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Traveling waves in lattice differential equations with distributed maturation delay

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Abstract

In this paper we derive a lattice model with infinite distributed delay to describe the growth of a single-species population in a 2D patchy environment with infinite number of patches connected locally by diffusion and global interaction. We consider the existence of traveling wave solutions when the birth rate is large enough that each patch can sustain a positive equilibrium. When the birth function is monotone, we prove that there exists a traveling wave solution connecting two equilibria with wave speed $c > c^*(\theta)$ by using the monotone iterative method and super and subsolution technique, where $\theta \in [0, 2\pi]$ is any fixed direction of propagation. When the birth function is non-monotone, we prove the existence of non-trivial traveling wave solutions by constructing two auxiliary systems satisfying quasi-monotonicity.

Keywords: lattice differential equation, infinite distributed delay, traveling wave solution.

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1 Introduction

In 1990, Aiello and Freedman [1] derived the following model to describe the growth of a single-species population:

$$\begin{cases} u_i'(t) = \alpha u_m(t) - \gamma u_i(t) - \alpha e^{-\gamma\tau} u_m(t-\tau), \\ u_m'(t) = \alpha e^{-\gamma\tau} u_m(t-\tau) - \beta u_m^2(t). \end{cases} \tag{1.1}$$

Here α, β and γ are positive constants, u_m and u_i denote the number of immature (juvenile) and mature (adult) members of the population, respectively; the delay $\tau > 0$ is the time taken from birth to maturity. In particular, they assumed that the maturation delay τ is known exactly and that all individuals take this amount of time to mature. They showed that the unique positive equilibrium of (1.1) is globally asymptotically stable.

However, as reported by Al-Omari and Gourley [2], the individuals do not necessarily mature at the same time, and it has been observed that distributed delays are more reasonable than discrete delays in modeling maturation periods. Therefore, they proposed a more general model:

$$\begin{cases} u_i'(t) = b(u_m(t)) - \gamma u_i(t) - \int_0^\infty b(u_m(t-s)) f(s) e^{-\gamma s} ds, \\ u_m'(t) = \int_0^\infty b(u_m(t-s)) f(s) e^{-\gamma s} ds - d(u_m(t)), \end{cases} \tag{1.2}$$

where the probability density function $f(s) \in L^1([0,\infty), \mathbb{R}^+)$ describes the probability of maturing at each age s and satisfies $\int_0^\infty f(s)ds = 1$, $b(\cdot)$ and $d(\cdot)$ are the more general birth rate and death rate functions, respectively. The authors also considered spatial effects and proposed the following nonlocal reaction-diffusion model with distributed delay

$$\begin{cases} \frac{\partial u_i(x,t)}{\partial t} = & D_i \Delta u_i(x,t) + b(u_m(x,t)) - \gamma u_i(x,t) \\ & - \int_0^\tau \int_\Omega G(x,y,s) f(s) e^{-\gamma s} b(u_m(y,t-s)) dy ds, \\ \frac{\partial u_m(x,t)}{\partial t} = & D_m \Delta u_m(x,t) - d(u_m(x,t)) \\ & + \int_0^\tau \int_\Omega G(x,y,s) f(s) e^{-\gamma s} b(u_m(y,t-s)) dy ds, \end{cases} \quad x \in \Omega,\ t > 0, \tag{1.3}$$

subject to homogeneous Neumann boundary conditions

$$\vec{n} \cdot \nabla u_i = \vec{n} \cdot \nabla u_m = 0 \ \text{ on } \partial\Omega,$$

where $\vec{n}$ is an outward normal to $\partial\Omega$, $\Omega \subset \mathbb{R}^{\mathbb{N}}$ is bounded. They proved that the positive equilibria of system (1.2) and (1.3) are stable under some assumptions about the functions b and d. In [3], Al-Omari and Gourley further studied the following system

$$\begin{cases} \frac{\partial u_i}{\partial t} = & D_i \frac{\partial^2 u_i}{\partial x^2} + \alpha u_m - \gamma u_i \\ & -\alpha \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi D_i s}} e^{-\frac{(x-y)^2}{4D_i s}} u_m(y, t-s) e^{-\gamma s} f(s) dy ds, \\ \frac{\partial u_m}{\partial t} = & D_m \frac{\partial^2 u_m}{\partial x^2} - \beta u_m^2 \\ & +\alpha \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi D_i s}} e^{-\frac{(x-y)^2}{4D_i s}} u_m(y, t-s) e^{-\gamma s} f(s) dy ds, \end{cases} \tag{1.4}$$

where $x \in (-\infty, \infty)$. When $f(s) = (s/\tau^2)e^{-s/\tau}$, they proved that system (1.4) admits traveling waves connecting two equilibria. Weng and Wu [31] also studied the existence of traveling waves for the second equation of system (1.4).

Another important single-species model with diffusion and stage-structure is the following equation:

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \epsilon \int_{-\infty}^{\infty} b\big(w(y, t-r_0)\big) \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{(x-y)^2}{4\alpha}} dy, \tag{1.5}$$

which was derived by So et al. [25]. In (1.5), $w(x,t)$ is the total matured population at time $t > 0$ and position $x \in \mathbb{R}$; D_m and d_m are the diffusion and death rates for the mature population respectively; r_0 is the maturation time and $b(\cdot)$ is the birth function. There have been many studies on the existence and stability of traveling waves of (1.5), see [7, 12, 20, 23, 28]. However, the mature time of the individuals in (1.5) are the same, which is not realistic as mentioned in the previous. Therefore, Gourley and So [10] further proposed and studied the following model with distributed mature time

$$\frac{\partial w}{\partial t} = D \frac{\partial^2 w}{\partial x^2} - dw + \int_0^\infty f(a) e^{-da} \int_{-\infty}^\infty b(w(y, t-a)) \frac{1}{\sqrt{4\pi Da}} e^{-\frac{(x-y)^2}{4Da}} dy da. \tag{1.6}$$

For more details on the studies of traveling wave solutions of (1.4), (1.5) and (1.6), we refer to [7, 9, 10, 12, 16, 18, 22, 23, 27, 28, 29] and the references therein.

For the model (1.5), its discrete version was firstly proposed by Weng et al. [30]. They considered the growth of a single-species population living in a patch environment consisting of all integer nodes of a 1D lattice. They divided the population into two ages classes: immature and mature, and assumed that the mature periods of all individuals are same as those in So et al. [25]. By the discrete Fourier transform, Weng et al. [30] obtained the following lattice differential equations

$$\begin{aligned}\frac{dw_j(t)}{dt} =&D_m[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - d_m w_j(t)\\ &+ \frac{\mu}{2\pi}\sum_{k=-\infty}^{\infty}\beta_\alpha(j-k)b\big(w_k(t-r)\big),\end{aligned}\tag{1.7}$$

where $\beta_\alpha(l) = 2e^{-\nu}\int_0^\pi \cos(l\omega)e^{\nu\cos\omega}d\omega$ for $\nu := 2\alpha$. They established the spreading speed and the existence of monotone traveling waves for (1.7) when the birth function is monotone. They further showed that the minimal wave speed coincides with the spreading speed. For more studies on (1.7), we refer to [6, 11, 12, 15, 16, 19, 21, 22, 30, 32].

Cheng et al. [4] extended the work of Weng et al. [30] and proposed the following lattice equation

$$\begin{aligned}\frac{dw_{k,j}(t)}{dt} =&D_m[w_{k+1,j}(t) + w_{k-1,j}(t) + w_{k,j+1}(t) + w_{k,j-1}(t) - 4w_{k,j}(t)]\\ &- d_m w_{k,j}(t) + \frac{\mu}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b(w_{k+l,j+q}(t-r)),\end{aligned}\tag{1.8}$$

which models the growth of a single-species population with two age classes distributed over a patchy environment consisting of all integer nodes of a 2D lattice. They studied the well-posedness of the initial-value problem and established the existence of monotone traveling waves for wave speed $c \geq c^*(\theta) > 0$, where θ is any fixed direction of propagation. They further showed that the minimal wave speed $c^*(\theta)$ coincides with the asymptotic speed of spread for any fixed direction $\theta \in [0, 2\pi]$. They showed that the asymptotic speed of propagation depends on not only the maturation period and the diffusion rate of mature population monotonically but also the direction of propagation, which is different from the case when the spatial variable is continuous. In Cheng et al. [5], the authors established the asymptotic stability of traveling wave fronts for equation (1.8) when immature population is not mobile.

The aim of the current paper is to modify (1.8) to allow for the fact that the time from birth to maturity may be rather imperfectly known, or it might vary from individual to individual, as done by Gourley and So [10]. Therefore, in this paper we firstly derive a lattice differential equations with distribution mature delay in 2D lattice and then establish the well-posedness of the nontrivial traveling wave solutions for the equations. This paper is organized as follows. In Section 2, we derive the lattice differential equations (2.7) and show some properties of equation

(2.7). The existence of non-trivial traveling waves of equation (2.7) is obtained in Section 3, where we consider two cases, namely, the monotone birth function and the non-monotone birth function.

2 Model derivation

Let $u_{k,j}(t,a) \geq 0$ denote the population density of the species of the $(k,j)-th$ patch at time $t \geq 0$ and age $a \geq 0$. Assume that the patches are located at the integer nodes of a $2D$ lattice and spatial diffusion occurs only at the nearest neighborhood. From Metz and Diekmann [24], we can obtain the following model:

$$\begin{aligned}\frac{\partial}{\partial t}u_{k,j}(t,a) + \frac{\partial}{\partial a}u_{k,j}(t,a) =& D(a)[u_{k+1,j}(t,a) + u_{k-1,j}(t,a) + u_{k,j+1}(t,a) \\ &+ u_{k,j-1}(t,a) - 4u_{k,j}(t,a)] - d(a)u_{k,j}(t,a), \\ &t > 0, (k,j) \in \mathbb{Z} \times \mathbb{Z},\end{aligned} \tag{2.1}$$

where $D(a)$ and $d(a)$ are the diffusion and death rates at age a, respectively. Assume that $u_{k,j}(t,\infty) = 0$ for $t \geq 0, (k,j) \in \mathbb{Z} \times \mathbb{Z}$. We want an expression for $w_{k,j}(t)$, that is the total matured population at time t and $(k,j) \in \mathbb{Z} \times \mathbb{Z}$. Let

$$f(r)dr = \text{probability of maturing between the ages } r \text{ and } r + dr, \tag{2.2}$$

where $f(r)$ is the probability of maturing at each age r. Note that the probability of maturing before age a is

$$F(a) := \int_0^a f(r)dr. \tag{2.3}$$

Since f is a probability density function, we assume $f(r) \geq 0$, and of course, $\int_0^\infty f(r)dr = 1$. Of the total number of mature adults, the number that matured between age r and $r + dr$ is

(number of age at least r)$\times$(probability of having matured between ages r and $r+dr$),

or from (2.2),

$$\left(\int_r^\infty u_{k,j}(t,a)da\right) \cdot f(r)dr.$$

Thus, the total number of matures is

$$w_{k,j}(t) = \int_0^\infty \left(\int_r^\infty u_{k,j}(t,a)da\right) \cdot f(r)dr.$$

Reversing the order of integration yields the following alternative expression:

$$w_{k,j}(t) = \int_0^\infty u_{k,j}(t,a)\left(\int_0^a f(r)dr\right)da = \int_0^\infty u_{k,j}(t,a)F(a)da, \tag{2.4}$$

where $F(a)$ is given by (2.3).

In the following, we want to find a differential equation satisfied by $w_{k,j}(t)$. Differentiating (2.4) with respect to t, together with (2.1), one sees that

$$\begin{aligned}\frac{dw_{k,j}(t)}{dt} &= \int_0^\infty \frac{\partial u_{k,j}(t,a)}{\partial t}F(a)da \\ &= \int_0^\infty \Big\{ -\frac{\partial u_{k,j}(t,a)}{\partial a} + D(a)[u_{k+1,j}(t,a) + u_{k-1,j}(t,a) + u_{k,j+1}(t,a) \\ &\quad + u_{k,j-1}(t,a) - 4u_{k,j}(t,a)] - d(a)u_{k,j}(t,a)\Big\}F(a)da. \end{aligned} \tag{2.5}$$

In this paper we shall assume that the diffusion coefficient and death rate are age independent. i.e. $D(a) = D$ and $d(a) = d$ for $a \in [0,\infty)$, where D and d are positive constants. From (2.5),

$$\begin{aligned}\frac{dw_{k,j}(t)}{dt} =& -\int_0^\infty F(a)\frac{\partial u_{k,j}(t,a)}{\partial a}da + D\big[w_{k+1,j}(t) + w_{k-1,j}(t) \\ &+ w_{k,j+1}(t) + w_{k,j-1}(t) - 4w_{k,j}(t)\big] - dw_{k,j}(t).\end{aligned}$$

Integrating by parts on the first term, and using $F(0) = 0$ and $u_{k,j}(t,\infty) = 0$, we obtain

$$\begin{aligned}\frac{dw_{k,j}(t)}{dt} =& \int_0^\infty f(a)u_{k,j}(t,a)da + D\big[w_{k+1,j}(t) + w_{k-1,j}(t) + w_{k,j+1}(t) \\ &+ w_{k,j-1}(t) - 4w_{k,j}(t)\big] - dw_{k,j}(t).\end{aligned} \tag{2.6}$$

By an argument similar to that of Cheng et al. [4], using discrete Fourier transformation and inverse discrete Fourier transformation(see [8, 26]), we obtain a closed system as follows:

$$\begin{aligned}\frac{dw_{k,j}(t)}{dt} =& D[w_{k+1,j}(t) + w_{k-1,j}(t) + w_{k,j+1}(t) + w_{k,j-1}(t) - 4w_{k,j}(t)] - dw_{k,j}(t) \\ &+ \frac{1}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q)b\big(w_{k+l,j+q}(t-a)\big)e^{-da}f(a)da\right],\end{aligned} \tag{2.7}$$

where $\alpha = Da, b : \mathbb{R}^+ \to \mathbb{R}^+$ is a birth function and

$$\beta_\alpha(l) = \mathrm{Re} \int_{-\pi}^{\pi} e^{il\omega_1 - 4\alpha \sin^2 \frac{\omega_1}{2}} d\omega_1 = 2e^{-\nu} \int_0^{\pi} \cos(l\omega_1) e^{\nu \cos \omega_1} d\omega_1, \ (\nu := 2\alpha), \quad (2.8)$$

$$\gamma_\alpha(l) = \mathrm{Re} \int_{-\pi}^{\pi} e^{il\omega_2 - 4\alpha \sin^2 \frac{\omega_2}{2}} d\omega_2 = 2e^{-\nu} \int_0^{\pi} \cos(l\omega_2) e^{\nu \cos \omega_2} d\omega_2, \ (\nu := 2\alpha), \quad (2.9)$$

for any $l \in \mathbb{Z}$.

The following lemma gives the properties of β_α and γ_α, see [30, 4].

Lemma 2.1 *Let β_α and γ_α be given in* (2.8) *and* (2.9)*, respectively. Then the following holds:*

(1) $\beta_\alpha(l) = \beta_\alpha(|l|), \gamma_\alpha(l) = \gamma_\alpha(|l|), \forall l \in \mathbb{Z}$. i.e. $\beta_\alpha(l)$ and $\gamma_\alpha(l)$ are isotropic functions for any $\alpha \geq 0$;

(2) $\frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \beta_\alpha(l) = 1, \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \gamma_\alpha(l) = 1$;

(3) $\beta_\alpha(l) \geq 0, \ \gamma_\alpha(l) \geq 0$ if $\alpha = 0$ and $l \in \mathbb{Z}$; $\beta_\alpha(l) > 0, \ \gamma_\alpha(l) > 0$ if $\alpha > 0$ and $l \in \mathbb{Z}$.

3 Existence of traveling waves

In this section, we establish the existence of traveling waves for the lattice differential equation (2.7) when it has a positive equilibrium. We will consider two cases: (a) the birth function $b(\cdot)$ is monotone; (b) the birth function $b(\cdot)$ is nonmonotone.

3.1 Monotone birth functions

We assume that the birth function $b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (H_b).

($\mathbf{H}_b$) b is local Lipschitz continuous and $b'(0)$ exists. Furthermore, b satisfies the following:

(1) $b(0) = 0$, $b'(0)\bar{f}(d) > d$, $b(w) \leq b'(0)w$ for any $w \in \mathbb{R}_+$, where $\bar{f}(d) = \int_0^\infty e^{-da} f(a)da$;

(2) $\bar{f}(d)b(w) = dw$ admits a unique positive solution $w^* > 0$ and b is nondecreasing in $[0, w^*]$, where $\bar{f}(d) = \int_0^\infty e^{-da} f(a)da$;

(3) There exist constants $\rho \in (0, 1]$, $M_0 > 0$ and $\eta \in (0, w^*)$ such that $b'(0)w - b(w) < M_0 w^{1+\rho}$ for any $w \in (0, \eta)$;

(4) $b(w)\bar{f}(d) > dw$ for $w \in (0, w^*)$.

A traveling wave solution of lattice differential equation (2.7) is a solution with the form

$$w_{k,j}(t) = \phi(k\cos\theta + j\sin\theta + ct),$$

where $\theta \in [0, 2\pi]$ is any fixed direction of propagation, $c > 0$ is the wave speed. Denote

$$s = k\cos\theta + j\sin\theta + ct.$$

Substituting it into (2.7), we have that $\phi(\cdot)$ satisfies

$$\begin{aligned} & c\phi'(s) + (4D + d)\phi(s) \\ =& D\big[\phi(s + \cos\theta) + \phi(s - \cos\theta) + \phi(s + \sin\theta) + \phi(s - \sin\theta)\big] \\ &+ \frac{1}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q) b\big(\phi(s + l\cos\theta + q\sin\theta - ca)\big)e^{-da} f(a)da\right]. \end{aligned} \tag{3.1}$$

In view of the symmetry and periodicity in equation (3.1), it is sufficient to consider $\theta \in [0, \frac{\pi}{2}]$. Let $\theta \in [0, \frac{\pi}{2}]$. Denoting $C = C(\mathbb{R}, [0, K])$, we define

$$S = \left\{ \phi \in C \;\middle|\; \begin{array}{l} (i)\ \phi(s) \text{ is nondecreasing, for any } s \in \mathbb{R}; \\ (ii)\ \lim_{s\to-\infty}\phi(s) = 0,\ \lim_{s\to\infty}\phi(s) = w^*. \end{array} \right\}$$

Define two operators A and H on C by

$$\begin{aligned} A(\phi)(s) &= \phi(s + \cos\theta) + \phi(s - \cos\theta) + \phi(s + \sin\theta) + \phi(s - \sin\theta), \\ H(\phi)(s) &= \frac{1}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q) b\big(\phi(s + l\cos\theta + q\sin\theta - ca)\big)e^{-da} f(a)da\right]. \end{aligned}$$

Definition 3.1 *A function $U \in C$ is called an upper solution of* (3.1) *if U is differentiable almost everywhere on $\mathbb{R}$ and satisfies the following inequality*

$$cU'(s) \geq D[A(U)(s) - 4U(s)] - dU(s) + H(U)(s).$$

A lower solution can be defined similarly by reversing the inequality above.

Linearize (3.1) at the trivial equilibrium $w^0 = 0$, and we denote the characteristic equation by $\Delta(\lambda, c, \theta) = 0$. It can be seen that

$$\begin{aligned} \Delta(\lambda, c, \theta) =& -c\lambda + D[e^{\lambda\cos\theta} + e^{-\lambda\cos\theta} + e^{\lambda\sin\theta} + e^{-\lambda\sin\theta} - 4] - d \\ &+ b'(0)\int_0^\infty \left[\frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\beta_\alpha(l)e^{\lambda l\cos\theta}\right]\left[\frac{1}{2\pi}\sum_{q=-\infty}^{\infty}\gamma_\alpha(q)e^{\lambda q\sin\theta}\right]e^{-(c\lambda+d)a} f(a)da. \end{aligned}$$

Let

$$S(\alpha)=\frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\beta_\alpha(l)e^{\lambda l\cos\theta},\ T(\alpha)=\frac{1}{2\pi}\sum_{q=-\infty}^{\infty}\gamma_\alpha(q)e^{\lambda q\sin\theta}.$$

Since $S(0)=1$ and

$$\frac{dS(\alpha)}{d\alpha}=S(\alpha)(e^{\lambda\cos\theta}+e^{-\lambda\cos\theta}-2),$$

we have

$$S(\alpha)=\exp\big\{(e^{\lambda\cos\theta}+e^{-\lambda\cos\theta}-2)\alpha\big\}=\exp\big\{2\alpha[\cosh(\lambda\cos\theta)-1]\big\}.$$

Similarly,

$$T(\alpha)=\exp\big\{2\alpha[\cosh(\lambda\sin\theta)-1]\big\}.$$

Using the expressions of $S(\alpha)$ and $T(\alpha)$, we rewrite $\Delta(\lambda,c,\theta)$ as

$$\begin{aligned}\Delta(\lambda,c,\theta)=&b'(0)\int_0^\infty \exp\{2\alpha[\cosh(\lambda\cos\theta)+\cosh(\lambda\sin\theta)-2]-(c\lambda+d)a\}f(a)da\\&-c\lambda+D\big(e^{\lambda\cos\theta}+e^{-\lambda\cos\theta}+e^{\lambda\sin\theta}+e^{-\lambda\sin\theta}-4\big)-d.\end{aligned}$$

It is obvious that $\lim_{\lambda\to+\infty}\Delta(\lambda,c,\theta)=+\infty$ and

$$\Delta(0,c,\theta)=b'(0)\int_0^\infty e^{-da}f(a)da-d=b'(0)\bar{f}(d)-d>0 \text{ for } c\in\mathbb{R}.$$

By simple computations, we have

$$\begin{aligned}\frac{\partial\Delta(\lambda,c,\theta)}{\partial\lambda}=&-c+D\left[\cos\theta e^{\lambda\cos\theta}-\cos\theta e^{-\lambda\cos\theta}+\sin\theta e^{\lambda\sin\theta}-\sin\theta e^{-\lambda\sin\theta}\right]\\&+b'(0)\int_0^\infty \exp\big\{2\alpha[\cosh(\lambda\cos\theta)+\cosh(\lambda\sin\theta)-2]-(c\lambda+d)a\big\}\\&\times\big\{2\alpha[\sinh(\lambda\cos\theta)\cdot\cos\theta+\sinh(\lambda\sin\theta)\cdot\sin\theta]-ca\big\}f(a)da,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\Delta(\lambda,c,\theta)}{\partial\lambda^2}=&D\left[\cos^2\theta e^{\lambda\cos\theta}+\cos^2\theta e^{-\lambda\cos\theta}+\sin^2\theta e^{\lambda\sin\theta}+\sin^2\theta e^{-\lambda\sin\theta}\right]\\&+b'(0)\int_0^\infty \exp\big\{2\alpha[\cosh(\lambda\cos\theta)+\cosh(\lambda\sin\theta)-2]-(c\lambda+d)a\big\}\\&\times\Big\{2\alpha\Big[\sinh(\lambda\cos\theta)\cdot\cos\theta+\sinh(\lambda\sin\theta)\cdot\sin\theta]-ca\Big\}^2f(a)da\\&+b'(0)\int_0^\infty \exp\big\{2\alpha[\cosh(\lambda\cos\theta)+\cosh(\lambda\sin\theta)-2\Big]-(c\lambda+d)a\big\}\end{aligned}$$

$$\times 2\alpha \left[\cosh(\lambda\cos\theta)\cdot\cos^2\theta + \cosh(\lambda\sin\theta)\cdot\sin^2\theta\right] f(a)da$$
$$>0$$

and

$$\frac{\partial\Delta(\lambda,c,\theta)}{\partial\lambda}|_{\lambda=0} = -c - cb'(0)\int_0^\infty ae^{-da}f(a)da < 0 \quad \text{for } c>0.$$

Then $\Delta(\lambda,c,\theta)$ is convex with respect to λ. Differentiating $\Delta(\lambda,c,\theta)$ with respect to $c>0$, we obtain that

$$\begin{aligned}\frac{\partial\Delta(\lambda,c,\theta)}{\partial c} =& -\lambda\Big\{1 + b'(0)\int_0^\infty \exp\big\{2\alpha[\cosh(\lambda\cos\theta)+\cosh(\lambda\sin\theta)-2]\\ &-(c\lambda+d)a\big\}af(a)da\Big\} < 0\end{aligned}$$

for $\lambda>0$. Furthermore, it is easy to show that $\Delta(\lambda,0,\theta)>0$ and $\lim_{c\to+\infty}\Delta(\lambda,c,\theta) = -\infty$ for any given $\lambda>0$.

Summarizing the above discussion, we have the following assertion.

Lemma 3.2 *For any fixed $\theta\in[0,\frac{\pi}{2}]$, there exists a pair of $c_*(\theta)$ and λ_*, such that*

(1) $\Delta(\lambda_*,c_*(\theta),\theta)=0$, $\frac{\partial}{\partial\lambda}\Delta(\lambda_*,c_*(\theta),\theta)=0$;

(2) $\Delta(\lambda,c,\theta)>0$ *for* $0<c<c_*(\theta)$ *and any* $\lambda>0$;

(3) *For any* $c>c_*(\theta)$, *equation* $\Delta(\lambda,c,\theta)=0$ *has two positive real solutions* $0<\lambda_1<\lambda_2$ *such that* $\Delta(\cdot,c,\theta)<0$ *in* (λ_1,λ_2) *and* $\Delta(\cdot,c,\theta)>0$ *in* $\mathbb{R}/[\lambda_1,\lambda_2]$.

Define

$$\phi^+(s) = \begin{cases} w^*, & s\geq 0,\\ e^{\lambda_1 s}w^*, & s\leq 0,\end{cases} \tag{3.2}$$

and

$$\phi^-(s) = \begin{cases} 0, & s\geq -\frac{1}{\varepsilon}\ln M,\\ w^*(1-Me^{\varepsilon s})e^{\lambda_1 s}, & s\leq -\frac{1}{\varepsilon}\ln M,\end{cases} \tag{3.3}$$

where $0<\varepsilon<\frac{1}{2}\lambda_1\rho$, $\varepsilon<\lambda_2-\lambda_1$, λ_1 and λ_2 are given in Lemma 3.2. Choose $M>1$ large enough so that $\phi^-(s)<\eta$ for $s\in\mathbb{R}$, where η is a given constant by the assumption (H_b).

Lemma 3.3 *For functions $\phi^+(s)$ and $\phi^-(s)$ given by* (3.2) *and* (3.3), *if $M>1$ is large enough, then $\phi^+(s)$ and $\phi^-(s)$ are a pair of upper and lower solutions of* (3.1).

Proof. We first prove that $\phi^+(s)$ is an upper solution of (3.1).

For $s>0$, $\phi^+(s)=w^*$. By Lemma 2.1 (2), and the monotonicity of $b(\cdot)$, we have

$$\begin{aligned}&-c\phi^{+\prime}(s)+D\big[A\left(\phi^+\right)(s)-4\phi^+(s)\big]-d\phi^+(s)+H(\phi^+)(s)\\ \leq& D(w^*+w^*+w^*+w^*-4w^*)-dw^*+b(w^*)\bar{f}(d)=0.\end{aligned}$$

For $s\leq 0$, $\phi^+(s)=e^{\lambda_1 s}w^*$. Since $\phi^+(s)\leq e^{\lambda_1 s}w^*$ for $s\in\mathbb{R}$ and $b(w)\leq b'(0)w$ for $w\geq 0$, we have

$$\begin{aligned}&-c\phi^{+\prime}(s)+D\big[A\left(\phi^+\right)(s)-4\phi^+(s)\big]-d\phi^+(s)+H(\phi^+)(s)\\ \leq& e^{\lambda_1 s}w^*\Big[-c\lambda_1+D\big(e^{\lambda_1\cos\theta}+e^{-\lambda_1\cos\theta}+e^{\lambda_1\sin\theta}+e^{-\lambda_1\sin\theta}-4\big)-d\Big]\\ &+\frac{b'(0)}{(2\pi)^2}e^{\lambda_1 s}w^*\int_0^\infty\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)e^{\lambda_1 l\cos\theta+\lambda_1 q\sin\theta}e^{-(c\lambda_1+d)a}f(a)da=0.\end{aligned}$$

Hence, $\phi^+(s)$ is an upper solution of (3.1).

Next, we prove that $\phi^-(s)$ is a lower solution of (3.1). Obviously, $\phi^-(s)\geq 0$ and $H(\phi^-)(s)\geq 0$ for $s\in\mathbb{R}$. If $s\geq -\frac{1}{\varepsilon}\ln M$, then $\phi^-(s)=0$. It follows that

$$\begin{aligned}&-c\phi^{-\prime}(s)+D\big[A\left(\phi^-\right)(s)-4\phi^-(s)\big]-d\phi^-(s)+H(\phi^-)(s)\\ =&D\big[\phi^-(s+\cos\theta)+\phi^-(s-\cos\theta)+\phi^-(s+\sin\theta)+\phi^-(s-\sin\theta)\big]+H(\phi^-)(s)\geq 0.\end{aligned}$$

Notice that $w^*(1-Me^{\varepsilon s})e^{\lambda_1 s}\leq\phi^-(s)\leq w^*e^{\lambda_1 s}$ and $0\leq\phi^-(s)\leq\eta$ for all $s\in\mathbb{R}$. Then, if $s\leq-\frac{1}{\varepsilon}\ln M$, we have $\phi^-(s)=w^*(1-Me^{\varepsilon s})e^{\lambda_1 s}$. In view of $\Delta(\lambda_1+\varepsilon,c,\theta)<0$, one sees that

$$\begin{aligned}&-c\phi^{-\prime}(s)+D\big[A\left(\phi^-\right)(s)-4\phi^-(s)\big]-d\phi^-(s)+H(\phi^-)(s)\\ \geq\;&-c\phi^{-\prime}(s)+D\big[A\left(\phi^-\right)(s)-4\phi^-(s)\big]-d\phi^-(s)\\ &+\frac{b'(0)}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left[\int_0^\infty\beta_\alpha(l)\gamma_\alpha(q)\phi^-(s+l\cos\theta+q\sin\theta-ca)e^{-da}f(a)da\right]\\ &-\frac{M_0}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left[\int_0^\infty\beta_\alpha(l)\gamma_\alpha(q)\left(\phi^-(s+l\cos\theta+q\sin\theta-ca)\right)^{1+\rho}e^{-da}f(a)da\right]\\ \geq\;& w^*e^{\lambda_1 s}\Delta(\lambda_1,c,\theta)-w^*Me^{(\lambda_1+\varepsilon)s}\Delta(\lambda_1+\varepsilon,c,\theta)-\frac{M_0\left(w^*\right)^{1+\rho}}{(2\pi)^2}e^{\lambda_1(1+\rho)s}\\ &\times\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left[\int_0^\infty\beta_\alpha(l)\gamma_\alpha(q)\exp\left\{\lambda_1(1+\rho)(l\cos\theta+q\sin\theta-ca)\right\}e^{-da}f(a)da\right]\\ \geq\;&-w^*Me^{(\lambda_1+\varepsilon)s}\Delta(\lambda_1+\varepsilon,c,\theta)-\frac{M_0\left(w^*\right)^{1+\rho}}{(2\pi)^2}\left(\frac{1}{M}\right)^{\frac{\lambda_1\rho-\varepsilon}{\varepsilon}}e^{(\lambda_1+\varepsilon)s}\end{aligned}$$

$$\times \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q) \exp\left\{\lambda_1(1+\rho)(l\cos\theta + q\sin\theta - ca)\right\} e^{-da} f(a)da \right]$$
$$\geq 0$$

provided that $M > 1$ large enough. Hence, $\phi^-(s)$ is a lower solution of (3.1). This completes the proof. □

Let $\delta = 4D + d$. Define an operator $\mathcal{F} : S \to C$ by

$$\mathcal{F}(\phi)(s) = \frac{1}{c} e^{-\frac{\delta s}{c}} \int_{-\infty}^{s} e^{\frac{\delta \tau}{c}} \Big\{ DA(\phi)(\tau) + H(\phi)(\tau) \Big\} d\tau,$$

It is easy to find that $\mathcal{F}$ is well-defined and a fixed point of $\mathcal{F}$ is a solution of (3.1).

Lemma 3.4 (1) *$\phi^-(s) \leq \mathcal{F}(\phi^-)(s) \leq \mathcal{F}(\phi^+)(s) \leq \phi^+(s)$ for any $s \in \mathbb{R}$;*

(2) *If $\phi(s) \leq \psi(s)$ for any $\phi, \psi \in C(\mathbb{R}, [0, w^*])$ with $s \in \mathbb{R}$, then $\mathcal{F}(\phi)(s) \leq \mathcal{F}(\psi)(s)$ for any $s \in \mathbb{R}$;*

(3) *$\mathcal{F}(\phi) \in S$ for any $\phi \in S$;*

(4) *There exists $K_S > 0$ such that for any $\phi \in S$, it holds that $\|\mathcal{F}(\phi)(\cdot)\|_{C^{1,1}} \leq K_S$.*

By Lemma 3.3 and the monotonicity of H, we can easily obtain (1)-(3) of Lemma 3.4, see also Ma [17]. Differentiating with $\mathcal{F}(\phi)(s)$ and using the Lipschitz continuity of b, we can prove Lemma 3.4 (4).

Define an iteration sequence $\phi^n = \mathcal{F}\phi^{n-1}, n \geq 1, \phi^0 = \phi^+$. It follows from Lemma 3.4 that,

$$\phi^-(s) \leq \cdots \leq \phi^n(s) \leq \phi^{n-1}(s) \leq \cdots \leq \phi^1(s) \leq \phi^+(s), \forall s \in \mathbb{R}. \tag{3.4}$$

By (3.4) and Lemma 3.4 (4), there exists a function $\phi^* \in C^1(\mathbb{R}, [0, w^*])$ such that $\phi^n(\cdot)$ converge to ϕ^* in C^1_{loc}. It follows from the Lebesgue dominated convergence theorem that ϕ^* is a fixed point of $\mathcal{F}$, which is also a solution of (3.1). Furthermore, ϕ^* is nondecreasing and satisfies

$$\phi^-(s) \leq \phi^*(s) \leq \phi^+(s), \ s \in \mathbb{R}.$$

It is easy to show that $\lim_{s \to -\infty} \phi^*(s) = 0$ and $\lim_{s \to \infty} \phi^*(s) = u^* > 0$, where u^* is a constant. Especially, $0 < u^* \leq w^*$. By the standard discussion (see Ma [17], Wu and Zou [33]), we get $u^* = w^*$. From the above arguments, we have proved the following theorem.

Theorem 3.5 *Assume that $b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (H_b). For each $\theta \in [0, \frac{\pi}{2}]$, there exists $c^*(\theta) > 0$ defined in Lemma 3.2 such that for any $c > c^*(\theta)$, (2.7) has a monotone traveling wave $\phi^* : \mathbb{R} \to \mathbb{R}$ such that*

$$\lim_{s\to-\infty} \phi^*(s) = 0 \quad \text{and} \quad \lim_{s\to\infty} \phi^*(s) = w^*.$$

3.2 Nonmonotone birth functions

In this subsection, we establish the existence of nontrivial traveling waves when the birth function $b(\cdot)$ is nonmonotone. The main method is to construct two auxiliary lattice differential equations with monotone birth functions and then apply Schauder's fixed-point theorem.

At first, let b satisfy the following assumption:

($\mathbf{H}_b'$) $b : \mathbb{R}_+ \to \mathbb{R}_+$ is local Lipschitz continuous, and
(1) $b(0) = 0, b'(0)\bar{f}(d) > d, b''(0)$ exists, $b(w) \le b'(0)w$ for any $w > 0$, where $\bar{f}(d) = \int_0^\infty e^{-da} f(a)da$;
(2) $\bar{f}(d)b_+(w) = dw$ has a unique positive solution w_+^*, where $\bar{f}(d) = \int_0^\infty e^{-da} f(a)da$;
(3) there exist positive constants $\rho \in (0, 1]$, $M_0 > 0$ and $\eta \in (0, w_+^*)$ such that $b'(0)w - b(w) < M_0 w^{1+\rho}$ and $b_\pm(w) = b(w)$ for any $w \in (0, \eta)$, where $b_\pm(w)$ are defined as follows:

$$b_+(w) := \max_{v\in[0,w]} b(v), \; b_-(w) := \min_{v\in[w,w_+^*]} b(v).$$

It is obvious that b_- and b_+ are nondecreasing and satisfy

$$b_-(w) \le b(w) \le b_+(w) \text{ for } w \in [0, w_+^*].$$

If b is nondecreasing, then $b_\pm = b$ and (H_b') reduces into (H_b). If $\bar{f}(d)b(w) = dw$ has a unique positive solution w^*, then

$$\bar{f}(d)b(w) > dw \text{ for } 0 < w < w^*$$

and

$$\bar{f}(d)b(w) < dw \text{ for } w > w^*.$$

Thus $(H_b')(2)$ holds. $(H_b')(3)$ implies that both $\bar{f}(d)b_-(w) = dw$ and $\bar{f}(d)b(w) = dw$ have minimum positive solutions in $(0, w_+^*]$, denoting by w_-^* and w^* respectively. Obviously, if b satisfies (H_b'), then $b_\pm$ satisfies (H_b). Hereafter, we assume that w_-^*

and w^* are the minimum positive solutions of $\bar{f}(d)b_-(w) = dw$ and $\bar{f}(d)b(w) = dw$ in $[0, w_+^*]$, respectively. In particular, $b_+'(0) = b_-'(0) = b'(0)$.

We consider the following two equations:

$$\begin{aligned}\frac{dw_{k,j}(t)}{dt} =& D\big[w_{k+1,j}(t) + w_{k-1,j}(t) + w_{k,j+1}(t) + w_{k,j-1}(t) - 4w_{k,j}(t)\big] - dw_{k,j}(t)\\ &+ \frac{1}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\Big[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q)b_+\big(w_{k+l,j+q}(t-a)\big)e^{-da}f(a)da\Big]\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\frac{dw_{k,j}(t)}{dt} =& D\big[w_{k+1,j}(t) + w_{k-1,j}(t) + w_{k,j+1}(t) + w_{k,j-1}(t) - 4w_{k,j}(t)\big] - dw_{k,j}(t)\\ &+ \frac{1}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\Big[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q)b_-\big(w_{k+l,j+q}(t-a)\big)e^{-da}f(a)da\Big].\end{aligned} \tag{3.6}$$

The traveling wave equations of (3.5) and (3.6) are

$$\begin{aligned}c\phi'(s) =& D\big[A(\phi)(s) - 4\phi(s)\big] - d\phi(s)\\ &+ \frac{1}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\Big[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q)b_+\big(\phi(s + l\cos\theta + q\sin\theta - ca)\big)e^{-da}f(a)da\Big]\end{aligned}$$

and

$$\begin{aligned}c\phi'(s) =& D\big[A(\phi)(s) - 4\phi(s)\big] - d\phi(s)\\ &+ \frac{1}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\Big[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q)b_-\big(\phi(s + l\cos\theta + q\sin\theta - ca)\big)e^{-da}f(a)da\Big],\end{aligned}$$

respectively.

The following lemma is the immediate consequence of Theorem 3.5.

Lemma 3.6 *Let b satisfy (H_b'). Then for any $\theta \in [0, \frac{\pi}{2}]$, there exists $c^*(\theta) > 0$ defined in Lemma 3.2 such that for any $c > c^*(\theta)$, both (3.5) and (3.6) admit monotone traveling wave solutions $\psi_+(s)$ and $\psi_-(s)$, respectively, such that*

$$\lim_{s\to-\infty}\psi_+(s) = \lim_{s\to-\infty}\psi_-(s) = 0,$$

$$\lim_{s\to\infty}\psi_+(s) = w_+^*, \quad \lim_{s\to\infty}\psi_-(s) = w_-^*$$

and

$$\lim_{s\to\infty}\psi_+(s) \le w_+^* e^{\lambda_1 s}, \quad \lim_{s\to\infty}\psi_-(s) \le w_-^* e^{\lambda_1 s}, \ \forall s \in \mathbb{R}.$$

Define $\bar{b}(w) := \frac{1}{d} b(w)$. Assume

(P) For any $u, v \in [w_-^*, w_+^*]$, if $u \le w^* \le v$, $u \ge \bar{b}(v)$ and $v \le \bar{b}(u)$, then $u = v$.

Similar to the discussion of Hus and Zhao [14, Lemma2.1] (see Fang et al. [6]), it follows that one of the following is the sufficient condition for (P):

(P1) $wb(w)$ is strictly increasing for $w \in [w_-^*, w_+^*]$;

(P2) $b(w)$ is nonincreasing for $w \in [w^*, w_+^*]$ and $\frac{\hat{b}(w)}{w}$ is strictly decreasing for $w \in (0, w^*]$, where $\hat{b}(w) = \bar{b}(\bar{b}(w))$.

Theorem 3.7 *Let b satisfy (H_b'). Then for any $\theta \in [0, \frac{\pi}{2}]$, there exists $c^*(\theta) > 0$ defined in Lemma* 3.2 *such that for any $c > c^*(\theta)$,* (2.7) *has a traveling wave solution $\phi(s)$ satisfying*

$$\phi(-\infty) = 0,\ \ w_-^* \le \liminf_{s\to\infty} \phi(s) \le \limsup_{s\to\infty} \phi(s) \le w_+^*.$$

Furthermore, if $b(w)/w$ is strictly decreasing in $w \in [w_-^*, w_+^*]$ and (P) holds, then $\phi(+\infty) = w^*$.

Proof. It is obvious that traveling wave equation $\phi(s)$ of (3.1) is a solution of the following equation:

$$\phi'(s) + \frac{\delta}{c}\phi(s) = \widehat{H}(\phi)(s), \tag{3.7}$$

where

$$\begin{aligned}\widehat{H}(\phi)(s) =& \frac{D}{c}\big[\phi(s+\cos\theta) + \phi(s-\cos\theta) + \phi(s+\sin\theta) + \phi(s-\sin\theta)\big] \\ &+ \frac{1}{c(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \Big[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q) b\big(\phi(s + l\cos\theta \\ &\qquad + q\sin\theta - ca)\big) e^{-da} f(a)da \Big].\end{aligned} \tag{3.8}$$

We define $\widehat{H}_+$ and $\widehat{H}_-$ by replacing b in (3.8) with b_+ and b_-, respectively. It is easy to show that $\widehat{H}_\pm$ are nondecreasing and satisfy

$$\widehat{H}(w^*) = \frac{\delta}{c} w^*,\ \ \widehat{H}_+(w_+^*) = \frac{\delta}{c} w_+^*,\ \ \widehat{H}_-(w_-^*) = \frac{\delta}{c} w_-^*$$

and

$$\widehat{H}(0) = \widehat{H}_+(0) = \widehat{H}_-(0) = 0.$$

Notice that equation (3.7) is equivalent to

$$\phi(s) = e^{-\frac{\delta}{c}s} \int_{-\infty}^{s} e^{\frac{\delta}{c}\tau} \widehat{H}(\phi)(\tau) d\tau.$$

We can define an operator $\mathcal{T} : C\left(\mathbb{R}, [0, w_+^*]\right) \to C(\mathbb{R})$ as follows:

$$\mathcal{T}(\phi)(s) = e^{-\frac{\delta}{c}s} \int_{-\infty}^{s} e^{\frac{\delta}{c}\tau} \widehat{H}(\phi)(\tau) d\tau. \tag{3.9}$$

Similarly, define $\mathcal{T}_+$ and $\mathcal{T}_-$ by replacing $\widehat{H}$ in (3.9) with $\widehat{H}_+$ and $\widehat{H}_-$, respectively. It is not difficult to show that $\mathcal{T}_\pm$ are nondecreasing and satisfy

$$\mathcal{T}(w^*) = w^*,\ \mathcal{T}_+(w_+^*) = w_+^*, \mathcal{T}_-(w_-^*) = w_-^*$$

and

$$\mathcal{T}_-(\varphi) \leq \mathcal{T}(\varphi) \leq \mathcal{T}_+(\varphi),\ \forall \varphi \in C\left(\mathbb{R}, [0, w_+^*]\right).$$

For any $c > c^*(\theta)$, let $\lambda_1 = \lambda_1(c)$ (see Lemma 3.2). Define

$$\bar{\phi}(s) = \begin{cases} w_+^*, & s \geq 0, \\ e^{\lambda_1 s} w_+^*, & s < 0, \end{cases} \quad \underline{\phi}(s) = \phi_-(s),\ s \in \mathbb{R}.$$

where $\phi_-(s)$ is a traveling wave solution of equation (3.6). By Lemma 3.4, it is easy to show

$$\mathcal{T}_-(\underline{\phi})(s) = \underline{\phi}(s),\ \mathcal{T}_+(\bar{\phi})(s) \leq \bar{\phi}(s).$$

For any given $\lambda > 0$, let

$$X_\lambda := \left\{ \varphi \in C(\mathbb{R}, \mathbb{R}) \,\middle|\, \sup_{x \in \mathbb{R}} |\varphi(x)| e^{-\lambda x} < \infty \right\}$$

and $\|\varphi\|_\lambda = \sup_{x \in \mathbb{R}} |\varphi(x)| e^{-\lambda x}$. Then $(X_\lambda, \|\cdot\|_\lambda)$ is a Banach space. Notice that both $\underline{\phi}$ and $\bar{\phi}$ are elements of X_λ for any given $\lambda \in (0, \lambda_1)$. Define

$$Y := \left\{ \varphi \in X_\lambda \,\middle|\, \underline{\phi} \leq \varphi \leq \bar{\phi} \right\}.$$

Obviously, Y is a convex and closed subset of X_λ. For any $\varphi \in Y$, it holds that

$$\underline{\phi} = \mathcal{T}_-(\underline{\phi}) \leq \mathcal{T}_-(\varphi) \leq \mathcal{T}(\varphi) \leq \mathcal{T}_+(\varphi) \leq \mathcal{T}_+(\bar{\phi}) \leq \bar{\phi}.$$

Thus, $\mathcal{T}(Y) \subset Y$.

Next, we show that $\mathcal{T}$ is completely continuous on Y. For any $\varphi, \psi \in Y$,

$$
\begin{aligned}
&\|\widehat{H}(\varphi) - \widehat{H}(\psi)\|_\lambda = \sup_{x\in\mathbb{R}} |\widehat{H}(\varphi)(x) - \widehat{H}(\psi)(x)| e^{-\lambda x} \\
\leq & \Big\{ \frac{D}{c} \big(e^{\lambda \cos\theta} + e^{-\lambda\cos\theta} + e^{\lambda \sin\theta} + e^{-\lambda \sin\theta} \big) \\
& + \frac{L_b}{c(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \Big[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q) e^{\lambda(l\cos\theta + q\sin\theta - ca)} e^{-da} f(a) da \Big] \Big\} \|\varphi - \psi\|_\lambda \\
:=& M' \|\varphi - \psi\|_\lambda,
\end{aligned}
$$

where L_b is a Lipschitz constant of $b(w)$ in $[0, w_+^*]$. It follows that

$$
\begin{aligned}
& \|\mathcal{T}(\varphi) - \mathcal{T}(\psi)\|_\lambda \\
\leq\ & \sup_{x\in\mathbb{R}} \left| e^{-\frac{\delta}{c}x} \int_{-\infty}^{x} e^{\frac{\delta}{c}y} \widehat{H}(\varphi)(y) dy - e^{-\frac{\delta}{c}x} \int_{-\infty}^{x} e^{\frac{\delta}{c}y} \widehat{H}(\psi)(y) dy \right| e^{-\lambda x} \\
\leq\ & \sup_{x\in\mathbb{R}} e^{-(\frac{\delta}{c}+\lambda)x} \int_{-\infty}^{x} e^{(\frac{\delta}{c}+\lambda)y} \left\| \widehat{H}(\varphi) - \widehat{H}(\psi) \right\|_\lambda dy \\
\leq\ & \sup_{x\in\mathbb{R}} e^{-(\frac{\delta}{c}+\lambda)x} \int_{-\infty}^{x} M' e^{(\frac{\delta}{c}+\lambda)y} \|\varphi - \psi\|_\lambda dy = \frac{cM'}{\delta + c\lambda} \|\varphi - \psi\|_\lambda.
\end{aligned}
$$

Therefore, $\mathcal{T}$ is continuous on Y.

Notice that $\widehat{H}$ is uniformly bounded on Y. Then there exists $M'' > 0$ such that

$$
\left| \frac{d}{ds} \mathcal{T}(\varphi)(s) \right| = \left| -\frac{\delta}{c} e^{-\frac{\delta}{c}s} \int_{-\infty}^{s} e^{\frac{\delta}{c}\tau} \widehat{H}(\varphi)(\tau) d\tau + \widehat{H}(\varphi)(s) \right| \leq M''
$$

for any $\varphi \in Y$. Therefore, $\mathcal{T}(Y)$ is a family of uniformly bounded and equicontinuous functions on $\mathbb{R}$. For any given sequence $\{\psi_n\}_{n\geq 1}$ in $\mathcal{T}(Y)$, there exists $\psi \in C(\mathbb{R}, \mathbb{R})$ such that $\psi_n(x) \to \psi(x)$ uniformly for any bounded subset of $\mathbb{R}$ (by passing to a subsequence if necessary, still denoted by $\{\psi_n\}_{n\geq 1}$). Since $\underline{\phi} \leq \psi_n \leq \bar{\phi}$, we have $\underline{\phi} \leq \psi \leq \bar{\phi}$ and $\psi \in Y$. Since $\lim\limits_{|x|\to\infty} |\bar{\phi}(x) - \underline{\phi}(x)| e^{-\lambda x} = 0$, for any given $\varepsilon > 0$, there exist $B > 0$ and $N > 1$ such that $0 \leq |\bar{\phi}(x) - \underline{\phi}(x)| e^{-\lambda x} < \varepsilon$ if $|x| \geq B$, and $|\psi_n(x) - \psi(x)| e^{-\lambda x} < \varepsilon$ if $|x| \leq B$ with $n \geq N$. Thus, for any $n \geq N$ we have $\|\psi_n - \psi\|_\lambda < \varepsilon$. Therefore, $T(Y)$ is compact in X_λ. By the Schauder's fixed-point theorem, the operator $\mathcal{T}$ has a fixed point ϕ in Y. Clearly, ϕ is non-trivial and satisfies $\phi(-\infty) = 0$ and

$$
\phi_-(s) \leq \phi(s) \leq \min\left\{ w_+^*, w_+^* e^{\lambda_1 s} \right\}.
$$

Similar to the proof of Hus and Zhao [14, Theorem 3.1], we can get that $b(w)/w$ is strictly decreasing in $w \in [w_-^*, w_+^*]$, and $\phi(+\infty) = w^*$ if (P) holds. This completes the proof. $\square$

3.3 An example

In this subsection, we illustrate the results of the previous by examining an equation derived by Gurney et. al. [13] as a model of the population dynamics of a species of fly. Namely, we take $b(w) = pwe^{-rw}$ and $f(a) = \frac{2}{\sqrt{\pi}}e^{-a^2}$, where p and r are positive constants. Then (2.7) becomes

$$\begin{aligned} \frac{dw_{k,j}(t)}{dt} =& D[w_{k+1,j}(t) + w_{k-1,j}(t) + w_{k,j+1}(t) + w_{k,j-1}(t) - 4w_{k,j}(t)] \\ & - dw_{k,j}(t) + \frac{1}{(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left[\int_0^\infty \beta_\alpha(l)\gamma_\alpha(q) \right. \\ & \left. \times \left(pw_{k+l,j+q}(t-a)e^{-rw_{k+l,j+q}(t-a)} \right) e^{-da} \left(\frac{2}{\sqrt{\pi}} e^{-a^2} \right) da \right]. \end{aligned} \tag{3.10}$$

Let

$$\bar{f}(d) = \int_0^\infty e^{-da} f(a) da = e^{\frac{d^2}{4}} \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{d}{2}} e^{-x^2} dx \right)$$

and

$$w^* = \frac{1}{r} \ln \left(\frac{p}{d} \bar{f}(d) \right).$$

Then we consider two cases:

Case (i): $1 < \frac{p}{d}\bar{f}(d) \le e$. In this case we can confirm that the assumption (H_b) holds. Consequently, applying Theorem 3.5 yields that for each $\theta \in [0, \frac{\pi}{2}]$, there exists $c^*(\theta) > 0$ such that for any $c > c^*(\theta)$, (3.10) has a monotone traveling wave solution connecting two equilibria 0 and w^*.

The remainder is to show that (H_b) holds for $b(w) = pwe^{-rw}$ if $1 < \frac{p}{d}\bar{f}(d) \le e$. Firstly, $b(w) = pwe^{-rw}$ is Lipschitz continuous in $w \in [0, \infty)$. By a simple computation, we obtain $b(0) = 0$ and $b'(0) = p$. Since $\frac{p}{d}\bar{f}(d) > 1$, we have $b'(0)\bar{f}(d) > d$. In addition, we have $b(w) = pwe^{-rw} \le pw = b'(0)w$ for $w \ge 0$. Thus, $(H_b)(1)$ holds. It is easy to calculate that $w^* = \frac{1}{r}\ln(\frac{p}{d}\bar{f}(d))$ is the unique positive solution of $\bar{f}(d)b(w) = dw$. Suppose $1 < \frac{p}{d}\bar{f}(d) \le e$, then $b(w)$ is nondecreasing in $[0, w^*]$ and $\bar{f}(d)b(w) > dw$ for $w \in (0, w^*)$, which implies that $(H_b)(2)$ and $(H_b)(4)$ hold. From the fact that $b \in C^2([0, +\infty))$, it is easy to prove that $(H_b)(3)$ holds. This completes the proof.

Case (ii): $\frac{p}{d}\bar{f}(d) > e$. We note that the function $b(w) = pwe^{-rw}$ is increasing in $w \in \left(0, \frac{1}{r}\right)$ and is decreasing in $w \in \left(\frac{1}{r}, +\infty\right)$. Define

$$b_+(w) = \begin{cases} b(w), & 0 \le w \le \frac{1}{r}, \\ b(\frac{1}{r}) = \frac{p}{er}, & w > \frac{1}{r} \end{cases}$$

and

$$b_-(w) = \begin{cases} b(w), & 0 \le w \le w_0^*, \\ b(w_0^*) := b(w_+^*) = \frac{p^2}{erd}\bar{f}(d)e^{-\frac{p}{de}\bar{f}(d)}, & w > w_0^*, \end{cases}$$

where $w_+^* = \frac{p}{der}\bar{f}(d) > \frac{1}{r}$ is the positive root of $\bar{f}(d)b_+(w) = dw$ and $w_0^* \in \left(0, \frac{1}{r}\right)$ satisfies $b(w_0^*) = b(w_+^*)$. Let $w_-^* = \frac{p^2}{erd^2}(\bar{f}(d))^2 e^{-\frac{p}{de}\bar{f}(d)}$ be the positive root of $\bar{f}(d)b_-(w) = dw$. Then we have $w_0^* < w_-^* < w_+^*$. Similar to Case (i), in this case we can easily show that the assumption (H_b') holds with $b_\pm(w)$ defined above. Applying Theorem 3.7, we have that for each $\theta \in [0, \frac{\pi}{2}]$, there exists $c^*(\theta) > 0$ such that for any $c > c^*(\theta)$, (3.10) has a traveling wave solution $w_{k,j}(t) = \phi(k\cos\theta + j\sin\theta + ct)$ satisfying $\phi(-\infty) = 0$ and $w_-^* \le \liminf_{s\to\infty}\phi(s) \le \limsup_{s\to\infty}\phi(s) \le w_+^*$.

References

[1] W. G. Aiello, H. I. Freedman, A time-delay model of single-species growth with stage structure, *Math. Biosci.* **101** (1990) 139-153.

[2] J. F. M. Al-Omari, S.A. Gourley, A nonlocal reaction-diffusion model for a single species with stage structure and distributed maturation delay, *European J. Appl. Math.* **16** (2005) 37-51.

[3] J. F. M. Al-Omari, S.A. Gourley, Monotone wave-fronts in a structured population model with distributed maturation delay, *IMA J. Appl. Math.* **70** (2005) 858-879.

[4] C.-P. Cheng, W.-T. Li, Z.-C. Wang, Spreading speeds and traveling waves in a delayed population model with stage structure on a 2D spatial lattice, *IMA J. Appl. Math.* **73** (2008) 592-618.

[5] C.-P. Cheng, W.-T. Li, Z.-C. Wang, Asymptotic stability of traveling wavefronts in a delayed population model with stage structure on a two-dimensional spatial lattice, *Discrete Contin. Dyn. Syst. Ser. B* **13** (2010) 559-575.

[6] J. Fang, J. Wei, X.-Q. Zhao, Spreading speeds and travelling waves for non-monotone time-delayed lattice equations, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **466** (2010) 1919-1934.

[7] T. Faria, W. Huang, J. Wu, Travelling waves for delayed reaction-diffusion equations with global response, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **462** (2006) 229-261.

[8] R. R. Goldberg, Fourier transform. New York: Cambridge University Press, 1965.

[9] S. A. Gourley, Y. Kuang, Wavefronts and global stability in a time-delayed population model with stage structure, *Proc. Roy. Soc. Lond. A* **459** (2003) 1563-1579.

[10] S. A. Gourley, J. W.-H. So, Extinction and wavefront propagation in a reaction-diffusion model of a structured population with distributed maturation delay, *Proc. Roy. Soc. Edinburgh Sect. A* **133** (2003) 527-548.

[11] S. A. Gourley, J. Wu, Extinction and periodic oscillations in an age-structured population model in a patchy environment, *J. Math. Anal. Appl.* **289** (2004) 431-445.

[12] S. A. Gourley, J. Wu, Delayed non-local diffusive systems in biological invasion and disease spread, *Fields Inst. Commun.* **48** (2006) 137-200.

[13] W. Gurney, S. Blythe, R. Nisbet, Nicholson's blowflies revisited, *Nature* **287** (1980) 17-21.

[14] S.-B. Hsu, X.-Q. Zhao, Spreading speeds and traveling waves for non-monotone integrodifference equations, *SIAM J. Math. Anal.* **40** (2008) 776-789.

[15] Y. Kyrychko, S. A. Gourley, M. V. Bartuccelli, Dynamics of a stage-structured population model on an isolated finite lattice, *SIAM J. Math. Anal.* **37** (2006) 1688-1708.

[16] X. Liang, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.* **60** (2007) 1-40.

[17] S. Ma, Travelling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, *J. Differential Equations* **171** (2001) 294-314.

[18] S. Ma, Traveling waves for non-local delayed diffusion equations via auxiliary equations, *J. Differential Equations* **237** (2007) 259-277.

[19] S. Ma, P. Weng, X. Zou, Asymptotic speed of propagation and traveling wavefronts in a non-local delayed lattice differential equation, *Nonlinear Anal.* **65** (2006) 1858-1890.

[20] S. Ma, J. Wu, Existence, uniqueness and asymptotic stability of traveling wavefronts in a non-local delayed diffusion equation, *J. Dynam. Differential Equations* **19** (2007) 391-436.

[21] S. Ma, X. Zou, Propagation and its failure in a lattice delayed differential equation with global interaction, *J. Differential Equations* **212** (2005) 129-190.

[22] S. Ma, X. Zou, Existence, uniqueness and stability of traveling waves in a discrete reaction-diffusion monostable equation with delay, *J. Differential Equations* **217** (2005) 54-87.

[23] M. Mei, C.-K. Lin, C.-T. Lin, J. W.-H. So, Traveling wavefronts for time-delayed reaction-diffusion equation: (II) Nonlocal nonlinearity, *J. Differential Equations* **247** (2009) 511-529.

[24] J. A. J. Metz, O. Diekmann, The dynamics of physiologically structured populations, Springer, Berlin, 1986.

[25] J. W.-H. So, J. Wu, X. Zou, A reaction-diffusion model for a single species with age structure 1, Traveling wavefronts on unbounded domains, *Proc. Roy. Soc. Lond. A* **457** (2001) 1841-1853.

[26] E. C. Titchmarsh, Introduction to the theory of Fourier integrals, Oxford University Press, 1937.

[27] Z.-C. Wang, W.-T. Li, Dynamics of a non-local delayed reaction-diffusion equation without quasi-monotonicity, *Proc. Roy. Soc. Edinburgh Sect. A* **140** (2010) 1081-1109.

[28] Z.-C. Wang, W.-T. Li, S. Ruan, Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay, *J. Differential Equations* **238** (2007) 153-200.

[29] P. Weng, Extinction, stable pattern and their transition in a diffusive single species population with distributed maturity, *European J. Appl. Math.* **19** (2008) 285-309.

[30] P. Weng, H. Huang, J. Wu, Asymptotic speed of propagation of wave fronts in a lattice delay differential equation with global interaction, *IMA J. Appl. Math.* **68** (2003) 409-439.

[31] P. Weng, J. Wu, Wavefronts for a non-local reaction-diffusion population model with general distributive maturity, *IMA J. Appl. Math.* **73** (2008) 477-495.

[32] P. Weng, J. Wu, H. Huang, J. Ling, Asymptotic speed of propagation of wave fronts in a 2D lattice delay differential equation with global interaction, *Can. Appl. Math. Q.* **11** (2003) 377-414.

[33] J. Wu, X. Zou, Travelling wave fronts of reaction-diffusion systems with delay, *J. Dynam. Differential Equations* **13** (2001) 651-687.

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