_1(s)p(s)(\|u\| + \|v\|)\,ds = \lambda\nu\nu_1 \frac{1}{\varepsilon} A \|(u,v)\|_Y \ge \alpha_1' \|(u,v)\|_Y.$$

Therefore, $\|Q_1(u,v)\| \ge Q_1(u,v)(\sigma) \ge \alpha_1'\|(u,v)\|_Y$. In a similar manner, we conclude

$$Q_2(u,v)(\sigma) \ge \mu\nu_2 \int_\sigma^{1-\sigma} J_2(s)q(s)g(s,u(s),v(s))\,ds \ge \mu\nu_2 \int_\sigma^{1-\sigma} J_2(s)q(s)\frac{1}{\varepsilon}(u(s)+v(s))\,ds$$
$$\ge \mu\nu\nu_2 \tfrac{1}{\varepsilon} \int_\sigma^{1-\sigma} J_2(s)q(s)(\|u\| + \|v\|)\,ds = \mu\nu\nu_2 \frac{1}{\varepsilon} C \|(u,v)\|_Y \ge \alpha_2' \|(u,v)\|_Y.$$

So, $\|Q_2(u,v)\| \ge Q_2(u,v)(\sigma) \ge \alpha_2'\|(u,v)\|_Y$.

Thus, for an arbitrary element $(u,v) \in P \cap \partial\Omega_3$, we deduce $\|\mathcal{Q}(u,v)\|_Y \ge (\alpha_1' + \alpha_2')\|(u,v)\|_Y = \|(u,v)\|_Y$.

Now, we define the functions f^*, $g^* : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$, $f^*(t,x) = \max\limits_{0 \le u+v \le x} f(t,u,v)$, $g^*(t,x) = \max\limits_{0 \le u+v \le x} g(t,u,v)$, $t \in [0,1]$, $x \in \mathbb{R}_+$. Then $f(t,u,v) \le f^*(t,x)$, $g(t,u,v) \le g^*(t,x)$ for all $t \in [0,1]$, $u \ge 0$, $v \ge 0$ and $u+v \le x$. The functions $f^*(t,\cdot)$, $g^*(t,\cdot)$ are nondecreasing for every $t \in [0,1]$, and they satisfy the conditions

$$\lim_{x\to\infty} \max_{t\in[0,1]} \frac{f^*(t,x)}{x} = 0, \quad \limsup_{x\to\infty} \max_{t\in[0,1]} \frac{g^*(t,x)}{x} \le g_\infty^s.$$

Therefore, for $\varepsilon > 0$, there exists $\bar{R}_4 > 0$ such that for all $x \ge \bar{R}_4$ and $t \in [0,1]$, we have

$$\frac{f^*(t,x)}{x} \le \lim_{x\to\infty} \max_{t\in[0,1]} \frac{f^*(t,x)}{x} + \varepsilon = \varepsilon, \quad \frac{g^*(t,x)}{x} \le \limsup_{x\to\infty} \max_{t\in[0,1]} \frac{g^*(t,x)}{x} + \varepsilon \le g_\infty^s + \varepsilon,$$

and so $f^*(t,x) \le \varepsilon x$ and $g^*(t,x) \le (g_\infty^s + \varepsilon)x$.

We consider $R_4 = \max\{2R_3, \bar{R}_4\}$ and we denote by $\Omega_4 = \{(u,v) \in Y,\ \|(u,v)\|_Y < R_4\}$. Let $(u,v) \in P \cap \partial\Omega_4$. By the definitions of f^* and g^*, we conclude

$$f(t,u(t),v(t)) \le f^*(t, \|(u,v)\|_Y), \quad g(t,u(t),v(t)) \le g^*(t, \|(u,v)\|_Y), \quad \forall\, t \in [0,1].$$

Then for all $t \in [0,1]$, we obtain

$$
\begin{aligned}
&Q_1(u,v)(t) \le \lambda \int_0^1 J_1(s)p(s)f(s,(u(s),v(s))\,ds \le \lambda \int_0^1 J_1(s)p(s)f^*(s,\|(u,v)\|_Y)\,ds \\
&\le \lambda\varepsilon \int_0^T J_1(s)p(s)\|(u,v)\|_Y\,ds = \lambda\varepsilon B\|(u,v)\|_Y \le \widetilde{\alpha}_1'\|(u,v)\|_Y,
\end{aligned}
$$

and so, $\|Q_1(u,v)\| \le \widetilde{\alpha}_1'\|(u,v)\|_Y$.

In a similar manner, we deduce

$$
\begin{aligned}
&Q_2(u,v)(t) \le \mu \int_0^1 J_2(s)q(s)g(s,u(s),v(s))\,ds \le \mu \int_0^1 J_2(s)q(s)g^*(s,\|(u,v)\|_Y)\,ds \\
&\le \mu(g_\infty^s+\varepsilon) \int_0^1 J_2(s)q(s)\|(u,v)\|_Y\,ds = \mu(g_\infty^s+\varepsilon)D\|(u,v)\|_Y \le \widetilde{\alpha}_2'\|(u,v)\|_Y,
\end{aligned}
$$

and so, $\|Q_2(u,v)\| \le \widetilde{\alpha}_2'\|(u,v)\|_Y$.

Therefore, for $(u,v) \in P\cap\partial\Omega_4$, it follows that $\|\mathcal{Q}(u,v)\|_Y \le (\widetilde{\alpha}_1'+\widetilde{\alpha}_2')\|(u,v)\|_Y = \|(u,v)\|_Y$. By using Lemma 3.1 and Theorem 1.1 ii), we conclude that $\mathcal{Q}$ has a fixed point $(u,v) \in P\cap(\bar{\Omega}_4\setminus\Omega_3)$ such that $R_3 \le \|(u,v)\|_Y \le R_4$. □

In what follows, we shall determine intervals for λ and μ for which there exists no positive solution of problem $(S)-(BC)$.

Theorem 3.3 *Assume that (I1)–(I6) hold. If f_0^s, f_∞^s, g_0^s, $g_\infty^s < \infty$, then there exist positive constants λ_0, μ_0 such that for every $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, the boundary value problem $(S)-(BC)$ has no positive solution.*

Proof. Since f_0^s, f_∞^s, g_0^s, $g_\infty^s < \infty$, we deduce that there exist M_1, $M_2 > 0$ such that

$$
f(t,u,v) \le M_1(u+v),\;\; g(t,u,v) \le M_2(u+v),\;\; \forall\, u,\, v \ge 0,\;\; t \in [0,1].
$$

We define $\lambda_0 = \frac{1}{2M_1B}$ and $\mu_0 = \frac{1}{2M_2D}$, where $B = \int_0^1 J_1(s)p(s)\,ds$ and $D = \int_0^1 J_2(s)q(s)\,ds$. We shall show that for every $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, the problem $(S)-(BC)$ has no positive solution.

Let $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$. We suppose that $(S)-(BC)$ has a positive solution $(u(t),v(t))$, $t \in [0,1]$. Then, we have

$$
\begin{aligned}
u(t) =& Q_1(u,v)(t) = \lambda \int_0^1 G_1(t,s)p(s)f(s,u(s),v(s))\,ds \\
\le& \lambda \int_0^1 J_1(s)p(s)f(s,u(s),v(s))\,ds \le \lambda M_1 \int_0^1 J_1(s)p(s)(u(s)+v(s))\,ds \\
\le& \lambda M_1(\|u\|+\|v\|) \int_0^1 J_1(s)p(s)\,ds = \lambda M_1 B\|(u,v)\|_Y,\;\; \forall\, t \in [0,1].
\end{aligned}
$$

Therefore, we conclude

$$\|u\| \le \lambda M_1 B \|(u,v)\|_Y < \lambda_0 M_1 B \|(u,v)\|_Y = \frac{1}{2}\|(u,v)\|_Y.$$

In a similar manner, we have $v(t) \le \mu M_2 D \|(u,v)\|_Y$ for all $t \in [0,1]$, and so

$$\|v\| \le \mu M_2 D \|(u,v)\|_Y < \mu_0 M_2 D \|(u,v)\|_Y = \frac{1}{2}\|(u,v)\|_Y.$$

Hence, $\|(u,v)\|_Y = \|u\| + \|v\| < \frac{1}{2}\|(u,v)\|_Y + \frac{1}{2}\|(u,v)\|_Y = \|(u,v)\|_Y$, which is a contradiction. So, the boundary value problem $(S)-(BC)$ has no positive solution. □

4 An example

Let $T=1$, $a(t)=1$, $b(t)=4$, $c(t)=1$, $d(t)=1$, $p(t)=1$, $q(t)=1$ for all $t \in (0,1)$, $\alpha=1$, $\beta=3$, $\gamma=1$, $\delta=1$, $\widetilde{\alpha}=3$, $\widetilde{\beta}=2$, $\widetilde{\gamma}=1$, $\widetilde{\delta}=3/2$,

$$H_1(t)=t^2,\ \ H_2(t)=\begin{cases} 0, & t\in[0,1/3), \\ 7/2, & t\in[1/3,2/3), \\ 11/2, & t\in[2/3,1], \end{cases} \quad K_1(t)=\begin{cases} 0, & t\in[0,1/2), \\ 4/3, & t\in[1/2,1], \end{cases} \quad K_2(t)=t^3.$$

Then $\int_0^1 u(s)\,dH_2(s)=\frac{7}{2}u\left(\frac{1}{3}\right)+2u\left(\frac{2}{3}\right)$, $\int_0^1 u(s)\,dK_1(s)=\frac{4}{3}u\left(\frac{1}{2}\right)$, $\int_0^1 u(s)\,dH_1(s)=2\int_0^1 su(s)\,ds$, $\int_0^1 u(s)\,dK_2(s)=3\int_0^1 s^2u(s)\,ds$.

We consider the second-order differential system

$$(S_0) \qquad \begin{cases} u''(t)-4u(t)+\lambda f(t,u(t),v(t))=0, \ \ t\in(0,1), \\ v''(t)-v(t)+\mu g(t,u(t),v(t))=0, \ \ t\in(0,1), \end{cases}$$

with the boundary conditions

$$(BC_0) \qquad \begin{cases} u(0)-3u'(0)=2\int_0^1 su(s)\,ds, \ \ u(1)+u'(1)=\frac{7}{2}u\left(\frac{1}{3}\right)+2u\left(\frac{2}{3}\right), \\ 3v(0)-2v'(0)=\frac{4}{3}v\left(\frac{1}{2}\right), \ \ v(1)+\frac{3}{2}v'(1)=3\int_0^1 s^2v(s)\,ds, \end{cases}$$

where the functions f and g are given by

$$f(t,u,v)=\frac{e^t[p_1(u+v)+1](u+v)(q_1+\sin v)}{u+v+1},$$
$$g(t,u,v)=\frac{e^{-t}[p_2(u+v)+1](u+v)(q_2+\cos u)}{u+v+1},$$

for all $t\in[0,1]$, u, $v\ge 0$, with p_1, $p_2>0$ and q_1, $q_2>1$. For $\sigma=1/4$, we deduce $f_0^s=eq_1$, $g_0^s=q_2+1$, $f_\infty^i=e^{1/4}p_1(q_1-1)$, $g_\infty^i=e^{-3/4}p_2(q_2-1)$.

The functions ψ and ϕ from Section 2 are the solutions of the following problems

$$\begin{cases} \psi''(t)-4\psi(t)=0, \ \ 0<t<1, \\ \psi(0)=3, \ \ \psi'(0)=1, \end{cases} \qquad \begin{cases} \phi''(t)-4\phi(t)=0, \ \ 0<t<1, \\ \phi(1)=1, \ \ \phi'(1)=-1. \end{cases}$$

We obtain $\psi(t)=\dfrac{7e^{4t}+5}{4e^{2t}}$ and $\phi(t)=\dfrac{1+3e^{4-4t}}{4e^{2-2t}}$ for all $t\in[0,1]$, $\tau_1=\dfrac{21e^4-5}{4e^2}$,

$$\begin{aligned}
\Lambda_1 :=&\tau_1-\int_0^1 \psi(s)\,dH_2(s)=\tau_1-\left(\frac{7}{2}\psi\left(\frac{1}{3}\right)+2\psi\left(\frac{2}{3}\right)\right)\\
=&(42e^4-28e^{10/3}-49e^{8/3}-35e^{4/3}-20e^{2/3}-10)/(8e^2)\approx 10.51047404>0,\\
\Lambda_2 :=&\tau_1-\int_0^1 \phi(s)\,dH_1(s)=\tau_1-2\int_0^1 s\phi(s)\,ds\\
=&(21e^4-5)/(4e^2)-2\int_0^1 s(1+3e^{4-4s})/(4e^{2-2s})\,ds\approx 36.83556247>0,\\
\Lambda_3 :=&\int_0^1 \psi(s)\,dH_1(s)=2\int_0^1 s\psi(s)\,ds=2\int_0^1 s(7e^{4s}+5)/(4e^{2s})\,ds\approx 7.71167043,\\
\Lambda_4 :=&\int_0^1 \phi(s)\,dH_2(s)=\frac{7}{2}\phi\left(\frac{1}{3}\right)+2\phi\left(\frac{2}{3}\right)\\
=&7(1+3e^{8/3})/(8e^{4/3})+2(1+3e^{4/3})/(4e^{2/3})\approx 13.36733534,\\
\Delta_1 =&\Lambda_1\Lambda_2-\Lambda_3\Lambda_4\approx 284.07473844>0.
\end{aligned}$$

The functions g_1 and J_1 are given by

$$g_1(t,s)=\frac{1}{\tau_1}\begin{cases}\phi(t)\psi(s), & s\le t,\\ \phi(s)\psi(t), & t\le s,\end{cases}\quad \text{or}\quad g_1(t,s)=\frac{4e^2}{21e^4-5}\begin{cases}\dfrac{(1+3e^{4-4t})(7e^{4s}+5)}{16e^{2-2t+2s}}, & s\le t,\\ \dfrac{(1+3e^{4-4s})(7e^{4t}+5)}{16e^{2-2s+2t}}, & t\le s,\end{cases}$$

$$\begin{aligned}
J_1(s)=&g_1(s,s)+\frac{1}{\Delta_1}(\Lambda_4\psi(1)+\Lambda_1\phi(0))\times 2\int_0^1 \tau g_1(\tau,s)\,d\tau\\
&+\frac{1}{\Delta_1}(\Lambda_2\psi(1)+\Lambda_3\phi(0))\left[\frac{7}{2}g_1\left(\frac{1}{3},s\right)+2g_1\left(\frac{2}{3},s\right)\right]\\
=&g_1(s,s)+\frac{1}{\Delta_1}(\Lambda_4\psi(1)+\Lambda_1\phi(0))\times 2\left(\int_0^s \tau g_1(\tau,s)\,d\tau+\int_s^1 \tau g_1(\tau,s)\,d\tau\right)\\
&+\frac{1}{\Delta_1}(\Lambda_2\psi(1)+\Lambda_3\phi(0))\left[\frac{7}{2}g_1\left(\frac{1}{3},s\right)+2g_1\left(\frac{2}{3},s\right)\right]
\end{aligned}$$

$$=\begin{cases}
\dfrac{1}{\tau_1}\left\{\phi(s)\psi(s)+\dfrac{1}{\Delta_1}(\Lambda_4\psi(1)+\Lambda_1\phi(0))\times 2\left(\phi(s)\displaystyle\int_0^s \tau\psi(\tau)\,d\tau+\psi(s)\int_s^1 \tau\phi(\tau)\,d\tau\right)\right.\\
\left.+\dfrac{1}{\Delta_1}(\Lambda_2\psi(1)+\Lambda_3\phi(0))\left(\dfrac{7}{2}\phi\left(\dfrac{1}{3}\right)\psi(s)+2\phi\left(\dfrac{2}{3}\right)\psi(s)\right)\right\}, & 0\le s<\frac{1}{3},\\
\dfrac{1}{\tau_1}\left\{\phi(s)\psi(s)+\dfrac{1}{\Delta_1}(\Lambda_4\psi(1)+\Lambda_1\phi(0))\times 2\left(\phi(s)\displaystyle\int_0^s \tau\psi(\tau)\,d\tau+\psi(s)\int_s^1 \tau\phi(\tau)\,d\tau\right)\right.\\
\left.+\dfrac{1}{\Delta_1}(\Lambda_2\psi(1)+\Lambda_3\phi(0))\left(\dfrac{7}{2}\phi(s)\psi\left(\dfrac{1}{3}\right)+2\phi\left(\dfrac{2}{3}\right)\psi(s)\right)\right\}, & \frac{1}{3}\le s<\frac{2}{3},\\
\dfrac{1}{\tau_1}\left\{\phi(s)\psi(s)+\dfrac{1}{\Delta_1}(\Lambda_4\psi(1)+\Lambda_1\phi(0))\times 2\left(\phi(s)\displaystyle\int_0^s \tau\psi(\tau)\,d\tau+\psi(s)\int_s^1 \tau\phi(\tau)\,d\tau\right)\right.\\
\left.+\dfrac{1}{\Delta_1}(\Lambda_2\psi(1)+\Lambda_3\phi(0))\left(\dfrac{7}{2}\phi(s)\psi\left(\dfrac{1}{3}\right)+2\phi(s)\psi\left(\dfrac{2}{3}\right)\right)\right\}, & \frac{2}{3}\le s\le 1.
\end{cases}$$

The functions $\widetilde{\psi}$ and $\widetilde{\phi}$ from Section 2 are the solutions of the following problems

$$\begin{cases} \widetilde{\psi}''(t)-\widetilde{\psi}(t)=0, & 0<t<1,\\ \widetilde{\psi}(0)=2, \ \widetilde{\psi}'(0)=3, \end{cases} \qquad \begin{cases} \widetilde{\phi}''(t)-\widetilde{\phi}(t)=0, & 0<t<1,\\ \widetilde{\phi}(1)=\frac{3}{2}, \ \widetilde{\phi}'(1)=-1. \end{cases}$$

We obtain $\widetilde{\psi}(t)=\dfrac{5e^{2t}-1}{2e^t}$ and $\widetilde{\phi}(t)=\dfrac{1+5e^{2-2t}}{4e^{1-t}}$ for all $t\in[0,1]$, $\tau_2=\dfrac{25e^2+1}{4e}$,

$$\begin{aligned}
\widetilde{\Lambda}_1 &:= \tau_2-\int_0^1 \widetilde{\psi}(s)\,dK_2(s)=\tau_2-3\int_0^1 s^2\widetilde{\psi}(s)\,ds\\
&=(25e^2+1)/(4e)-3\int_0^1 s^2(5e^{2s}-1)/(2e^s)\,ds \approx 11.93502177>0,\\
\widetilde{\Lambda}_2 &:= \tau_2-\int_0^1 \widetilde{\phi}(s)\,dK_1(s)=\tau_2-\frac{4}{3}\widetilde{\phi}\left(\frac{1}{2}\right)\\
&=(25e^2+1)/(4e)-(1+5e)/(3e^{1/2})\approx 14.13118562>0,\\
\widetilde{\Lambda}_3 &:= \int_0^1 \widetilde{\psi}(s)\,dK_1(s)=\frac{4}{3}\widetilde{\psi}\left(\frac{1}{2}\right)=(10e-2)/(3e^{1/2})\approx 5.09138379,\\
\widetilde{\Lambda}_4 &:= \int_0^1 \widetilde{\phi}(s)\,dK_2(s)=3\int_0^1 s^2\widetilde{\phi}(s)\,ds=3\int_0^1 s^2(1+5e^{2-2s})/(4e^{1-s})\,ds\approx 1.83529455,\\
\Delta_2 &= \widetilde{\Lambda}_1\widetilde{\Lambda}_2-\widetilde{\Lambda}_3\widetilde{\Lambda}_4\approx 159.31181898>0.
\end{aligned}$$

The functions g_2 and J_2 are given by

$$g_2(t,s)=\frac{1}{\tau_2}\begin{cases}\widetilde{\phi}(t)\widetilde{\psi}(s), & s\le t,\\ \widetilde{\phi}(s)\widetilde{\psi}(t), & t\le s,\end{cases} \quad \text{or} \quad g_2(t,s)=\frac{4e}{25e^2+1}\begin{cases}\dfrac{(1+5e^{2-2t})(5e^{2s}-1)}{8e^{1-t+s}}, & s\le t,\\ \dfrac{(1+5e^{2-2s})(5e^{2t}-1)}{8e^{1-s+t}}, & t\le s,\end{cases}$$

$$\begin{aligned}
J_2(s) &= g_2(s,s)+\frac{1}{\Delta_2}(\widetilde{\Lambda}_4\widetilde{\psi}(1)+\widetilde{\Lambda}_1\widetilde{\phi}(0))\frac{4}{3}g_2\left(\frac{1}{2},s\right)\\
&\quad+\frac{1}{\Delta_2}(\widetilde{\Lambda}_2\widetilde{\psi}(1)+\widetilde{\Lambda}_3\widetilde{\phi}(0))\times 3\int_0^1\tau^2 g_2(\tau,s)\,d\tau\\
&= g_2(s,s)+\frac{1}{\Delta_2}(\widetilde{\Lambda}_4\widetilde{\psi}(1)+\widetilde{\Lambda}_1\widetilde{\phi}(0))\frac{4}{3}g_2\left(\frac{1}{2},s\right)\\
&\quad+\frac{1}{\Delta_2}(\widetilde{\Lambda}_2\widetilde{\psi}(1)+\widetilde{\Lambda}_3\widetilde{\phi}(0))\times 3\left(\int_0^s\tau^2 g_2(\tau,s)\,d\tau+\int_s^1\tau^2 g_2(\tau,s)\,d\tau\right)
\end{aligned}$$

$$=\begin{cases}
\dfrac{1}{\tau_2}\Big\{\widetilde{\phi}(s)\widetilde{\psi}(s)+\dfrac{1}{\Delta_2}\left(\widetilde{\Lambda}_4\widetilde{\psi}(1)+\widetilde{\Lambda}_1\widetilde{\phi}(0)\right)\times\dfrac{4}{3}\widetilde{\phi}\left(\dfrac{1}{2}\right)\widetilde{\psi}(s)\\
+\dfrac{1}{\Delta_2}\left(\widetilde{\Lambda}_2\widetilde{\psi}(1)+\widetilde{\Lambda}_3\widetilde{\phi}(0)\right)\times 3\left(\widetilde{\phi}(s)\displaystyle\int_0^s\tau^2\widetilde{\psi}(\tau)\,d\tau+\widetilde{\psi}(s)\int_s^1\tau^2\widetilde{\phi}(\tau)\,d\tau\right)\Big\}, & 0\le s<\frac{1}{2},\\
\dfrac{1}{\tau_2}\Big\{\widetilde{\phi}(s)\widetilde{\psi}(s)+\dfrac{1}{\Delta_2}\left(\widetilde{\Lambda}_4\widetilde{\psi}(1)+\widetilde{\Lambda}_1\widetilde{\phi}(0)\right)\times\dfrac{4}{3}\widetilde{\phi}(s)\widetilde{\psi}\left(\dfrac{1}{2}\right)\\
+\dfrac{1}{\Delta_2}\left(\widetilde{\Lambda}_2\widetilde{\psi}(1)+\widetilde{\Lambda}_3\widetilde{\phi}(0)\right)\times 3\left(\widetilde{\phi}(s)\displaystyle\int_0^s\tau^2\widetilde{\psi}(\tau)\,d\tau+\widetilde{\psi}(s)\int_s^1\tau^2\widetilde{\phi}(\tau)\,d\tau\right)\Big\}, & \frac{1}{2}\le s\le 1.
\end{cases}$$

We also have $\nu=\nu_1=(1+3e)e^{3/2}/(1+3e^4)$, $\nu_2=(5e^{1/2}-1)e^{3/4}/(5e^2-1)$. After some computations, we deduce $A=\int_{1/4}^{3/4}J_1(s)\,ds\approx 1.35977188$, $B=\int_0^1 J_1(s)\,ds\approx 2.51890379$, $C=$

$\int_{1/4}^{3/4} J_2(s)\,ds \approx 0.48198213$, $D = \int_0^1 J_2(s)\,ds \approx 0.93192847$. For α_1, $\alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$, we consider $\widetilde{\alpha}_1 = \alpha_1$, $\widetilde{\alpha}_2 = \alpha_2$. Then, we obtain

$$L_1 = \frac{\alpha_1(1+3e^4)^2}{e^{13/4}(1+3e)^2 p_1(q_1-1)A}, \qquad L_2 = \frac{\alpha_1}{eq_1 B},$$
$$L_3 = \frac{\alpha_2(1+3e^4)(5e^2-1)}{e^{3/2}(1+3e)(5e^{1/2}-1)p_2(q_2-1)C}, \quad L_4 = \frac{\alpha_2}{(q_2+1)D}.$$

The conditions $L_1 < L_2$ and $L_3 < L_4$ become

$$\frac{p_1(q_1-1)}{q_1} > \frac{(1+3e^4)^2 eB}{(1+3e)^2 e^{13/4} A}, \quad \frac{p_2(q_2-1)}{q_2+1} > \frac{(1+3e^4)(5e^2-1)D}{(1+3e)(5e^{1/2}-1)e^{3/2} C}.$$

If $p_1(q_1-1)/q_1 \geq 64$ and $p_2(q_2-1)/(q_2+1) \geq 39$, then the above conditions are satisfied. For example, if $\alpha_1 = \alpha_2 = 1/2$, $p_1 = 128$, $q_1 = 2$, $p_2 = 117$, $q_2 = 2$, we obtain $L_1 \approx 0.03609275$, $L_2 \approx 0.03651185$, $L_3 \approx 0.17672178$, $L_4 \approx 0.17884062$. Therefore, by Theorem 3.1 1), for each $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$, there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for the problem $(S_0) - (BC_0)$.

Because $f_0^s = eq_1$, $f_\infty^s = ep_1(q_1+1)$, $g_0^s = q_2 + 1$, $g_\infty^s = p_2(q_2+1)$ are finite, we can also apply Theorem 3.3. Using the same values for p_1, q_1, p_2, q_2 as above, that is $p_1 = 128$, $q_1 = 2$, $p_2 = 117$, $q_2 = 2$, we deduce

$$M_1 = \sup_{u,v\geq 0} \max_{t\in[0,1]} \frac{f(t,u,v)}{u+v} = e \sup_{u,v\geq 0} \frac{[p_1(u+v)+1](q_1+\sin v)}{u+v+1} \approx 1043.82022212,$$
$$M_2 = \sup_{u,v\geq 0} \max_{t\in[0,1]} \frac{g(t,u,v)}{u+v} = \sup_{u,v\geq 0} \frac{[p_2(u+v)+1](q_2+\cos u)}{u+v+1} = 351,$$

Then, we obtain $\lambda_0 = \frac{1}{2M_1 B} \approx 0.00019016$ and $\mu_0 = \frac{1}{2M_2 D} \approx 0.00152855$. Therefore, by Theorem 3.3, we conclude that for every $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$, the problem $(S_0) - (BC_0)$ has no positive solution.

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