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On Lyapunov-type inequality for a class of quasilinear systems

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Abstract. In this paper, we establish a new Lyapunov-type inequality for quasilinear systems. Our result in special case reduces to the result of Watanabe et al. [J. Inequal. Appl. 242(2012), 1–8]. As an application, we also obtain lower bounds for the eigenvalues of corresponding systems.

Keywords: Lyapunov-type inequality, quasilinear system, lower bound.

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1 Introduction

In 1907, Lyapunov [23] obtained the following remarkable inequality

$$\frac{4}{b-a} \leq \int_a^b f_1(z)\, dz, \tag{1.1}$$

if Hill's equation

$$u_1'' + f_1(x)\, u_1 = 0 \tag{1.2}$$

has a real nontrivial solution $u_1(x)$ such that the Dirichlet boundary conditions

$$u_1(a) = 0 = u_1(b) \tag{1.3}$$

hold, where $a, b \in \mathbb{R}$ with $a < b$ consecutive zeros, u_1 is not identically zero on $[a, b]$, and f_1 is a real-valued positive continuous function defined on $\mathbb{R}$. We know that the constant 4 on the left-hand side of inequality (1.1) cannot be replaced by a larger number (see [19, p. 345]).

Since the appearance of Lyapunov's fundamental paper, various proofs and generalizations or improvements have appeared in the literature under the Dirichlet boundary conditions. For example, for authors who are interested in the Lyapunov-type inequalities, we refer to Eliason [16], Harris and Kong [18], Hartman [19], Kwong [21], and Reid [33]. We should also mention here that inequality (1.1) has been generalized to second order nonlinear differential

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equations by Eliason [16] and Pachpatte [26,27], to delay differential equations of the second order by Dahiya and Singh [13] and Eliason [17], to third order differential equations by Parhi and Panigrahi [29], to certain higher order differential equations by Çakmak [7], He and Tang [20], Pachpatte [25], Panigrahi [28], Parhi and Panigrahi [30], Yang [38], and Yang and Lo [39], and to systems by Aktaş [3], Aktaş et al. [4], Bonder and Pinasco [5], Çakmak and Tiryaki [8,9], Çakmak [10], Çakmak et al. [11], Napoli and Pinasco [24], Tang and He [34], Tiryaki et al. [35–37], and Yang et al. [40,41]. Lyapunov-type inequalities can be found in Pachpatte's paper [27] for the Emden-Fowler type equations, and were obtained for the first time by Elbert [15] for the half-linear equation, but the proof of its extension can be found in the book of Došlý and Řehák [14]. Lyapunov-type inequalities for the half-linear equation have been rediscovered by Lee et al. [22] and Pinasco [31,32].

Recently, Aktaş et al. [2], Çakmak [12] and Wang [42] obtained the Lyapunov-type inequalities under the anti-periodic boundary conditions.

More recently, by using the clamped-free boundary conditions, Watanabe et al. [43] obtained a new Lyapunov-type inequality for $2n$-th order differential equation as follows.

Theorem A [43, Theorem 1]. *If $f_1 \in C([-s,s],\mathbb{R})$ and $u_1(x)$ is a nontrivial solution on $[-s,s]$ for the following $2n$-th order differential equation*

$$(-1)^n u_1^{(2n)} = f_1(x)u_1 \tag{1.4}$$

with the clamped-free boundary conditions

$$u_1^{(i)}(-s) = 0 = u_1^{(n+i)}(s) \tag{1.5}$$

for $i = 0,1,\ldots,n-1$, then the inequality

$$\frac{[(n-1)!]^2(2n-1)}{(2s)^{2n-1}} < \int_{-s}^{s} f_1^+(z)dz \tag{1.6}$$

holds, where $f_1^+(x) = \max\{0, f_1(x)\}$ is the nonnegative part of $f_1(x)$.

In this paper, we prove a new Lyapunov-type inequality for the following system

$$\left(r_k(x)\,\phi_{p_k}(u_k')\right)' + f_k(x)\phi_{\alpha_{kk}}(u_k)\prod_{\substack{i=1\\ i\neq k}}^{n}|u_i|^{\alpha_{ki}} = 0, \tag{1.7}$$

where $n \in \mathbb{N}$, $\phi_\gamma(u) = |u|^{\gamma-2}u$, $\gamma > 1$, $f_k, r_k \in C([-s,s],\mathbb{R})$, $r_k(x) > 0$ for $k = 1,2,\ldots,n$ and $x \in \mathbb{R}$, $(u_1(x), u_2(x), \ldots, u_n(x))$ is a real nontrivial solution of system (1.7) such that the boundary conditions

$$u_k(-s) = 0 = u_k'(s) \tag{1.8}$$

hold for $k = 1,2,\ldots,n$, u_k for $k = 1,2,\ldots,n$ are not identically zero on $[-s,s]$, $1 < p_k < \infty$ and α_{ki} for $k,i = 1,2,\ldots,n$ are nonnegative constants.

As an application, we have also investigated the lower bounds on the generalized eigenvalue $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of the following problem

$$\left(r_k(x)\,\phi_{p_k}(u_k')\right)' + \lambda_k r(x)\phi_{\alpha_{kk}}(u_k)\prod_{\substack{i=1\\ i\neq k}}^{n}|u_i|^{\alpha_{ki}} = 0 \tag{1.9}$$

with the boundary conditions (1.8) for $k = 1, 2, \ldots, n$ and $r(x) \in C([-s, s], \mathbb{R})$.

As usual, it is easier to find upper bounds for eigenvalues than lower bounds. In fact, they can be obtained by using elementary inequalities. Finding the estimated lower bounds is based on giving a suitable Lyapunov inequality for the corresponding systems. For readers who are interested in the existence of the generalized eigenvalues for the special case of system (1.9), we refer to the paper by Napoli and Pinasco [24].

Note that if we take $\alpha_{kk} = p_k$, $k = 1, 2, \ldots, n$, and for $i \neq k$, $\alpha_{ki} = 0$ for $i = 1, 2, \ldots, n$, then we obtain uncoupled equations, i.e. the half-linear second order differential equations

$$\left(r_k(x)\,\phi_{p_k}\left(u_k'\right)\right)' + f_k(x)\phi_{p_k}(u_k) = 0 \tag{1.10}$$

for $k = 1, 2, \ldots, n$ from system (1.7). However, the equation (1.4), which was considered by Watanabe et al. [43], does not reduce to the equation (1.10). Moreover, when $n = 1$ in the problem (1.7)–(1.8) with $r_1(x) = 1$ and $p_1 = 2$ or (1.4)–(1.5), we have the following linear problem

$$\begin{cases} u_1'' + f_1(x)u_1 = 0, \\ u_1(-s) = 0 = u_1'(s). \end{cases} \tag{1.11}$$

Thus, we obtain the following inequality

$$\frac{1}{2s} < \int_{-s}^{s} f_1^+(z)dz \tag{1.12}$$

from Theorem A with $n = 1$ given by Watanabe et al. [43].

In this paper, our motivation comes from the recent papers of Çakmak and Tiryaki [9], Yang et al. [40], and Watanabe et al. [43]. We prove a new Lyapunov-type inequality for system (1.7) with the boundary conditions (1.8).

Since our attention is restricted to the Lyapunov-type inequality for the quasilinear systems of differential equations, we shall assume the existence of the nontrivial solution of system (1.7). For readers who are interested in the existence of the solution of this type of systems, we refer to the paper by Afrouzi and Heidarkhani [1].

2 Main results

We prove a lemma which we will use in the proof of our main result.

Lemma 2.1. *If $(u_1(x), u_2(x), \ldots, u_n(x))$ is a nontrivial solution of system (1.7) satisfying the condition $u_k(-s) = 0 = u_k'(s)$ for $k = 1, 2, \ldots, n$, then we have*

$$|u_k(z)| < \left(\int_{-s}^{s} r_k^{1/(1-p_k)}(v)\,dv\right)^{(p_k-1)/p_k} \left(\int_{-s}^{s} r_k(v)\left|u_k'(v)\right|^{p_k} dv\right)^{1/p_k} \tag{2.1}$$

for $z \in [-s, s]$ and $k = 1, 2, \ldots, n$.

Proof. Let $u_k(-s) = 0 = u_k'(s)$ for $k = 1, 2, \ldots, n$ where $n \in \mathbb{N}$ and u_k for $k = 1, 2, \ldots, n$ are not identically zero on $[-s, s]$. By using $u_k(-s) = 0$ and Hölder's inequality, we get

$$\begin{aligned} |u_k(z)| &= \left|\int_{-s}^{z} u_k'(v)\,dv\right| \le \int_{-s}^{z} \left|u_k'(v)\right| dv \le \int_{-s}^{s} \left|u_k'(v)\right| dv = \int_{-s}^{s} r_k^{-1/p_k}(v)\,r_k^{1/p_k}(v)\left|u_k'(v)\right| dv \\ &\le \left(\int_{-s}^{s} r_k^{-1/(p_k-1)}(v)\,dv\right)^{(p_k-1)/p_k} \left(\int_{-s}^{s} r_k(v)\left|u_k'(v)\right|^{p_k} dv\right)^{1/p_k} \end{aligned} \tag{2.2}$$

for $z \in [-s,s]$ and $k = 1,2,\dots,n$. We claim that

$$|u_k(z)|^{p_k} < \left(\int_{-s}^{s} r_k^{-1/(p_k-1)}(v)\,dv\right)^{p_k-1}\left(\int_{-s}^{s} r_k(v)\left|u_k'(v)\right|^{p_k}dv\right) \tag{2.3}$$

for $z \in [-s,s]$ and $k = 1,2,\dots,n$. In fact, if (2.3) is not true, then it follows from (2.2) that

$$\begin{aligned}&\left(\int_{-s}^{s}\left|u_k'(v)\right|dv\right)^{p_k}\\ &\quad= \left(\int_{-s}^{s} r_k^{-1/(p_k-1)}(v)\,dv\right)^{p_k-1}\left(\int_{-s}^{s} r_k(v)\left|u_k'(v)\right|^{p_k}dv\right),\ k = 1,2,\dots,n,\end{aligned} \tag{2.4}$$

which, together with the Hölder's inequality, implies that there exists a constant c such that

$$r_k(x)\left|u_k'(x)\right|^{p_k} = c r_k^{-1/(p_k-1)}(x) \tag{2.5}$$

for $-s \le x \le s$ and $k = 1,2,\dots,n$. If $c = 0$, then $u_k'(x) = 0$ for $x \in [-s,s]$, it follows from (2.2) that $u_k(z) = 0$, which contradicts the fact that $u_k(z) \ne 0$ for $z \in [-s,s]$ and $k = 1,2,\dots,n$. If $c \ne 0$, then $\left|u_k'(x)\right| > 0$ for $x \in [-s,s]$, it follows that $u_k'(z) \ne 0$ for $z \in [-s,s]$ and $k = 1,2,\dots,n$, which contradicts the fact that $u_k'(s) = 0$ for $k = 1,2,\dots,n$. Therefore, the inequality (2.1) for $z \in [-s,s]$ and $k = 1,2,\dots,n$ holds. $\square$

Now, we give the main result of this paper.

Theorem 2.2. *Assume that there exist nontrivial solutions $(e_1, e_2, \dots, e_n)$ of the following linear homogeneous system*

$$e_k\left(1 - \frac{\alpha_{kk}}{p_k}\right) - \sum_{\substack{i=1\\ i\ne k}}^{n}\frac{\alpha_{ik}}{p_k}e_i = 0, \tag{2.6}$$

where $e_k \ge 0$ for $k = 1,2,\dots,n$. If $f_k \in C([-s,s],\mathbb{R})$ for $k = 1,2,\dots,n$ and $(u_1(x), u_2(x), \dots, u_n(x))$ is a nontrivial solution on $[-s,s]$ for problem (1.7)-(1.8), then the inequality

$$1 < \prod_{k=1}^{n}\left[\int_{-s}^{s} f_k^{+}(z)\prod_{i=1}^{n}\left(\int_{-s}^{s} r_i^{1/(1-p_i)}(v)\,dv\right)^{\alpha_{ki}(p_i-1)/p_i}dz\right]^{e_k} \tag{2.7}$$

holds, where $f_k^{+}(x) = \max\{0, f_k(x)\}$ for $k = 1,2,\dots,n$.

Proof. Let $u_k(-s) = 0 = u_k'(s)$ for $k = 1,2,\dots,n$ where $n \in \mathbb{N}$ and u_k for $k = 1,2,\dots,n$ are not identically zero on $[-s,s]$. Multiplying the k-th equation of system (1.7) by u_k, integrating from $-s$ to s, and by using boundary conditions (1.8), we get

$$\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k}dz = \int_{-s}^{s} f_k(z)\prod_{i=1}^{n}|u_i(z)|^{\alpha_{ki}}dz \tag{2.8}$$

for $k=1,2,\ldots,n$. By using the inequality (2.1) in (2.8), we obtain

$$\begin{aligned}
&\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz \\
&\quad\le \int_{-s}^{s} f_k^{+}(z)\prod_{i=1}^{n}\left|u_i(z)\right|^{\alpha_{ki}} dz \\
&\quad< \int_{-s}^{s} f_k^{+}(z)\prod_{i=1}^{n}\left[\left(\int_{-s}^{s} r_i^{1/(1-p_i)}(v)\,dv\right)^{\alpha_{ki}(p_i-1)/p_i}\left(\int_{-s}^{s} r_i(v)\left|u_i'(v)\right|^{p_i} dv\right)^{\alpha_{ki}/p_i}\right] dz \\
&\quad= \left[\prod_{i=1}^{n}\left(\int_{-s}^{s} r_i(z)\left|u_i'(z)\right|^{p_i} dz\right)^{\alpha_{ki}/p_i}\right] \\
&\qquad\times\left[\int_{-s}^{s} f_k^{+}(z)\prod_{i=1}^{n}\left(\int_{-s}^{s} r_i^{1/(1-p_i)}(v)\,dv\right)^{\alpha_{ki}(p_i-1)/p_i} dz\right] \qquad (2.9)
\end{aligned}$$

for $k=1,2,\ldots,n$. Now, we prove that $0<\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz$ for $k=1,2,\ldots,n$. If the inequality $0<\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz$ is not true, then $\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz=0$ for $k=1,2,\ldots,n$. If $\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz=0$, then it follows that

$$u_k'(x)\equiv 0 \qquad (2.10)$$

for $-s\le x\le s$ and $k=1,2,\ldots,n$. Combining (2.2) with (2.10), we obtain that $u_k(z)=0$, which contradicts $u_k(z)\neq 0$ for $z\in[-s,s]$ and $k=1,2,\ldots,n$. Therefore,

$$0<\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz \qquad (2.11)$$

for $k=1,2,\ldots,n$ holds. Thus, from (2.9) and (2.11), we have

$$\begin{aligned}
\left(\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz\right)^{1-\frac{\alpha_{kk}}{p_k}} &< \left[\prod_{\substack{i=1\\ i\neq k}}^{n}\left(\int_{-s}^{s} r_i(z)\left|u_i'(z)\right|^{p_i} dz\right)^{\frac{\alpha_{ki}}{p_i}}\right] \\
&\quad\times\left[\int_{-s}^{s} f_k^{+}(z)\prod_{i=1}^{n}\left(\int_{-s}^{s} r_i^{1/(1-p_i)}(v)\,dv\right)^{\alpha_{ki}(p_i-1)/p_i} dz\right] \qquad (2.12)
\end{aligned}$$

for $k=1,2,\ldots,n$. Raising both sides of the inequality (2.12) to the power e_k for each $k=1,2,\ldots,n$, respectively, and multiplying the resulting inequalities side by side, we obtain

$$\begin{aligned}
\prod_{k=1}^{n}\left(\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz\right)^{\left(1-\frac{\alpha_{kk}}{p_k}\right)e_k} &< \prod_{k=1}^{n}\left[\prod_{\substack{i=1\\ i\neq k}}^{n}\left(\int_{-s}^{s} r_i(z)\left|u_i'(z)\right|^{p_i} dz\right)^{\frac{\alpha_{ki}}{p_i}}\right]^{e_k} \\
&\quad\times\prod_{k=1}^{n}\left[\int_{-s}^{s} f_k^{+}(z)\prod_{i=1}^{n}\left(\int_{-s}^{s} r_i^{1/(1-p_i)}(v)\,dv\right)^{\alpha_{ki}(p_i-1)/p_i} dz\right]^{e_k} \qquad (2.13)
\end{aligned}$$

and hence

$$\begin{aligned}
\prod_{k=1}^{n}\left(\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz\right)^{\left(1-\frac{\alpha_{kk}}{p_k}\right)e_k} &< \left[\prod_{k=1}^{n}\left(\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz\right)^{\sum\limits_{\substack{i=1\\ i\neq k}}^{n}\frac{\alpha_{ik}}{p_k}e_i}\right] \\
&\quad\times\prod_{k=1}^{n}\left[\int_{-s}^{s} f_k^{+}(z)\prod_{i=1}^{n}\left(\int_{-s}^{s} r_i^{1/(1-p_i)}(v)\,dv\right)^{\alpha_{ki}(p_i-1)/p_i} dz\right]^{e_k}. \qquad (2.14)
\end{aligned}$$

Thus, we have

$$\prod_{k=1}^{n}\left(\int_{-s}^{s} r_k(z)\left|u_k'(z)\right|^{p_k} dz\right)^{\theta_k} < \prod_{k=1}^{n}\left[\int_{-s}^{s} f_k^{+}(z)\prod_{i=1}^{n}\left(\int_{-s}^{s} r_i^{1/(1-p_i)}(v)\,dv\right)^{\alpha_{ki}(p_i-1)/p_i} dz\right]^{e_k}, \tag{2.15}$$

where

$$\theta_k = e_k\left(1-\frac{\alpha_{kk}}{p_k}\right)-\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{\alpha_{ik}}{p_k}e_i$$

for $k=1,2,\ldots,n$. By assumption, system (2.6) has nontrivial solutions $(e_1,e_2,\ldots,e_n)$ such that $\theta_k = 0$ for $k=1,2,\ldots,n$, where $e_k \geq 0$ for $k=1,2,\ldots,n$ and at least one $e_j > 0$ for $j=\{1,2,\ldots,n\}$. Choosing one of the solutions $(e_1,e_2,\ldots,e_n)$, we obtain the inequality (2.7) from (2.15). This completes the proof. □

The proof of the following result proceeds along the lines of that of Corollary 1 in Yang et al. [40] and hence is omitted.

Corollary 2.3. *Assume that*

$$\sum_{i=1}^{n}\alpha_{ik} = p_k \tag{2.16}$$

for $k=1,2,\ldots,n$. *If* $f_k \in C([-s,s],\mathbb{R})$ *for* $k=1,2,\ldots,n$ *and* $(u_1(x),u_2(x),\ldots,u_n(x))$ *is a nontrivial solution on* $[-s,s]$ *for problem* (1.7)–(1.8), *then the inequality*

$$1 < \prod_{k=1}^{n}\left[\int_{-s}^{s} f_k^{+}(z)\prod_{i=1}^{n}\left(\int_{-s}^{s} r_i^{1/(1-p_i)}(v)\,dv\right)^{\alpha_{ki}(p_i-1)/p_i} dz\right] \tag{2.17}$$

holds, where $f_k^{+}(x)=\max\{0,f_k(x)\}$ *for* $k=1,2,\ldots,n$.

Remark 2.4. If we take $n=1$ and $\alpha_{11}=p_1$ in the problem (1.7)–(1.8), then we obtain the following half-linear problem

$$\begin{cases} \left(r_1(x)\,\phi_{p_1}(u_1')\right)' + f_1(x)\phi_{p_1}(u_1) = 0, \\ u_1(-s) = 0 = u_1'(s). \end{cases} \tag{2.18}$$

Thus, we have the following inequality

$$\left(\int_{-s}^{s} r_1^{1/(1-p_1)}(v)\,dv\right)^{1-p_1} < \int_{-s}^{s} f_1^{+}(z)\,dz \tag{2.19}$$

from the inequality (2.17) in Corollary 2.3. In addition to this, if we take $p_1=2$ and $r_1(x)=1$ in the problem (2.18), then the inequality (2.19) reduces to the inequality (1.12) given by Watanabe et al. [43].

Now, we present an application of the Lyapunov-type inequality obtained for system (1.7). We obtain the following result which gives lower bounds for the n-th component of any generalized eigenvalue $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ of problem (1.9)–(1.8). The proof of the following theorem is based on above generalization of the Lyapunov-type inequality, as in that of Theorem 9 of Çakmak and Tiryaki [9] and hence is omitted.

Theorem 2.5. *Assume that there exist nontrivial solutions* $(e_1, e_2, \ldots, e_n)$ *of system* (2.6)*. Then there exists a function* $h_1(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ *such that*

$$h_1(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) < |\lambda_n| \tag{2.20}$$

for any generalized eigenvalue $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *of problem* (1.9)–(1.8)*, where*

$$\begin{aligned} &h_1(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \\ &= \left\{ \left[\prod_{k=1}^{n-1} |\lambda_k|^{e_k} \right] \left[\prod_{k=1}^{n} \left(\int_{-s}^{s} |r(z)| \prod_{i=1}^{n} \left(\int_{-s}^{s} r_i^{1/(1-p_i)}(v)\, dv \right)^{\alpha_{ki}(p_i-1)/p_i} dz \right)^{e_k} \right] \right\}^{-\frac{1}{e_n}}. \end{aligned} \tag{2.21}$$

Remark 2.6. Since h_1 is a continuous function, then $h_1(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \to +\infty$ as any component of eigenvalue $\lambda_k \to 0$ for $k = 1, 2, \ldots, n-1$. Therefore, there exists a ball centered in the origin such that the generalized spectrum is contained in its exterior. Also, by rearranging terms in (2.20) we obtain

$$\prod_{k=1}^{n} \left[\int_{-s}^{s} |r(z)| \prod_{i=1}^{n} \left(\int_{-s}^{s} r_i^{1/(1-p_i)}(v)\, dv \right)^{\alpha_{ki}(p_i-1)/p_i} dz \right]^{-e_k} < \prod_{k=1}^{n} |\lambda_k|^{e_k}. \tag{2.22}$$

It is clear that when the interval collapses, left-hand side of (2.22) goes to infinity.

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References

[1] G. A. Afrouzi, S. Heidarkhani, Existence of three solutions for a class of Dirichlet quasilinear elliptic systems involving the $(p_1, p_2, \ldots, p_n)$-Laplacian, *Nonlinear Anal.* **70**(2009) 135–143. MR2468224; url

[2] M. F. Aktaş, D. Çakmak, A. Tiryaki, Lyapunov-type inequality for quasilinear systems with anti-periodic boundary conditions, *J. Math. Inequal.*, accepted.

[3] M. F. Aktaş, Lyapunov-type inequalities for a certain class of n-dimensional quasilinear systems, *Electron. J. Differential Equations* **2013**, No. 67, 1–8. MR3040644

[4] M. F. Aktaş, D. Çakmak, A. Tiryaki, A note on Tang and He's paper, *Appl. Math. Comput.* **218**(2012), 4867–4871. MR2870011; url

[5] J. F. Bonder, J. P. Pinasco, Estimates for eigenvalues of quasilinear elliptic systems. Part II, *J. Differential Equations* **245**(2008), 875–891. MR2427399; url

[6] S. S. Cheng, A discrete analogue of the inequality of Lyapunov, *Hokkaido Math. J.* **12**(1983), 105–112. MR0689261

[7] D. Çakmak, Lyapunov-type integral inequalities for certain higher order differential equations, *Appl. Math. Comput.* **216**(2010), 368–373. MR2601503; url

[8] D. ÇAKMAK, A. TIRYAKI, On Lyapunov-type inequality for quasilinear systems, *Appl. Math. Comput.* **216**(2010), 3584–3591. MR2661715; url

[9] D. ÇAKMAK, A. TIRYAKI, Lyapunov-type inequality for a class of Dirichlet quasilinear systems involving the $(p_1, p_2, \dots, p_n)$-Laplacian, *J. Math. Anal. Appl.* **369**(2010), 76–81. MR2643847; url

[10] D. ÇAKMAK, On Lyapunov-type inequality for a class of nonlinear systems, *Math. Inequal. Appl.* **16**(2013), 101–108. MR3060382; url

[11] D. ÇAKMAK, M. F. AKTAŞ, A. TIRYAKI, Lyapunov-type inequalities for nonlinear systems involving the $(p_1, p_2, \dots, p_n)$-Laplacian, *Electron. J. Differential Equations* **2013**, No. 128, 1–10. MR3084608

[12] D. ÇAKMAK, Lyapunov-type inequalities for two classes of nonlinear systems with anti-periodic boundary conditions, *Appl. Math. Comput.* **223**(2013), 237–242. MR3116259; url

[13] R. S. DAHIYA, B. SINGH, A Lyapunov inequality and nonoscillation theorem for a second order non-linear differential-difference equations, *J. Math. Phys. Sci.* **7**(1973), 163–170. MR0350151

[14] O. DOŠLÝ, P. ŘEHÁK, *Half-linear differential equations*, North-Holland Mathematics Studies, Vol. 202, Elsevier Science B.V., Amsterdam, 2005. MR2158903

[15] Á. ELBERT, A half-linear second order differential equation, In: *Qualitative theory of differential equations, Vol. I, II (Szeged, 1979)*, Colloq. Math. Soc. János Bolyai, Vol. 30, pp. 153–180, North-Holland, Amsterdam–New York, 1981. MR0680591

[16] S. B. ELIASON, A Lyapunov inequality for a certain nonlinear differential equation, *J. London Math. Soc.* **2**(1970), 461–466. MR0267191

[17] S. B. ELIASON, Lyapunov type inequalities for certain second order functional differential equations, *SIAM J. Appl. Math.* **27**(1974), 180–199. MR0350152; url

[18] B. J. HARRIS, Q. KONG, On the oscillation of differential equations with an oscillatory coefficient, *Trans. Amer. Math. Soc.* **347**(1995), 1831–1839. MR1283552; url

[19] P. HARTMAN, *Ordinary differential equations*, Reprint of the second edition. Birkhauser, Boston, 1982. MR0658490

[20] X. HE, X. H. TANG, Lyapunov-type inequalities for even order differential equations, *Commun. Pure Appl. Anal.* **11**(2012), 465–473. MR2861793; url

[21] M. K. KWONG, On Lyapunov's inequality for disfocality, *J. Math. Anal. Appl.* **83**(1981), 486–494. MR0641347; url

[22] C. LEE, C. YEH, C. HONG, R. P. AGARWAL, Lyapunov and Wirtinger inequalities, *Appl. Math. Letters* **17**(2004), 847–853. MR2072845; url

[23] A. LYAPUNOV, Problème général de la stabilité du mouvement, *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (2)* **9**(1907), 203–474. MR1508297

[24] P. L. NAPOLI, J. P. PINASCO, Estimates for eigenvalues of quasilinear elliptic systems, *J. Differential Equations* **227**(2006), 102–115. MR2233956; url

[25] B. G. Pachpatte, On Lyapunov-type inequalities for certain higher order differential equations, *J. Math. Anal. Appl.* **195**(1995), 527–536. MR1354560; url

[26] B. G. Pachpatte, Lyapunov type integral inequalities for certain differential equations, *Georgian Math. J.* **4**(1997), 139–148. MR1439592; url

[27] B. G. Pachpatte, Inequalities related to the zeros of solutions of certain second order differential equations, *Facta Universitatis, Ser. Math. Inform.* **16**(2001), 35–44. MR2016354

[28] S. Panigrahi, Liapunov-type integral inequalities for certain higher order differential equations, *Electron. J. Differential Equations* **2009**, No. 28, 1–14. MR2481102

[29] N. Parhi, S. Panigrahi, On Liapunov-type inequality for third-order differential equations, *J. Math. Anal. Appl.* **233**(1999), 445–460. MR1689641; url

[30] N. Parhi, S. Panigrahi, Liapunov-type inequality for higher order differential equations, *Math. Slovaca* **52**(2002), 31–46. MR1901012

[31] J. P. Pinasco, Lower bounds for eigenvalues of the one-dimensional p-Laplacian, *Abstr. Appl. Anal.* **2**(2004), 147–153. MR2058270; url

[32] J. P. Pinasco, Comparison of eigenvalues for the p-Laplacian with integral inequalities, *Appl. Math. Comput.* **182**(2006), 1399–1404. MR2282582; url

[33] T. W. Reid, A matrix Liapunov inequality, *J. Math. Anal. Appl.* **32**(1970), 424–434. MR0268457; url

[34] X. H. Tang, X. He, Lower bounds for generalized eigenvalues of the quasilinear systems, *J. Math. Anal. Appl.* **385**(2012), 72–85. MR2832075; url

[35] A. Tiryaki, D. Çakmak, M. F. Aktaş, Lyapunov-type inequalities for two classes of Dirichlet quasilinear systems, *Math. Inequal. Appl.* **17**(2014), 843–863. url

[36] A. Tiryaki, D. Çakmak, M. F. Aktaş, Lyapunov-type inequalities for a certain class of nonlinear systems, *Comput. Math. Appl.* **64**(2012), 1804–1811. MR2960804; url

[37] A. Tiryaki, M. Ünal, D. Çakmak, Lyapunov-type inequalities for nonlinear systems, *J. Math. Anal. Appl.* **332**(2007), 497–511. MR2319679; url

[38] X. Yang, On Liapunov-type inequality for certain higher-order differential equations, *Appl. Math. Comput.* **134**(2003), 307–317. MR1931541; url

[39] X. Yang, K. Lo, Lyapunov-type inequality for a class of even-order differential equations, *Appl. Math. Comput.* **215**(2010), 3884–3890. MR2578854; url

[40] X. Yang, Y. Kim, K. Lo, Lyapunov-type inequality for n-dimensional quasilinear systems, *Math. Inequal. Appl.* **16**(2013), 929–934. MR3134772; url

[41] X. Yang, Y. Kim, K. Lo, Lyapunov-type inequality for quasilinear systems, *Appl. Math. Comput.* **219**(2012), 1670–1673. MR2983874; url

[42] Y. Wang, Lyapunov-type inequalities for certain higher order differential equations with anti-periodic boundary conditions, *Appl. Math. Letters* **25**(2012), 2375–2380. MR2967847; url

[43] K. Watanabe, K. Takemura, Y. Kametaka, A. Nagai, H. Yamagishi, Lyapunov-type inequalities for 2Mth order equations under clamped-free boundary conditions, *J. Inequal. Appl.* **242**(2012), 1–8. MR3017139; url