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Quasilinear degenerated equations with L^1 datum and without coercivity in perturbation terms*

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Abstract

In this paper we study the existence of solutions for the generated boundary value problem, with initial datum being an element of $L^1(\Omega) + W^{-1,p'}(\Omega, w^*)$

$$-\text{div}a(x, u, \nabla u) + g(x, u, \nabla u) = f - \text{div}F$$

where $a(.)$ is a Carathéodory function satisfying the classical condition of type Leray-Lions hypothesis, while $g(x, s, \xi)$ is a non-linear term which has a growth condition with respect to ξ and no growth with respect to s, but it satisfies a sign condition on s.

1. Introduction

Let Ω be a bounded subset of $I\!R^N$ $(N \geq 2)$, $1 < p < \infty$, and $w = \{w_i(x);\ i = 0, ..., N\}$, be a collection of weight functions on Ω i.e., each w_i is a measurable and strictly positive function everywhere on Ω and satisfying some integrability conditions (see section 2). Let us consider the non-linear elliptic partial differential operator of order 2 given in divergence form

$$Au = -\text{div}(a(x, u, \nabla u)) \tag{1.1}$$

It is well known that equation $Au = h$ is solvable by Drabek, Kufner and Mustonen in [7] in the case where $h \in W^{-1,p'}(\Omega, w^*)$.
In this paper we investigate the problem of existence solutions of the following Dirichlet problem

$$Au + g(x, u, \nabla u) = \mu \ \text{ in } \ \Omega. \tag{1.2}$$

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where $\mu \in L^1(\Omega) + \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$.
In this context of nonlinear operators, if μ belongs to $W^{-1,p'}(\Omega, w^*)$ existence results for problem (1.2) have been proved in [2], where the authors have used the approach based on the strong convergence of the positive part u_ε^+ (resp. ngative part u_ε^-).
The case where $\mu \in L^1(\Omega)$ is investigated in [3] under the following coercivity condition,

$$|g(x,s,\xi)| \geq \beta \sum_{i=1}^{N} w_i |\xi_i|^p \quad \text{for} \quad |s| \geq \gamma, \tag{1.3}$$

Let us recall that the results given in [2, 3] have been proved under some additional conditions on the weight function σ and the parameter q introduced in Hardy inequality.
The main point in our study to prove an existence result for some class of problem of the kind (1.2), without assuming the coercivity condition (1.3). Moreover, we didn't supose any restriction for weight function σ and parameter q.
It would be interesting at this stage to refer the reader to our previous work [1]. For different appproach used in the setting of Orlicz Sobolev space the reader can refer to [4], and for same results in the L^p case, to [10].
The plan of this is as follows : in the next section we will give some preliminaries and some technical lemmas, section 3 is concerned with main results and basic assumptions, in section 4 we prove main results and we study the stability and the positivity of solution.

2. Preliminaries

Let Ω be a bounded open subset of $I\!R^N (N \geq 2)$. Let $1 < p < \infty$, and let $w = \{w_i(x); 0 \leq i \leq N\}$, be a vector of weight functions i.e. every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that for $0 \leq i \leq N$

$$w_i \in L^1_{loc}(\Omega) \quad \text{and} \quad w_i^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega). \tag{2.1}$$

We define the weighted space with weight γ in Ω as

$$L^p(\Omega, \gamma) = \{u(x) : u\gamma^{\frac{1}{p}} \in L^1(\Omega)\},$$

which is endowed with, we define the norm

$$\|u\|_{p,\gamma} = \left(\int_\Omega |u(x)|^p \gamma(x)\, dx \right)^{\frac{1}{p}}.$$

We denote by $W^{1,p}(\Omega, w)$ the weighted Sobolev space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \text{for all} \quad i = 1, ..., N.$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \left(\int_\Omega |u(x)|^p w_0\, dx + \sum_{i=1}^{N} \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx \right)^{\frac{1}{p}}. \tag{2.2}$$

To deal with the Dirichlet problem, we use the space

$$X = W_0^{1,p}(\Omega, w)$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.2). Note that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(X, \|.\|_{1,p,w})$ is a reflexive Banach space.
We recall that the dual of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}\}$, $i = 1, ..., N$ and p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$. For more details we refer the reader to [8].
We introduce the functional spaces, we will need later.
For $p \in (1, \infty)$, $\mathcal{T}_0^{1,p}(\Omega, w)$ is defined as the set of measurable functions $u : \Omega \longrightarrow I\!R$ such that for $k > 0$ the truncated functions $T_k(u) \in W_0^{1,p}(\Omega, w)$.
We give the following lemma which is a generalization of Lemma 2.1 [5] in weighted Sobolev spaces.

Lemma 2.1. *For every $u \in \mathcal{T}_0^{1,p}(\Omega, w)$, there exists a unique measurable function $v : \Omega \longrightarrow I\!R^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \quad \textit{almost everywhere in } \Omega, \quad \textit{for every } k > 0.$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Lemma 2.2. *Let $\lambda \in I\!R$ and let u and v be two functions which are finite almost everywhere, and which belongs to $\mathcal{T}_0^{1,p}(\Omega, w)$. Then,*

$$\nabla(u + \lambda v) = \nabla u + \lambda \nabla v \quad \textit{a.e. in} \quad \Omega,$$

where ∇u, ∇v and $\nabla(u + \lambda v)$ are the gradients of u, v and $u + \lambda v$ introduced in Lemma 2.1.

The proof of this lemma is similar to the proof of Lemma 2.12 [6] for the non weighted case.

Definition 2.1. *Let Y be a reflexive Banach space, a bounded operator B from Y to its dual Y^* is called pseudo-monotone if for any sequence $u_n \in Y$ with $u_n \rightharpoonup u$ weakly in Y. $Bu_n \rightharpoonup \chi$ weakly in Y^* and $\limsup_{n\to\infty}\langle Bu_n, u_n\rangle \leq \langle \chi, u\rangle$, we have*

$$Bu_n = Bu \quad \textit{and} \quad \langle Bu_n, u_n\rangle \to \langle \chi, u\rangle \quad \textit{as} \quad n \to \infty.$$

Now, we state the following assumptions.
$(\mathbf{H_1})$-The expression

$$\|u\|_X = \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}}, \tag{2.3}$$

is a norm defined on X and is equivalent to the norm (2.2). (Note that $(X, \|u\|_X)$ is a uniformly convex (and reflexive) Banach space.
-There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that the Hardy inequality

$$\left(\int_\Omega |u|^q \sigma(x)\, dx\right)^{\frac{1}{q}} \leq C\left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}}, \tag{2.4}$$

holds for every $u \in X$ with a constant $C > 0$ independent of u. Moreover, the imbeding

$$X \hookrightarrow L^q(\Omega, \sigma) \tag{2.5}$$

determined by the inequality (2.4) is compact.
We state the following technical lemmas which are needed later.

Lemma 2.3 *[2]. Let $g \in L^r(\Omega, \gamma)$ and let $g_n \in L^r(\Omega, \gamma)$, with $\|g_n\|_{\Omega,\gamma} \leq c, 1 < r < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^r(\Omega, \gamma)$.*

Lemma 2.4 *[2]. Assume that (H_1) holds. Let $F : I\!R \to I\!R$ be unifomly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^{1,p}(\Omega, w)$. Then $F(u) \in W_0^{1,p}(\Omega, w)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial F(u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \quad in \quad \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \quad in \quad \{x \in \Omega : u(x) \in D\}. \end{cases}$$

From the previous lemma, we deduce the following.

Lemma 2.5 *[2]. Assume that (H_1) holds. Let $u \in W_0^{1,p}(\Omega, w)$, and let $T_k(u), k \in I\!R^+$, be the usual truncation, then $T_k(u) \in W_0^{1,p}(\Omega, w)$. Moreover, we have*

$$T_k(u) \to u \quad \textit{strongly in} \quad W_0^{1,p}(\Omega, w).$$

3. Main results

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$. Consider the second order operator $A : W_0^{1,p}(\Omega, w) \longrightarrow W^{-1,p'}(\Omega, w^*)$ in divergence form

$$Au = -\mathrm{div}(a(x, u, \nabla u)),$$

where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function Satisfying the following assumptions:
$(\mathbf{H_2})$ For $i = 1, ..., N$

$$|a_i(x, s, \xi)| \leq \beta w_i^{\frac{1}{p}}(x)[k(x) + \sigma^{\frac{1}{p'}}|s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}], \tag{3.1}$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0 \ \text{ for all } \ \xi \neq \eta \in I\!R^N, \tag{3.2}$$

$$a(x, s, \xi)\xi \geq \alpha \sum_{i=1}^{N} w_i(x)|\xi_i|^p. \tag{3.3}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, β are positive constants.
Assume that $g : \Omega \times I\!R \times I\!R^N \longrightarrow I\!R$ is a Carathéodory function satisfying :
$(\mathbf{H_3})$ $g(x, s, \xi)$ is a Carathéodory function satisfying

$$g(x, s, \xi).s \geq 0, \tag{3.4}$$

$$|g(x,s,\xi)| \le b(|s|)(\sum_{i=1}^{N} w_i(x)|\xi_i|^p + c(x)), \tag{3.5}$$

where $b : I\!R^+ \to I\!R^+$ is a positive increasing function and $c(x)$ is a positive function which belong to $L^1(\Omega)$.
Furthermore we suppose that

$$\mu = f - \mathrm{div} F, \quad f \in L^1(\Omega), \ F \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}), \tag{3.6}$$

Consider the nonlinear problem with Dirichlet boundary condition

$$(P) \begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega, w), \qquad g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u) T_k(u-v)\,dx \\ \qquad\qquad \le \displaystyle\int_\Omega f T_k(u-v)\,dx + \int_\Omega F\nabla T_k(u-v)\,dx \\ \forall v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega) \ \forall k > 0. \end{cases}$$

We shall prove the following existence theorem

Theorem 3.1. *Assume that $(H_1) - (H_3)$ hold true. Then there exists at least one solution of the problèm (P).*

Remark 3.1. *If $w_i = \sigma = q = 1$, the result of the preceding theorem coincides with those of Porretta (see [10]).*

4. Proof of main results

In order to prove the existence theorem we need the following

Lemma 4.1 [2]. *Assume that (H_1) and (H_2) are satisfied, and let $(u_n)_n$ be a sequence in $W_0^{1,p}(\Omega, w)$ such that*

1) $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$,

2) $\displaystyle\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\,dx \to 0$

then,

$$u_n \to u \ \ in \ \ W_0^{1,p}(\Omega, w).$$

We give now the proof of theorem 3.1.
STEP 1. The approximate problem.
Let f_n be a sequence of smooth functions which strongly converges to f in $L^1(\Omega)$.
We Consider the sequence of approximate problems:

$$\begin{cases} u_n \in W_0^{1,p}(\Omega, w), \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla v\,dx + \int_\Omega g_n(x,u_n,\nabla u_n) v\,dx \\ \qquad\qquad = \displaystyle\int_\Omega f_n v\,dx + \int_\Omega F\nabla v\,dx \\ \forall v \in W_0^{1,p}(\Omega, w). \end{cases} \tag{4.1}$$

where $g_n(x,s,\xi)=\frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}\theta_n(x)$ with $\theta_n(x)=nT_{1/n}(\sigma^{1/q}(x))$.
Note that $g_n(x,s,\xi)$ satisfises the following conditions

$$g_n(x,s,\xi)s\geq 0,\quad |g_n(x,s,\xi)|\leq|g(x,s,\xi)| \text{ and } |g_n(x,s,\xi)|\leq n.$$

We define the operator $G_n\ : X\longrightarrow X^*$ by,

$$\langle G_n u,v\rangle=\int_\Omega g_n(x,u,\nabla u)v\,dx$$

and

$$\langle Au,v\rangle=\int_\Omega a(x,u,\nabla u)\nabla v\,dx$$

Thanks to Hölder's inequality, we have for all $u\in X$ and $v\in X$,

$$\begin{aligned}\left|\int_\Omega g_n(x,u,\nabla u)v\ dx\right| &\leq \left(\int_\Omega |g_n(x,u,\nabla u)|^{q'}\sigma^{-\frac{q'}{q}}\ dx\right)^{\frac{1}{q'}}\left(\int_\Omega |v|^q\sigma\ dx\right)^{\frac{1}{q}}\\ &\leq n\left(\int_\Omega \sigma^{q'/q}\sigma^{-q'/q}\ dx\right)^{\frac{1}{q'}}\|v\|_{q,\sigma}\\ &\leq C_n\|v\|_X,\end{aligned}\tag{4.2}$$

the last inequality is due to (2.3) and (2.4).

Lemma 4.2. *The operator $B_n=A+G_n$ from X into its dual X^* is pseudomonotone. Moreover, B_n is coercive, in the following sense:*

$$\frac{< B_n v,v>}{\|v\|_X}\longrightarrow+\infty\quad if\quad \|v\|_X\longrightarrow+\infty, v\in W_0^{1,p}(\Omega,w).$$

This Lemma will be proved below.
In view of Lemma 4.2, there exists at least one solution u_n of (4.1) (cf. Theorem 2.1 and Remark 2.1 in Chapter 2 of [11]).
STEP 2. A priori estimates.
Taking $v=T_k(u_n)$ as test function in (4.1), gives

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)\,dx\\ =\int_\Omega f_nT_k(u_n)\,dx+\int_\Omega F_n\nabla T_k(u_n)\,dx\end{aligned}$$

and by using in fact that $g_n(x,u_n,\nabla u_n)T_k(u_n)\geq 0$, we obtain

$$\int_{\{|u_n|\leq k\}}a(x,u_n,\nabla u_n)\nabla u_n\,dx\leq ck+\int_\Omega F_n\nabla T_k(u_n)\,dx.$$

Thank's to Young's inequality and (3.3), one easily has

$$\frac{\alpha}{2}\int_\Omega\sum_{i=1}^N\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p w_i(x)\ dx\leq c_1k.\tag{4.3}$$

STEP 3. Almost everywhere convergence of u_n.
We prove that u_n converges to some function u locally in measure (and therefore, we can

aloways assume that the convergence is a.e. after passing to a suitable subsequence). To prove this, we show that u_n is a Cauchy sequence in measure in any ball B_R.
Let $k > 0$ large enough, we have

$$\begin{aligned} k \text{ meas}(\{|u_n| > k\} \cap B_R) &= \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\,dx \le \int_{B_R} |T_k(u_n)|\,dx \\ &\le \left(\int_\Omega |T_k(u_n)|^p w_0\,dx\right)^{\frac{1}{p}} \left(\int_{B_R} w_0^{1-p'}\,dx\right)^{\frac{1}{q'}} \\ &\le c_0 \left(\int_\Omega \sum_{i=1}^N \Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p w_i(x)\,dx\right)^{\frac{1}{p}} \\ &\le c_1 k^{\frac{1}{p}}. \end{aligned}$$

Which implies that

$$\text{meas}(\{|u_n| > k\} \cap B_R) \le \frac{c_1}{k^{1-\frac{1}{p}}} \quad \forall k > 1. \tag{4.4}$$

Moreover, we have, for every $\delta > 0$,

$$\begin{aligned} \text{meas}(\{|u_n - u_m| > \delta\} \cap B_R) \le\; & \text{meas}(\{|u_n| > k\} \cap B_R) + \text{meas}(\{|u_m| > k\} \cap B_R) \\ & + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned} \tag{4.5}$$

Since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega, w)$, there exists some $v_k \in W_0^{1,p}(\Omega, w)$, such that

$$\begin{array}{lll} T_k(u_n) \rightharpoonup v_k & \text{weakly in } W_0^{1,p}(\Omega, w) \\ T_k(u_n) \to v_k & \text{strongly in } L^q(\Omega, \sigma) \text{ and } a.e. \text{ in } \Omega. \end{array}$$

Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.
Let $\varepsilon > 0$, then, by (4.4) and (4.5), there exists some $k(\varepsilon) > 0$ such that $\text{meas}(\{|u_n - u_m| > \delta\} \cap B_R) < \varepsilon$ for all $n, m \ge n_0(k(\varepsilon), \delta, R)$. This proves that $(u_n)_n$ is a Cauchy sequence in measure in B_R, thus converges almost everywhere to some measurable function u. Then

$$\begin{array}{lll} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^{1,p}(\Omega, w), \\ T_k(u_n) \to T_k(u) & \text{strongly in } L^q(\Omega, \sigma) \text{ and } a.e. \text{ in } \Omega. \end{array}$$

STEP 4. Strong convergence of truncations.
We fix $k > 0$, and let $h > k > 0$.
We shall use in (4.1) the test function

$$\begin{cases} v_n = \phi(w_n) \\ w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)), \end{cases} \tag{4.6}$$

with $\phi(s) = se^{\gamma s^2}, \gamma = (\frac{b(k)}{\alpha})^2$.
It is well known that

$$\phi'(s) - \frac{b(k)}{\alpha}|\phi(s)| \ge \frac{1}{2} \quad \forall s \in I\!R, \tag{4.7}$$

It follows that

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n)\nabla w_n \phi'(w_n)\,dx + \int_\Omega g_n(x, u_n, \nabla u_n)\phi(w_n)\,dx \\ = \int_\Omega f_n \phi(w_n)\,dx + \int_\Omega F\nabla\phi(w_n)\,dx. \end{aligned} \tag{4.8}$$

Since $\phi(w_n)g_n(x,u_n,\nabla u_n) > 0$ on the subset $\{x \in \Omega, |u_n(x)| > k\}$, we deduce from (4.8) that

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla w_n \phi'(w_n)\,dx + \int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\phi(w_n)\,dx \\ \le \int_\Omega f_n\phi(w_n)\,dx + \int_\Omega F\nabla\phi(w_n)\,dx.\end{aligned} \tag{4.9}$$

Denote by $\varepsilon_h^1(n), \varepsilon_h^2(n), ...$ various sequences of real numbers which converge to zero as n tends to infinity for any fixed value of h.
We will deal with each term of (4.9). First of all, observe that

$$\int_\Omega f_n\phi(w_n)\,dx = \int_\Omega f\phi(T_{2k}(u - T_h(u)))\,dx + \varepsilon_h^1(n) \tag{4.10}$$

and

$$\int_\Omega F\nabla\phi(w_n)\,dx = \int_\Omega F\nabla T_{2k}(u - T_h(u))\phi'(T_{2k}(u - T_h(u)))\,dx + \varepsilon_h^2(n). \tag{4.11}$$

Splitting the first integral on the left hand side of (4.9) where $|u_n| \le k$ and $|u_n| > k$, we can write,

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla w_n\phi'(w_n)\,dx \\ &= \int_{\{|u_n|\le k\}} a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)\,dx \\ &\quad + \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla w_n\phi'(w_n)\,dx.\end{aligned} \tag{4.12}$$

Setting $m = 4k + h$, using $a(x,s,\xi)\xi \ge 0$ and the fact that $\nabla w_n = 0$ on the set where $|u_n| > m$, we have

$$\begin{aligned}&\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla w_n\phi'(w_n)\,dx \\ &\ge -\phi'(2k)\int_{\{|u_n|>k\}} |a(x,T_m(u_n),\nabla T_m(u_n))||\nabla T_k(u)|\,dx,\end{aligned} \tag{4.13}$$

and since $a(x,s,0) = 0 \quad \forall s \in I\!R$, we have

$$\begin{aligned}&\int_{\{|u_n|\le k\}} a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)\,dx \\ &= \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)\,dx.\end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14), we get

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla w_n\phi'(w_n)\,dx \\ &\ge \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)\,dx \\ &\quad -\phi'(2k)\int_{\{|u_n|>k\}} |a(x,T_m(u_n),\nabla T_m(u_n))||\nabla T_k(u)|\,dx.\end{aligned} \tag{4.15}$$

The second term of the right hand side of the last inequality tends to 0 as n tends to infinity. Indeed. Since the sequence $(a(x, T_m(u_n), \nabla T_m(u_n)))_n$ is bounded in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$ while $\nabla T_k(u)\chi_{|u_n|>k}$ tends to 0 strongly in $\prod_{i=1}^{N} L^p(\Omega, w_i)$, which yields

$$\begin{aligned}\int_\Omega & a(x, u_n, \nabla u_n)\nabla w_n \phi'(w_n)\,dx \\ &\geq \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)\,dx + \varepsilon_h^3.\end{aligned} \tag{4.16}$$

On the other hand, the term of the right hand side of (4.16) reads as

$$\begin{aligned}\int_\Omega & a(x, T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)\,dx \\ &= \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\ &\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)\,dx \\ &\quad + \int_\Omega a(x, T_k(u_n), \nabla T_k(u))\nabla T_k(u_n)\phi'(T_k(u_n) - T_k(u))\,dx \\ &\quad - \int_\Omega a(x, T_k(u_n), \nabla T_k(u))\nabla T_k(u)\phi'(w_n)\,dx.\end{aligned} \tag{4.17}$$

Since $a_i(x, T_k(u_n), \nabla T_k(u))\phi'(T_k(u_n) - T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))\phi'(0)$ strongly in $L^{p'}(\Omega, w_i^{1-p'})$ by using the continuity of the Nymetskii operator, while $\frac{\partial(T_k(u_n))}{\partial x_i} \rightharpoonup \frac{\partial(T_k(u))}{\partial x_i}$ weakly in $L^p(\Omega, w_i)$, we have

$$\begin{aligned}\int_\Omega & a(x, T_k(u_n), \nabla T_k(u))\nabla T_k(u_n)\phi'(T_k(u_n) - T_k(u))\,dx \\ &= \int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\phi'(0)\,dx + \varepsilon_h^4(n).\end{aligned} \tag{4.18}$$

In the same way, we have

$$\begin{aligned}-\int_\Omega & a(x, T_k(u_n), \nabla T_k(u))\nabla T_k(u)\phi'(w_n)\,dx \\ &= -\int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\phi'(0)\,dx + \varepsilon_h^5(n).\end{aligned} \tag{4.19}$$

Combining (4.16)-(4.19), we get

$$\begin{aligned}\int_\Omega & a(x, u_n, \nabla u_n)\nabla w_n \phi'(w_n)\,dx \\ &\geq \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\ &\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)\,dx + \varepsilon_h^6(n).\end{aligned} \tag{4.20}$$

The second term of the left hand side of (4.9), can be estimated as

$$\begin{aligned}\left|\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\phi(w_n)\,dx\right| &\leq \int_{\{|u_n|\leq k\}} b(k)\left(c(x) + \sum_{i=1}^{N} w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p\right)|\phi(w_n)|\,dx \\ &\leq b(k)\int_\Omega c(x)|\phi(w_n)|\,dx \\ &\quad + \tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\phi(w_n)|\,dx.\end{aligned} \tag{4.21}$$

Since $c(x)$ belongs to $L^1(\Omega)$ it is easy to see that

$$b(k)\int_\Omega c(x)|\phi(w_n)|\,dx = b(k)\int_\Omega c(x)|\phi(T_{2k}(u-T_h(u)))|\,dx + \varepsilon_h^7(n). \tag{4.22}$$

On the other side, we have

$$\begin{aligned}
&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\phi(w_n)|\,dx \\
&\quad= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\
&\qquad\qquad \times[\nabla T_k(u_n)-\nabla T_k(u)]|\phi(w_n)|\,dx \\
&\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\phi(w_n)|\,dx \\
&\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]|\phi(w_n)|\,dx.
\end{aligned} \tag{4.23}$$

As above, by letting n go to infinity, we can easily see that each one of last two integrals of the right-hand side of the last equality is of the form $\varepsilon_h^8(n)$ and then

$$\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\phi(w_n)\,dx\right| \\
&\quad\le \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\
&\qquad\qquad \times[\nabla T_k(u_n)-\nabla T_k(u)]|\phi(w_n)|\,dx \\
&\qquad + b(k)\int_\Omega c(x)|\phi(T_{2k}(u-T_h(u)))|\,dx + \varepsilon_h^9(n).
\end{aligned} \tag{4.24}$$

(4.9)-(4.11), (4.20) and (4.24), we get

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\
&\qquad \times[\nabla T_k(u_n)-\nabla T_k(u)](\phi'(w_n) - \tfrac{b(k)}{\alpha}|\phi(w_n)|)\,dx \\
&\qquad \le b(k)\int_\Omega c(x)|\phi(T_{2k}(u-T_h(u)))|\,dx + \int_\Omega f\phi(T_{2k}(u-T_h(u)))\,dx, \\
&\qquad\quad + \int_\Omega F\nabla T_{2k}(u-T_h(u))\phi'(T_{2k}(u-T_h(u)))\,dx + \varepsilon_h^{10}(n),
\end{aligned}$$

which and (4.7) implies that

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\
&\qquad \times[\nabla T_k(u_n)-\nabla T_k(u)]\,dx \\
&\qquad \le 2b(k)\int_\Omega c(x)|\phi(T_{2k}(u-T_h(u)))|\,dx + 2\int_\Omega f\phi(T_{2k}(u-T_h(u)))\,dx, \\
&\qquad\quad + 2\int_\Omega F\nabla T_{2k}(u-T_h(u))\phi'(T_{2k}(u-T_h(u)))\,dx + \varepsilon_h^{11}(n),
\end{aligned}$$

in which, we can pass to the limit as $n\to+\infty$ to obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx \\
&\qquad \le 2b(k)\int_\Omega c(x)|\phi(T_{2k}(u-T_h(u)))|\,dx + 2\int_\Omega f\phi(T_{2k}(u-T_h(u)))\,dx, \\
&\qquad\quad + 2\int_\Omega F\nabla T_{2k}(u-T_h(u))\phi'(T_{2k}(u-T_h(u)))\,dx.
\end{aligned} \tag{4.25}$$

It remains to show, for our purposes, that the all terms on the right hand side of (4.25) converge to zero as h goes to infinity. The only difficulty that exists is in the last term. For the other terms it suffices to apply Lebesgue's theorem.
We deal with this term. Let us observe that, if we take $\phi(T_{2k}(u_n - T_h(u_n)))$ as test function in (4.1) and use (3.3), we obtain

$$\begin{aligned}\alpha\int_{\{h\leq|u_n|\leq 2k+h\}}\sum_{i=1}^{N}|\frac{\partial u_n}{\partial x_i}|^p w_i\phi'(T_{2k}(u_n - T_h(u_n)))\,dx\\ +\int_\Omega g_n(x,u_n,\nabla u_n)\phi(T_{2k}(u_n - T_h(u_n)))\,dx\\ \leq \int_{\{h\leq|u_n|\leq 2k+h\}} F\nabla u_n\phi'(T_{2k}(u_n - T_h(u_n)))\,dx\\ \int_\Omega f_n\phi(T_{2k}(u_n - T_h(u_n)))\,dx,\end{aligned}$$

and thanks to the sign condition (3.4), we get

$$\begin{aligned}\alpha\int_{\{h\leq|u_n|\leq 2k+h\}}\sum_{i=1}^{N}|\frac{\partial u_n}{\partial x_i}|^p w_i\phi'(T_{2k}(u_n - T_h(u_n)))\,dx\\ \leq \int_{\{h\leq|u_n|\leq 2k+h\}} F\nabla u_n\phi'(T_{2k}(u_n - T_h(u_n)))\,dx\\ +\int_\Omega f_n\phi(T_{2k}(u_n - T_h(u_n)))\,dx.\end{aligned}$$

Using the Young inequality we have

$$\begin{aligned}\tfrac{\alpha}{2}\int_{\{h\leq|u_n|\leq 2k+h\}}\sum_{i=1}^{N}|\frac{\partial u_n}{\partial x_i}|^p w_i\phi'(T_{2k}(u_n - T_h(u_n)))\,dx\\ \leq \int_\Omega f_n\phi(T_{2k}(u_n - T_h(u_n)))\,dx + c_k\int_{\{h\leq|u_n|\}}|w^{\frac{-1}{p}}.F|^{p'}\,dx,\end{aligned}\tag{4.26}$$

so that, since $\phi'\geq 1$, we have

$$\begin{aligned}\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u-T_h(u))}{\partial x_i}\right|^p w_i\,dx\\ \leq \int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u-T_h(u))}{\partial x_i}\right|^p w_i\phi'(T_{2k}(u-T_h(u)))\,dx,\end{aligned}$$

again because the norm is lower semi-continuity, we get

$$\begin{aligned}\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u-T_h(u))}{\partial x_i}\right|^p w_i\phi'(T_{2k}(u-T_h(u)))\,dx\\ \leq c_k\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u-T_h(u))}{\partial x_i}\right|^p w_i\,dx\\ \leq \liminf_{n\to\infty}\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u_n-T_h(u_n))}{\partial x_i}\right|^p w_i\,dx\\ \leq c_k\liminf_{n\to\infty}\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u_n-T_h(u_n))}{\partial x_i}\right|^p w_i\phi'(T_{2k}(u_n-T_h(u_n)))\,dx\end{aligned}\tag{4.27}$$

Consequently, in view of (4.26) and (4.27), we obtain

$$\int_\Omega \sum_{i=1}^N \Big|\frac{\partial T_{2k}(u-T_h(u))}{\partial x_i}\Big|^p w_i \phi'(T_{2k}(u-T_h(u)))\,dx$$
$$\le \liminf_{n\to\infty} c_k \int_{\{h\le|u_n|\}} |w^{\frac{-1}{p}} F|^{p'}\,dx$$
$$+ \liminf_{n\to\infty} \int_\Omega f_n \phi(T_{2k}(u_n - T_h(u_n)))\,dx.$$

Finally, the strong convergence in $L^1(\Omega)$ of f_n, we have, as first n and then h tend to infinity,

$$\limsup_{h\to\infty} \int_{\{h\le|u|\le 2k+h\}} \sum_{i=1}^N |\frac{\partial u}{\partial x_i}|^p w_i \phi'(T_{2k}(u-T_h(u)))\,dx = 0,$$

hence

$$\lim_{h\to\infty} \int_\Omega F\nabla T_{2k}(u-T_h(u))\phi'(T_{2k}(u-T_h(u)))\,dx = 0.$$

Therefore by (4.25), letting h go to infinity, we conclude,

$$\lim_{n\to\infty} \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx = 0,$$

which and using Lemma 4.1 implies that

$$T_k(u_n) \to T_k(u) \;\text{ strongly in }\; W_0^{1,p}(\Omega,w) \;\; \forall k>0. \tag{4.28}$$

STEP 5. Passing to the limit.
By using $T_k(u_n - v)$ as test function in (4.1), with $v \in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$, we get

$$\int_\Omega a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n-v)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx$$
$$= \int_\Omega f_n T_k(u_n-v)\,dx + \int_\Omega F\nabla T_k(u_n-v)\,dx. \tag{4.29}$$

By Fatou's lemma and the fact that

$$a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))$$

weakly in $\prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ one easily sees that

$$\int_\Omega a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))\nabla T_k(u-v)\,dx$$
$$\le \liminf_{n\to\infty} \int_\Omega a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n-v)\,dx. \tag{4.30}$$

For the second term of the right hand side of (4.29), we have

$$\int_\Omega F\nabla T_k(u_n-v)\,dx \longrightarrow \int_\Omega F\nabla T_k(u-v)\,dx \;\text{ as }\; n\to\infty. \tag{4.31}$$

since $\nabla T_k(u_n - v) \rightharpoonup \nabla T_k(u - v)$ weakly in $\prod_{i=1}^{N} L^p(\Omega, w_i)$, while $F \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$.
On the other hand, we have

$$\int_\Omega f_n T_k(u_n - v)\, dx \longrightarrow \int_\Omega f T_k(u - v)\, dx \quad \text{as} \quad n \to \infty. \tag{4.32}$$

To conclude the proof of theorem, it only remains to prove

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \quad \text{strongly in} \quad L^1(\Omega), \tag{4.33}$$

in particular it is enough to prove the equiintegrable of $g_n(x, u_n, \nabla u_n)$. To this purpose, we take $T_{l+1}(u_n) - T_l(u_n)$ as test function in (4.1), we obtain

$$\int_{\{|u_n|>l+1\}} |g_n(x, u_n, \nabla u_n)|\, dx \leq \int_{\{|u_n|>l\}} |f_n|\, dx.$$

Let $\varepsilon > 0$. Then there exists $l(\varepsilon) \geq 1$ such that

$$\int_{\{|u_n|>l(\varepsilon)\}} |g_n(x, u_n, \nabla u_n)|\, dx < \varepsilon/2. \tag{4.34}$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned} \int_E |g_n(x, u_n, \nabla u_n)|\, dx \quad &\leq \quad \int_E b(l(\varepsilon)) \left(c(x) + \sum_{i=1}^{N} w_i \Big|\frac{\partial (T_{l(\varepsilon)}(u_n))}{\partial x_i}\Big|^p \right) dx \\ &\quad + \int_{\{|u_n|>l(\varepsilon)\}} |g_n(x, u_n, \nabla u_n)|\, dx. \end{aligned}$$

In view of (4.28) there exists $\eta(\varepsilon) > 0$ such that

$$\int_E b(l(\varepsilon)) \left(c(x) + \sum_{i=1}^{N} w_i \Big|\frac{\partial (T_{l(\varepsilon)}(u_n))}{\partial x_i}\Big|^p \right) dx < \varepsilon/2 \quad \text{for all E such that meas } E < \eta(\varepsilon). \tag{4.35}$$

Finally, by combining (4.34) and (4.35) one easily has

$$\int_E |g_n(x, u_n, \nabla u_n)|\, dx < \varepsilon \quad \text{for all} \quad E \quad \text{such that} \quad \text{meas } E < \eta(\varepsilon),$$

which shows that $g_n(x, u_n, \nabla u_n)$ are uniformly equintegrable in Ω as required.
Thanks to (4.30)-(4.33) we can pass to the limit in (4.29) and we obtain that u is a solution of the problem (P).
This completes the proof of Theorem 3.1.

Remark 4.1. *Note that, we obtain the existence result withowt assuming the coercivity condition. However one can overcome this difficulty by introduced the function* $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ *in the test function* (4.6).

Proof of Lemma 4.2.
From Hölder's inequality, the growth condition (3.1) we can show that A is bounded, and by using (4.2), we have B_n bounded. The coercivity folows from (3.3) and (3.4). it remain to

show that B_n is pseudo-monotone.
Let a sequence $(u_k)_k \in W_0^{1,p}(\Omega, w)$ such that

$$u_k \rightharpoonup u \ \text{ weakly in } \ W_0^{1,p}(\Omega, w),$$

$$B_n u_k \rightharpoonup \chi \ \text{ weakly in } \ W^{-1,p'}(\Omega, w^*),$$

and $\limsup_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi, u\rangle$.
We will prove that

$$\chi = B_n u \ \text{ and } \ \langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \ \text{ as } \ k \to +\infty.$$

Since $(u_k)_k$ is a bounded sequence in $W_0^{1,p}(\Omega, w)$, we deduce that $(a(x, u_k, \nabla u_k))_k$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$, then there exists a function $h \in \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ such that

$$a(x, u_k, \nabla u_k) \rightharpoonup h \ \text{ weakly in } \ \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'}) \ \text{ as } \ k \to \infty,$$

similarly, it is easy to see that $(g_n(x, u_k, \nabla u_k))_k$ is bounded in $L^{q'}(\Omega, \sigma^{1-q'})$, then there exists a function $k_n \in L^{q'}(\Omega, \sigma^{1-q'})$ such that

$$g_n(x, u_k, \nabla u_k) \rightharpoonup k_n \ \text{ weakly in } \ Ł^{q'}(\Omega, \sigma^{1-q'}) \ \text{ as } \ k \to \infty.$$

It is clear that, for all $w \in W_0^{1,p}(\Omega, w)$, we have

$$\begin{aligned}
\langle \chi, w\rangle &= \lim_{k\to+\infty} \langle B_n u_k, w\rangle \\
&= \lim_{k\to+\infty} \int_\Omega a(x, u_k, \nabla u_k)\nabla w \, dx \\
&\quad + \lim_{k\to+\infty} \int_\Omega g_n(x, u_k, \nabla u_k).w \, dx.
\end{aligned}$$

Consequently, we get

$$\langle \chi, w\rangle = \int_\Omega h\nabla w \, dx + \int_\Omega k_n.w \, dx \quad \forall w \in W_0^{1,p}(\Omega, w). \tag{4.36}$$

On the one hand, we have

$$\int_\Omega g_n(x, u_k, \nabla u_k).u_k \, dx \longrightarrow \int_\Omega k_n.u \, dx \ \text{ as } \ k \to \infty, \tag{4.37}$$

and, by hypotheses, we have

$$\begin{aligned}
\limsup_{k\to\infty} \Big\{ \int_\Omega a(x, u_k, \nabla u_k)\nabla u_k \, dx + \int_\Omega g_n(x, u_k, \nabla u_k).u_k \, dx \Big\} \\
\leq \int_\Omega h\nabla u \, dx + \int_\Omega k_n.u \, dx,
\end{aligned}$$

therefore

$$\limsup_{k\to\infty} \int_\Omega a(x, u_k, \nabla u_k)\nabla u_k \, dx \leq \int_\Omega h\nabla u \, dx. \tag{4.38}$$

By virtue of (3.2), we have

$$\int_\Omega (a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))(\nabla u_k-\nabla u)\,dx>0. \tag{4.39}$$

Consequently

$$\begin{aligned}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \;&\geq\; -\int_\Omega a(x,u_k,\nabla u)\nabla u\,dx+\int_\Omega a(x,u_k,\nabla u_k)\nabla u\,dx\\ &\quad +\int_\Omega a(x,u_k,\nabla u)\nabla u_k\,dx,\end{aligned}$$

hence

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx\geq\int_\Omega h\nabla u\,dx.$$

This implies by using (4.38)

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx=\int_\Omega h\nabla u\,dx. \tag{4.40}$$

By means of (4.36), (4.37) and (4.40), we obtain

$$\langle B_n u_k,u_k\rangle\to\langle\chi,u\rangle \;\text{ as }\; k\longrightarrow+\infty.$$

On the other hand, by (4.40) and the fact that $a(x,u_k,\nabla u)\to a(x,u,\nabla u)$ strongly in $\prod_{i=1}^N L^{p'}(\Omega,w_i^{1-p'})$ it can be easily seen that

$$\lim_{k\to+\infty}\int_\Omega(a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))(\nabla u_k-\nabla u)\,dx=0,$$

and so, thanks to Lemma 4.1

$$\nabla u_n\to\nabla u \;\text{ a.e. in }\; \Omega.$$

We deduce then that

$$a(x,u_k,\nabla u_k)\to a(x,u,\nabla u)\;\text{ weakly in }\;\prod_{i=1}^N L^{p'}(\Omega,w_i^{1-p'}),$$

$$\text{and}\quad g_n(x,u_k,\nabla u_k)\to g(x,u,\nabla u)\;\text{ weakly in }\; L^{q'}(\Omega,w_i^{1-q'}).$$

Thus implies that $\chi=B_n u$.

Corollary 4.1. *Let $1<p<\infty$. Assume that the hypothesis $(H_1)-(H_3)$ holds, let f_n be any sequence of functions in $L^1(\Omega)$ which converge to f weakly in $L^1(\Omega)$ and let u_n the solution of the following problem*

$$(P_n')\begin{cases} u_n\in\mathcal{T}_0^{1,p}(\Omega,w),\quad g(x,u_n,\nabla u_n)\in L^1(\Omega)\\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx+\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-v)\,dx\\ \qquad\qquad\qquad\displaystyle\leq\int_\Omega f_nT_k(u_n-v)\,dx,\\ \forall\;\; v\in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega),\;\forall k>0.\end{cases}$$

Then there exists a subsequence of u_n still denoted u_n such that u_n converges to u almost everywhere and $T_k(u_n)\rightharpoonup T_k(u)$ strongly in $W_0^{1,p}(\Omega,w)$, further u is a solution of the problem (P) (with $F=0$).

Proof. We give a brief proof.
Step 1. A priori estimates.
As before we take $v = 0$ as test function in (P'_n), we get

$$\int_\Omega \sum_{i=1}^{N} w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p dx \le C_1 k. \tag{4.41}$$

Hence, by the same method used in the first step in the proof of Theorem 3.1 there exists a function $u \in \mathcal{T}_0^{1,p}(\Omega, w)$ and a subsequence still denoted by u_n such that

$$u_n \to u \text{ a.e. in } \Omega, T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega, w), \ \forall k > 0.$$

Step 2. Strong convergence of truncation.
The choice of $v = T_h(u_n - \phi(w_n))$ as test function in (P'_n), we get, for all $l > 0$

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n)\nabla T_l(u_n - T_h(u_n - \phi(w_n)))\, dx + \int_\Omega g(x, u_n, \nabla u_n)T_l(u_n - T_h(u_n - \phi(w_n))\, dx \\ \le \int_\Omega f_n T_l(u_n - T_h(u_n - \phi(w_n)))\, dx.\end{aligned}$$

Which implies that

$$\begin{aligned}\int_{\{|u_n-\phi(w_n)|\le h\}} a(x, u_n, \nabla u_n)\nabla T_l(\phi(w_n)) \\ + \int_\Omega g(x, u_n, \nabla u_n)T_l(u_n - T_h(u_n - \phi(w_n)))\, dx \\ \le \int_\Omega f_n T_l(u_n - T_h(u_n - \phi(w_n)))\, dx.\end{aligned}$$

Letting h tend to infinity and choosing l large enough, we deduce

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n)\nabla \phi(w_n)\, dx + \int_\Omega g(x, u_n, \nabla u_n)\phi(w_n)\, dx \\ \le \int_\Omega f_n \phi(w_n)\, dx,\end{aligned}$$

the rest of the proof of this step is the same as in step 4 of the proof of Theorem 3.1.
Step 3. Passing to the limit.
This step is similar to the step 5 of the proof of Theorem 3.1, by using the Egorov's theorem in the last term of (P'_n).

Remark 4.2. *In the case where $F = 0$, if we suppose that the second member is nonnegative, then we obtain a nonnegative solution.*

Indeed. If we take $v = T_h(u^+)$ in (P), we have

$$\begin{aligned}\int_\Omega a(x, u, \nabla u)\nabla T_k(u - T_h(u^+))\, dx \\ + \int_\Omega g(x, u, \nabla u)T_k(u - T_h(u^+))\, dx \\ \le \int_\Omega f T_k(u - T_h(u^+))\, dx.\end{aligned}$$

Since $g(x, u, \nabla u) T_k(u - T_h(u^+)) \geq 0$, we deduce

$$\int_\Omega a(x, u, \nabla u) \nabla T_k(u - T_h(u^+)) \, dx \leq \int_\Omega f T_k(u - T_h(u^+)) \, dx,$$

and remark also that by using $f \geq 0$ we have

$$\int_\Omega f T_k(u - T_h(u^+)) \, dx \leq \int_{\{u \geq h\}} f T_k(u - T_h(u)) \, dx.$$

On the other hand, thanks to (3.3), we conclude

$$\alpha \int_\Omega \sum_{i=1}^N w_i \left| \frac{\partial T_k(u^-)}{\partial x_i} \right|^p dx \leq \int_{\{u \geq h\}} f T_k(u - T_h(u)) \, dx.$$

Letting h tend to infinity, we can easily deduce

$$T_k(u^-) = 0, \quad \forall k > 0,$$

which implies that

$$u \geq 0.$$

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