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ASYMPTOTIC BEHAVIOR OF POSITIVE LARGE SOLUTIONS OF SEMILINEAR DIRICHLET PROBLEMS

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ABSTRACT. Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$. This paper deals with the existence and the asymptotic behavior of positive solutions of the following problems

$$\Delta u = a(x)u^{\alpha}, \alpha > 1 \text{ and } \Delta u = a(x)e^{u},$$

with the boundary condition $u_{|\partial\Omega} = +\infty$. The weight function $a(x)$ is positive in $C^{\gamma}_{loc}(\Omega)$, $0 < \gamma < 1$, and satisfies an appropriate assumption related to Karamata regular variation theory.

Our arguments are based on the sub-supersolution method.

1. Introduction

Let Ω be a C^2 bounded domain in $\mathbb{R}^n$, $n \geq 2$. In this paper, we deal with existence and estimates of solutions to the following elliptic problems

$$\begin{cases} \Delta u = a(x)u^{\alpha}, \ x \in \Omega, \ \alpha > 1, \\ u > 0, \quad \text{in} \quad \Omega, \\ \lim\limits_{\delta(x) \to 0} u\,(x) = +\infty \end{cases} \tag{1.1}$$

and

$$\begin{cases} \Delta u = a(x)e^{u}, \ x \in \Omega, \\ u > 0, \quad \text{in} \quad \Omega, \\ \lim\limits_{\delta(x) \to 0} u\,(x) = +\infty. \end{cases} \tag{1.2}$$

Here, a is a positive function in $C^{\gamma}_{loc}(\Omega)$, $0 < \gamma < 1$, satisfying an appropriate condition related to Karamata regular variation theory and $\delta(x) = dist(x, \partial\Omega)$.

Problems like (1.1) and (1.2) have been largely studied and their solutions are sometimes called large (or explosive) solutions. We refer the reader to ([1], [2], [5-9], [11-15], [18-20], [22-24]).

In [14], motivated by certain geometric problems, Loewner and Nirenberg studied problem (1.1) with $a \equiv 1$ and $\alpha = \frac{n+2}{n-2}$, $n \geq 3$. Later in [1], Bandle and Marcus considered problem (1.1) with a is a positive continuous function in $\overline{\Omega}$ such that a and $1/a$ are both bounded. The authors described the asymptotic behavior near the boundary of the unique large solution of problem (1.1).

The study of large solutions of problem (1.2) in the case $a \equiv 1$ goes back to the pioneering Bieberbach's paper in 1916, for $n = 2$ and Rademacher's work in 1943, for $n = 3$, (see [2]

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and [19]). Later in [15], Lazer and McKenna considered problem (1.2) with a in $C(\overline{\Omega})$ and they gave estimates near the boundary on the unique solution of problem (1.2).
More recently, some results of existence and nonexistence of solutions to problems (1.1) and (1.2) are established when the weight $a(x)$ is unbounded near $\partial\Omega$ (see [5], [18], [22], [23], [24]). For instance, in [5] Chuaqui et al. considered problems (1.1) and (1.2), where the function a satisfies the following conditions.

(A_1) $a \in C^{\gamma}_{loc}(\Omega)$, $0 < \gamma < 1$,

(A_2) There exist positive constants C_1, C_2 such that

$$C_1\left(\delta\left(x\right)\right)^{-\lambda} \leq a(x) \leq C_2\left(\delta\left(x\right)\right)^{-\lambda},\ \lambda > 0.$$

Then they proved the following theorems.

Theorem 1. *Assume (A_1) and (A_2). Then problem (1.1) has no positive solutions if $\lambda \geq 2$, and it has a unique positive solution $u \in C^{2,\gamma}(\Omega)$ when $0 < \lambda < 2$. Moreover, there exist m, $M > 0$ such that*

$$m\left(\delta(x)\right)^{\frac{2-\lambda}{1-\alpha}} \leq u(x) \leq M\left(\delta(x)\right)^{\frac{2-\lambda}{1-\alpha}},\quad x \in \Omega. \tag{1.3}$$

Theorem 2. *Assume (A_1) and (A_2). Then problem (1.2) has no solutions if $\lambda \geq 2$, and it has a unique solution $u \in C^{2,\gamma}(\Omega)$ when $0 < \lambda < 2$. Moreover, there exist m_1, $M_1 > 0$ such that*

$$m_1\left(\delta(x)\right)^{\lambda-2} \leq e^{u(x)} \leq M_1\left(\delta(x)\right)^{\lambda-2},\quad x \in \Omega. \tag{1.4}$$

In this paper, we take up problems (1.1) and (1.2) respectively as an extension of the above results. More precisely, we prove the existence of classical solutions for both problems (1.1) and (1.2) and we establish the asymptotic behavior of such solutions where the function a is required to be in a large class of functions related to Karamata regular variation theory. To state our results in details, we need some notations.

For two nonnegative functions f and g defined on a set S, the notation $f\left(x\right) \approx g\left(x\right)$, $x \in S$, means that there exists $c > 0$ such that

$$\frac{1}{c}f(x) \leq g(x) \leq cf(x),\ \text{for all } x \in S.$$

We denote by φ_1 the positive normalized (i.e $\max\limits_{x\in\Omega} \varphi_1(x) = 1$) eigenfunction corresponding to the first positive eigenvalue λ_1 of the Laplace operator $(-\Delta)$. It is well known (see [20] for example) that φ_1 is a positive function in $C^2\left(\overline{\Omega}\right)$ and we have for $x \in \Omega$,

$$\varphi_1\left(x\right) \approx \delta(x), \tag{1.5}$$

and

$$\varphi_1^2\left(x\right) + \left|\bigtriangledown\varphi_1(x)\right|^2 \approx 1. \tag{1.6}$$

We shall use $\mathcal{K}$ to denote the set of Karamata functions L defined on $(0, \eta]$ by

$$L\left(t\right) := c\exp\left(\int_t^{\eta} \frac{z(s)}{s}ds\right),$$

for some $\eta > 0$, where $c > 0$ and $z \in C^1((0, \eta])$ such that $z(0) = 0$ and $\lim\limits_{t\to 0^+} tz'(t) = 0$.

Now, let us state our hypothesis on the function a.

($\mathbf{H}$) $a \in C^{\gamma}_{loc}(\Omega)$, $0 < \gamma < 1$ and satisfies for each $x \in \Omega$

$$a(x) \approx \delta(x)^{-\lambda} L(\delta(x)),$$

where $\lambda \leq 2$ and $L \in \mathcal{K}$ defined on $(0, \eta]$, $(\eta > diam(\Omega))$ such that $\int_0^\eta t^{1-\lambda} L(t)dt < \infty$. Our main results are the following.

Theorem 3. *Let a be a function satisfying (H). Then problem (1.1) has a positive solution $u \in C^{2,\gamma}(\Omega)$ satisfying for $x \in \Omega$,*

$$u(x) \approx (\delta(x))^{\frac{2-\lambda}{1-\alpha}} \theta(\delta(x)), \tag{1.7}$$

where θ is the function defined on $(0, \eta)$ by

$$\theta(t) := \begin{cases} (L(t))^{\frac{1}{1-\alpha}}, & \text{if } \lambda < 2, \\ \left(\int_0^t \frac{L(s)}{s} ds\right)^{\frac{1}{1-\alpha}}, & \text{if } \lambda = 2. \end{cases} \tag{1.8}$$

Theorem 4. *Let a be a function satisfying (H). Then problem (1.2) has a positive solution $u \in C^{2,\gamma}(\Omega)$ satisfying for $x \in \Omega$,*

$$e^{u(x)} \approx (\delta(x))^{\lambda-2} \Psi(\delta(x)), \tag{1.9}$$

where Ψ is the function defined on $(0, \eta)$ by

$$\Psi(t) := \begin{cases} (L(t))^{-1}, & \text{if } \lambda < 2, \\ \left(\int_0^t \frac{L(s)}{s} ds\right)^{-1}, & \text{if } \lambda = 2. \end{cases} \tag{1.10}$$

Remark 1. 1) *We need to verify condition $\int_0^\eta t^{1-\lambda} L(t)dt < \infty$, only if $\lambda = 2$. This is due to Lemma 3 below.*
2) *We note that the functions θ and Ψ defined respectively by (1.8) and (1.10) belong to $\mathcal{K}$(see Lemma 1 and Lemma 4 below).*
3) *If we take $L = 1$ and $\lambda < 2$ in hypothesis (H), then we find again estimates (1.3) and (1.4).*
4) *Our results extend Theorems 1 and 2. Namely, we obtain existence and asymptotic behavior of solutions of problems (1.1) and (1.2) when the weight a belongs to $C^{\gamma}_{loc}(\Omega)$ and satisfies*

$$a(x) \approx \delta(x)^{-2} L(\delta(x)),$$

where $L \in \mathcal{K}$ and $\int_0^\eta \frac{L(s)}{s} ds < \infty$.

The paper is organized as follows. In Section 2, we present some useful properties of Karamata functions. Section 3 deals with proofs of our main results. The last section is reserved to some applications.

2. The Karamata class $\mathcal{K}$

Our approach relies on the Karamata regular variation theory established by Karamata in 1930 which is a basic tool in stochastic process (see [3], [17], [21] and references therein). This theory has been applied to study the asymptotic behavior of solutions to differential equations. We refer the reader to [4], [6], [16], [20] and [24] for more details. In what follows, we recapitulate some basic properties of functions in $\mathcal{K}$.

Lemma 1. *Let $L_1, L_2 \in \mathcal{K}$, $p \in \mathbb{R}$ and $\varepsilon > 0$. Then we have*

$$L_1 L_2 \in \mathcal{K},\ L_1^p \in \mathcal{K}$$

and

$$\lim_{t \to 0^+} t^{\varepsilon} L_1(t) = 0.$$

Example 1. *Let $m \in \mathbb{N}^*$. Let $c > 0$, $(\mu_1, \mu_2, ..., \mu_m) \in \mathbb{R}^m$ and ω be a sufficiently large positive real number such that the function*

$$L(t) = c \prod_{k=1}^{m} \left(\log_k \left(\tfrac{\omega}{t}\right)\right)^{\mu_k}$$

is defined and positive on $(0, \eta]$, for some $\eta > 0$, where $\log_k x = \log \circ \log \circ ... \circ \log x$ (k times). Then $L \in \mathcal{K}$.

Lemma 2. *A function L is in $\mathcal{K}$ if and only if L is a positive function in $C^2((0, \eta])$ satisfying*

$$\lim_{t \to 0^+} \frac{tL'(t)}{L(t)} = \lim_{t \to 0^+} \frac{t^2 L''(t)}{L(t)} = 0. \tag{2.1}$$

Proof. Let $L \in \mathcal{K}$. Since $L(t) := c \exp(\int_t^{\eta} \frac{z(s)}{s} ds)$, then for $t \in (0, \eta]$, we have

$$\frac{tL'(t)}{L(t)} = -z(t) \quad \text{and} \quad \frac{t^2 L''(t)}{L(t)} = -tz'(t) + z(t) + z^2(t).$$

So, using the fact that $z(0) = 0$ and $\lim_{t \to 0^+} tz'(t) = 0$, we deduce (2.1).

Conversely, let L be a positive function in $C^2((0, \eta])$ satisfying (2.1). For $t \in (0, \eta]$, put

$$z(t) = -\frac{tL'(t)}{L(t)}, \tag{2.2}$$

then $z \in C^1((0, \eta])$ and $\lim_{t \to 0^+} z(t) = 0$. Moreover, we have

$$L(t) = L(\eta) \exp\left(\int_t^{\eta} \frac{z(s)}{s} ds\right).$$

Now, by derivation of (2.2), we obtain for $t \in (0, \eta]$,

$$tz'(t) = -\frac{tL'(t)}{L(t)} - \frac{t^2 L''(t)}{L(t)} + \left(\frac{tL'(t)}{L(t)}\right)^2.$$

Hence, by (2.1) we deduce that $\lim_{t\to 0^+} tz'(t) = 0$ and we conclude that $L \in \mathcal{K}$. ∎

Lemma 3. (*Karamata's theorem*) *Let $\mu \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0,\eta]$. Then the following statements hold true.*
(i) If $\mu < -1$, then $\int_0^\eta s^\mu L(s)ds$ diverges and

$$\int_t^\eta s^\mu L(s)ds \underset{t\to 0^+}{\sim} -\frac{t^{1+\mu}L(t)}{\mu+1}.$$

(ii) If $\mu > -1$, then $\int_0^\eta s^\mu L(s)ds$ converges and

$$\int_0^t s^\mu L(s)ds \underset{t\to 0^+}{\sim} \frac{t^{1+\mu}L(t)}{\mu+1}.$$

Lemma 4. *Let $L \in \mathcal{K}$ defined on $(0,\eta]$, then the function*

$$t \to \int_t^\eta \frac{L(s)}{s}ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(s)}{s}ds$ converges, then the function

$$t \to \int_0^t \frac{L(s)}{s}ds \in \mathcal{K}.$$

Lemma 5. *Let $L \in \mathcal{K}$ and φ_1 be the first eigenfunction of $(-\Delta)$ in Ω. Then we have*

$$L(\varphi_1(x)) \approx L(\delta(x)), \ x \in \Omega.$$

Proof. Let $L \in \mathcal{K}$, then there exist a constant $c > 0$ and $z \in C([0,\eta]) \cap C^1((0,\eta])$ such that $z(0) = 0$ and for $t \in (0,\eta]$, $\eta > diam(\Omega)$,

$$L(t) := c\exp\left(\int_t^\eta \frac{z(s)}{s}ds\right).$$

Let $\zeta = \sup_{s\in[0,\eta]} |z(s)|$. By using (1.5), there exists $c_1 > 1$ such that

$$\frac{1}{c_1}\delta(x) \le \varphi_1(x) \le c_1\delta(x).$$

Then we deduce that

$$\left|\int_{\delta(x)}^{\varphi_1(x)} \frac{z(s)}{s}ds\right| \le \zeta \log c_1.$$

Hence,

$$c_1^{-\zeta} L(\delta(x)) \le L(\varphi_1(x)) \le c_1^{\zeta} L(\delta(x)), \ x \in \Omega.$$

This ends the proof. ∎

3. Proofs of main results

To prove our existence results, we use the sub-supersolution method. For that, we consider the following more general problem

$$\left\{\begin{array}{l}\Delta u=f(x,u),\ \text{in } \Omega,\\ \lim\limits_{\delta(x)\to 0} u(x)=+\infty.\end{array}\right. \tag{3.1}$$

Definition 1. *A function $\underline{u}\in C^2(\Omega)$ is called a subsolution of (3.1) if*

$$\left\{\begin{array}{l}\Delta\underline{u}\geq f(x,\underline{u}),\ \text{in } \Omega,\\ \lim\limits_{\delta(x)\to 0} \underline{u}(x)=+\infty.\end{array}\right.$$

Definition 2. *A function $\overline{u}\in C^2(\Omega)$ is called a supersolution of (3.1) if*

$$\left\{\begin{array}{l}\Delta\overline{u}\leq f(x,\overline{u}),\ \text{in } \Omega,\\ \lim\limits_{\delta(x)\to 0} \overline{u}(x)=+\infty.\end{array}\right.$$

Lemma 6. ([23]) *Let $f(x,s)$ be locally Hölder continuous in $\Omega\times(0,\infty)$ and continuously differentiable with respect to the second variable. Suppose that (3.1) has an explosive supersolution $\overline{u}$ and an explosive subsolution $\underline{u}$ such that $\underline{u}\leq\overline{u}$ on Ω, then (3.1) has at least one solution $u\in C^{2,\gamma}(\Omega)$ satisfying $\underline{u}\leq u\leq\overline{u}$ on Ω.*

Now, we will prove our results. First, we note that the proof of Theorem 4 is essentially the same as the proof of Theorem 3 in the exponential case, so it will be omitted.

Proof of Theorem 3. Let a be a function satisfying (H). As an application of Lemma 6, we need to construct a supersolution $\overline{u}$ and a subsolution $\underline{u}$ which satisfy $\underline{u}\leq\overline{u}$ on Ω. For that, let $\lambda\leq 2, \tau=\frac{2-\lambda}{1-\alpha}$ and θ be the function given by (1.8). Put $v=\varphi_1^\tau\theta(\varepsilon\varphi_1)$, where φ_1 is the normalized eigenfunction of $-\Delta$ in Ω associated to the first eigenvalue $\lambda_1$and $\varepsilon>0$ is to be chosen small.
We claim that there exists $c>0$ such that $\frac{1}{c}v$ and cv are respectively a subsolution and a supersolution of problem (1.1).
Indeed, a straightforward computation shows that

$$\Delta v=\varphi_1^{\tau-2}\theta(\varepsilon\varphi_1)\left(\begin{array}{l}\lambda_1\varphi_1^2\left(-\tau-\dfrac{\varepsilon\varphi_1\theta'(\varepsilon\varphi_1)}{\theta(\varepsilon\varphi_1)}\right)\\ \qquad +|\triangledown\varphi_1|^2\left(\tau(\tau-1)+2\tau\dfrac{\varepsilon\varphi_1\theta'(\varepsilon\varphi_1)}{\theta(\varepsilon\varphi_1)}+\dfrac{(\varepsilon\varphi_1)^2\theta''(\varepsilon\varphi_1)}{\theta(\varepsilon\varphi_1)}\right)\end{array}\right).$$

So, we distinguish the following cases.

► $\tau < 0$

In this case, we have $\lambda < 2$. Using Lemma 1, the function θ belongs to $\mathcal{K}$, then it follows from (2.1) and Lemma 2 that

$$\lim_{\varepsilon \to 0} \left(\sup_{x\in\Omega} \frac{\varepsilon\varphi_1(x)\theta'(\varepsilon\varphi_1(x))}{\theta(\varepsilon\varphi_1(x))} \right) = \lim_{\varepsilon \to 0} \left(\sup_{x\in\Omega} \frac{(\varepsilon\varphi_1(x))^2 \theta''(\varepsilon\varphi_1(x))}{\theta(\varepsilon\varphi_1(x))} \right) = 0. \tag{3.2}$$

Hence, there exists $0 < \varepsilon < 1$ such that

$$-\frac{\tau}{2} < -\tau - \frac{\varepsilon\varphi_1\theta'(\varepsilon\varphi_1)}{\theta(\varepsilon\varphi_1)} < -\frac{3}{2}\tau, \text{ in } \Omega$$

and

$$\frac{\tau(\tau-1)}{2} \leq \tau(\tau-1) + 2\tau\frac{\varepsilon\varphi_1\theta'(\varepsilon\varphi_1)}{\theta(\varepsilon\varphi_1)} + \frac{(\varepsilon\varphi_1)^2\theta''(\varepsilon\varphi_1)}{\theta(\varepsilon\varphi_1)} \leq \frac{3}{2}\tau(\tau-1), \text{ in } \Omega.$$

Therefore, by using (1.6), we get for $x \in \Omega$

$$\Delta v(x) \approx (\varphi_1(x))^{\tau-2}\theta(\varepsilon\varphi_1(x)).$$

Since $\theta \in \mathcal{K}$, il follows by Lemma 5 and (1.5) that for $x \in \Omega$

$$\begin{aligned} \Delta v(x)\, v(x)^{-\alpha} &\approx (\delta(x))^{\tau(1-\alpha)-2}\theta^{1-\alpha}(\delta(x)), \\ &\approx \delta(x)^{-\lambda} L(\delta(x)), \\ &\approx a(x). \end{aligned}$$

Hence, there exists $M > 0$ such that for $x \in \Omega$

$$\frac{1}{M}a(x)v^{\alpha}(x) \leq \Delta v(x) \leq M a(x) v^{\alpha}(x). \tag{3.3}$$

By putting $c = M^{\frac{1}{\alpha-1}}$, it follows from (3.3) that $\underline{u} = \frac{1}{c}v$ and $\overline{u} = cv$ are respectively subsolution and supersolution of problem (1.1). Thus, we conclude by Lemma 6 that problem (1.1) has a positive solution u such that

$$\underline{u} \leq u \leq \overline{u}.$$

Applying Lemma 5 and (1.5), we obtain

$$u(x) \approx \delta(x)^{\frac{2-\lambda}{1-\alpha}}(L(\delta(x))^{\frac{1}{1-\alpha}}.$$

► $\tau = 0$

In this case, we have $\lambda = 2$ and for $x \in \Omega$, $v(x) = \theta(\varepsilon\varphi_1)(x) = \left(\int_0^{\varepsilon\varphi_1(x)} \frac{L(s)}{s}ds\right)^{\frac{1}{1-\alpha}}$.

Since $\int_0^{\eta} \frac{L(s)}{s}ds < \infty$, it follows from Lemma 4 and Lemma 1 that $\theta \in \mathcal{K}$.
Moreover, we have

$$L(\varepsilon\varphi_1) = (1-\alpha)\varepsilon\varphi_1\theta'(\varepsilon\varphi_1)\theta^{-\alpha}(\varepsilon\varphi_1) \tag{3.4}$$

and

$$\frac{\varepsilon\varphi_1\theta''(\varepsilon\varphi_1)}{\theta'(\varepsilon\varphi_1)} = -1 + \frac{\varepsilon\varphi_1 L'(\varepsilon\varphi_1)}{L(\varepsilon\varphi_1)} + \alpha\frac{\varepsilon\varphi_1\theta'(\varepsilon\varphi_1)}{\theta(\varepsilon\varphi_1)}.$$

Using the fact that L and θ are in $\mathcal{K}$, we deduce by (2.1) that

$$\lim_{\varepsilon\to 0}\left(\sup_{x\in\Omega}\left(\frac{(\varepsilon\varphi_1(x)\theta''(\varepsilon\varphi_1(x)))}{\theta'(\varepsilon\varphi_1(x))} + 1\right)\right) = 0.$$

This together with (1.6) implies that there exists $0 < \varepsilon < 1$ such that

$$\lambda_1\varphi_1^2 - |\triangledown\varphi_1|^2\,(\frac{\varepsilon\varphi_1\theta''(\varepsilon\varphi_1)}{\theta'(\varepsilon\varphi_1)}) \approx 1.$$

On the other hand, we have

$$\begin{aligned}\Delta v &= -\lambda_1\varepsilon\varphi_1\theta'(\varepsilon\varphi_1) + \varepsilon^2|\triangledown\varphi_1|^2\theta''(\varepsilon\varphi_1)\\ &= -\varepsilon\varphi_1^{-1}\theta'(\varepsilon\varphi_1)\left(-|\triangledown\varphi_1|^2\,(\frac{\varepsilon\varphi_1\theta''(\varepsilon\varphi_1)}{\theta'(\varepsilon\varphi_1)}) + \lambda_1\varphi_1^2\right), \text{ in } \Omega.\end{aligned}$$

Hence, we get

$$(\Delta v)\,v^{-\alpha} \approx -\varphi_1^{-1}\theta'(\varepsilon\varphi_1)\,v^{-\alpha}, \text{ in } \Omega.$$

Thus, by using (3.4), (1.5) and Lemma 5, we obtain for $x \in \Omega$

$$\begin{aligned}(\Delta v(x))\,v(x)^{-\alpha} &\approx \varphi_1^{-2}(x)\,L(\varepsilon\varphi_1(x)),\\ &\approx (\delta(x))^{-2}L(\delta(x)),\\ &\approx a(x).\end{aligned}$$

Hence, there exists $m > 1$, such that for $x \in \Omega$

$$\frac{1}{m}a(x)v^{\alpha}(x) \le \Delta v(x) \le m a(x)v^{\alpha}(x).$$

Put $c = m^{\frac{1}{\alpha-1}}$. It follows that $\underline{u} = \frac{1}{c}v$ and $\overline{u} = cv$ are respectively subsolution and supersolution of problem (1.1). Thus, we conclude by Lemma 6 that problem (1.1) has a positive solution u such that

$$\underline{u} \le u \le \overline{u}.$$

Since $\theta \in \mathcal{K}$, we deduce by Lemma 5 and (1.5) that

$$u(x) \approx \left(\int_0^{\delta(x)}\frac{L(s)}{s}ds\right)^{\frac{1}{1-\alpha}}.$$

4. Applications

4.1. **First application.** Let a be a function satisfying (H). In this paragraph, we are interested in the following problem

$$
\left\{
\begin{array}{l}
\Delta u-\frac{\beta}{u}\left|\nabla u\right|^{2}=a(x)u^{\alpha},\ \text{in }\Omega,\\
u>0,\ \text{in }\Omega,\\
\lim\limits_{\delta(x)\to0}u(x)=+\infty,
\end{array}
\right.
\tag{4.1}
$$

where $\alpha>1$ and $\beta\in\mathbb{R}$.
Consider the case $\beta\neq1$,then by putting $v=u^{1-\beta}$, we obtain by simple calculation the following.
► If $\beta<1$, then v satisfies

$$
\left\{
\begin{array}{l}
\Delta v=(1-\beta)a(x)v^{\frac{\alpha-\beta}{1-\beta}},\ \text{in }\Omega,\\
v>0,\ \text{in }\Omega,\\
\lim\limits_{\delta(x)\to0}v(x)=+\infty.
\end{array}
\right.
\tag{4.2}
$$

Since $\frac{\alpha-\beta}{1-\beta}>1$, it follows by Theorem 3 that problem (4.2) has a solution $v\in C^{2,\gamma}(\Omega)$ such that

$$
v(x)\approx\left\{
\begin{array}{ll}
\delta(x)^{\frac{(2-\lambda)(1-\beta)}{1-\alpha}}\left(L(\delta\left(x\right)\right)^{\frac{1-\beta}{1-\alpha}}, & \text{if }\lambda<2,\\
\left(\displaystyle\int_{0}^{\delta(x)}\frac{L(t)}{t}dt\right)^{\frac{1-\beta}{1-\alpha}}, & \text{if }\lambda=2.
\end{array}
\right.
$$

So, we deduce that problem (4.1) has a positive solution $u\in C^{2,\gamma}(\Omega)$ satisfying

$$
u(x)\approx\left\{
\begin{array}{ll}
\delta(x)^{\frac{2-\lambda}{1-\alpha}}\left(L(\delta\left(x\right)\right)^{\frac{1}{1-\alpha}}, & \text{if }\lambda<2,\\
\left(\displaystyle\int_{0}^{\delta(x)}\frac{L(t)}{t}dt\right)^{\frac{1}{1-\alpha}}, & \text{if }\ \lambda=2.
\end{array}
\right.
$$

► If $\beta>1$, then v satisfies

$$
\left\{
\begin{array}{l}
-\Delta v=(\beta-1)a(x)v^{\frac{\alpha-\beta}{1-\beta}},\ \text{in }\Omega,\\
v>0,\\
\lim\limits_{\delta(x)\to0}v(x)=0,
\end{array}
\right.
\tag{4.3}
$$

Since $\frac{\alpha-\beta}{1-\beta}<1$, then by ([16], theorem 1), problem (4.3) has a unique positive solution $v\in C(\overline{\Omega})\cap C^{2,\gamma}(\Omega)$ satisfying

$$
v(x)\approx\delta(x)^{\min(\frac{(2-\lambda)(\beta-1)}{\alpha-1},1)}\Phi\left(\delta\left(x\right)\right),
$$

where for $t \in (0, \eta], (\eta > diam(\Omega))$,

$$\Phi(t) = \begin{cases} \left(\int_0^t \frac{L(s)}{s} ds \right)^{\frac{1-\beta}{1-\alpha}}, & \text{if } \lambda = 2, \\ (L(t))^{\frac{1-\beta}{1-\alpha}}, & \text{if } 1 + \frac{\alpha-\beta}{1-\beta} < \lambda < 2, \\ \left(\int_t^\eta \frac{L(s)}{s} ds \right)^{\frac{1-\beta}{1-\alpha}}, & \text{if } \lambda = 1 + \frac{\alpha-\beta}{1-\beta}, \\ 1, & \text{if } \lambda < 1 + \frac{\alpha-\beta}{1-\beta}. \end{cases}$$

Hence, problem (4.1) has a unique positive solution $u \in C^{2,\gamma}(\Omega)$ satisfying

$$u(x) \approx \begin{cases} \left(\int_0^{\delta(x)} \frac{L(t)}{t} dt \right)^{\frac{1}{1-\alpha}}, & \text{if } \lambda = 2, \\ \delta(x)^{\frac{2-\lambda}{1-\alpha}} \left(L\left(\delta\left(x\right)\right)\right)^{\frac{1}{1-\alpha}}, & \text{if } 1 + \frac{\alpha-\beta}{1-\beta} < \lambda < 2, \\ \delta(x)^{\frac{1}{1-\beta}} \left(\int_{\delta(x)}^\eta \frac{L(t)}{t} dt \right)^{\frac{1}{1-\alpha}}, & \text{if } \lambda = 1 + \frac{\alpha-\beta}{1-\beta}, \\ \delta\left(x\right)^{\frac{1}{1-\beta}}, & \text{if } \lambda < 1 + \frac{\alpha-\beta}{1-\beta}. \end{cases}$$

Now, for $\beta = 1$, putting $v = \log u$, problem (4.1) becomes

$$\begin{cases} \Delta v = a(x) e^{(\alpha-1)v}, \text{ in } \Omega, \\ v > 0, \text{ in } \Omega, \\ \lim\limits_{\delta(x) \to 0} v(x) = +\infty. \end{cases}$$

Then, we deduce from Theorem 4 that problem (4.1) has a positive solution $u \in C^{2,\gamma}\left(\Omega\right)$ satisfying

$$u(x) \approx \begin{cases} \delta(x)^{\frac{2-\lambda}{1-\alpha}} \left(L(\delta\left(x\right)\right)^{\frac{1}{1-\alpha}}, & \text{if } \lambda < 2, \\ \left(\int_0^{\delta(x)} \frac{L(t)}{t} dt \right)^{\frac{1}{1-\alpha}}, & \text{if } \quad \lambda = 2. \end{cases}$$

4.2. **Second application.** Let a be a function satisfying (H) such that

$$a(x) \approx \delta\left(x\right)^{-\lambda} L_1\left(\delta\left(x\right)\right).$$

Let $b \in C^{\gamma}_{loc}\left(\Omega\right)$, $0 < \gamma < 1$, satisfying for each $x \in \Omega$,

$$b(x) \approx \delta\left(x\right)^{-\mu} L_2\left(\delta\left(x\right)\right),$$

where $\mu \leq \lambda$ and $L_2 \in \mathcal{K}$. We aim to study the following system

$$(4.4) \qquad \begin{cases} \Delta u = a(x)e^u, \text{ in } \Omega, \\ \Delta v = b(x)e^u e^v, \text{ in } \Omega, \\ \lim\limits_{\delta(x)\to 0} u(x) = +\infty, \ \lim\limits_{\delta(x)\to 0} v(x) = +\infty. \end{cases}$$

Using Theorem 4, there exists a function $u \in C^{2,\gamma}(\Omega)$ satisfying for $x \in \Omega$

$$e^{u(x)} \approx \begin{cases} (\delta(x))^{\lambda-2} (L_1(\delta(x)))^{-1}, & \text{if } \lambda < 2, \\ \left(\int_0^{\delta(x)} \frac{L_1(t)}{t} dt \right)^{-1}, & \text{if } \lambda = 2. \end{cases}$$

Hence, we will distinguish the following cases.

► If $\lambda < 2$, we have

$$b(x)e^{u(x)} \approx \delta(x)^{-\mu+\lambda-2} (L_1(\delta(x))^{-1} L_2(\delta(x)) := \delta(x)^{-\mu+\lambda-2} L(\delta(x)).$$

It follows from Lemma 1 that $L \in \mathcal{K}$. Now, suppose that $\int_0^\eta t^{\lambda-\mu-1} L(t)dt < \infty$, then by Theorem 4, we conclude that system (4.4) has a solution $(u, v) \in C^{2,\gamma}(\Omega) \times C^{2,\gamma}(\Omega)$ such that

$$e^{u(x)} \approx (\delta(x))^{\lambda-2} (L_1(\delta(x)))^{-1}$$

and

$$e^{v(x)} \approx \begin{cases} (\delta(x))^{\mu-\lambda} (L(\delta(x)))^{-1}, & \text{if } \mu < \lambda, \\ \left(\int_0^{\delta(x)} \frac{L(t)}{t} dt \right)^{-1}, & \text{if } \mu = \lambda. \end{cases}$$

► If $\lambda = 2$, we have

$$b(x)e^{u(x)} \approx \delta(x)^{-\mu} L_2(\delta(x)) \left(\int_0^{\delta(x)} \frac{L_1(t)}{t} dt \right)^{-1} := \delta(x)^{-\mu} L(\delta(x)).$$

It follows from Lemma 4 and Lemma 1 that $L \in \mathcal{K}$. Now suppose that $\int_0^\eta t^{1-\mu} L(t)dt < \infty$, then we conclude by Theorem 4 that system (4.4) has a solution $(u, v) \in C^{2,\gamma}(\Omega) \times C^{2,\gamma}(\Omega)$ such that

$$e^{u(x)} \approx \left(\int_0^{\delta(x)} \frac{L_1(t)}{t} dt \right)^{-1}$$

and

$$e^{v(x)} \approx \begin{cases} (\delta(x))^{\mu-2} (L(\delta(x)))^{-1}, & \text{if } \mu < 2, \\ \left(\int_0^{\delta(x)} \frac{L(t)}{t} dt \right)^{-1}, & \text{if } \mu = 2. \end{cases}$$

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