Electronic Journal of Qualitative Theory of Differential Equations
2005, No. **25**, 1-6; **http://www.math.u-szeged.hu/ejqtde/**

An existence result of asymptotically stable solutions for an integral equation of mixed type

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Abstract

In the present Note an existence result of asymptotically stable solutions for the integral equation

$$x(t) = q(t) + \int_0^t K(t,s,x(s))\,ds + \int_0^\infty G(t,s,x(s))\,ds$$

is presented.

2000 Mathematics Subject Classification: 47H10, 45D10.
Key words and phrases: fixed points, integral equations.

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1. Introduction

In this Note we will present an existence result of asymptotically stable solutions to the equation

$$x(t) = q(t) + \int_0^t K(t,s,x(s))\,ds + \int_0^\infty G(t,s,x(s))\,ds, \tag{1.1}$$

under hypotheses which will be given in Section 2. We call the integral equation (1.1) to be of mixed type, since within its form an operator of Volterra type and an operator of Uryson type appear. The notion of asymptotically stable solution to the functional equation

$$x = F(x) \tag{1.2}$$

has been recently introduced in [6] and reconsidered in a more general framework in [7].

Let $F : BC \to BC$ be an operator, where $BC := BC\left(\mathbb{R}_+, \mathbb{R}^d\right) = \{x : \mathbb{R}_+ \to \mathbb{R}^d,\ x$ bounded and continuous$\}$, $\mathbb{R}_+ := [0,\infty)$, $d \geq 1$. Let $x \in BC$ be a solution to Eq. (1.2).

Definition 1.1 *The function x is said to be an* **asymptotically stable solution** *of* (1.1) *if for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that for every $t \geq T$ and for every other solution y of* (1.1) *, then*

$$|x(t) - y(t)| \leq \varepsilon, \tag{1.3}$$

where $|\cdot|$ denotes a norm in $\mathbb{R}^d$.

Remark that in [6] the case $d = 1$ is considered, unlike [7] wherein the general case is treated.

In our papers [2]-[4] we studied the existence of asymptotically stable solutions for certain particular cases of Eq. (1.2) , in which integral operators appear. Eq. (1.1) considered in the present Note is more general than those of [2]-[4].

Notice that Definition 1.1 may be stated on other spaces of functions defined on $\mathbb{R}_+$, not necessarily bounded. Since the method used in all the works cited above consists in the application of Schauder's fixed point Theorem, it is enough to suppose Definition 1.1 fulfilled only on the set on which the fixed point theorem is applied.

2. Notations and preliminaries

Let $|\cdot|$ be an arbitrary norm in $\mathbb{R}^d$, $\Delta := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+,\ s \le t\}$. Admit that $q : \mathbb{R}_+ \to \mathbb{R}^d$, $K : \Delta \times \mathbb{R}^d \to \mathbb{R}^d$, $G : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous functions.

The proof of the existence of asymptotically stable solutions to Eq. (1.1) is divided in two steps. First, we show that (1.1) admits solutions and next we prove that there exist solutions fulfilling Definition 1.1. Consider the functional space

$$C_c := \left\{x : \mathbb{R}_+ \to \mathbb{R}^d,\ x \text{ continuous}\right\},$$

equipped with the numerable families of seminorms

$$|x|_n := \sup_{t\in[0,n]} \{|x(t)|\},\ n \ge 1, \tag{2.1}$$

or

$$|x|_{\lambda_n} := \sup_{t\in[0,n]} \left\{|x(t)|\, e^{-\lambda_n t}\right\},\ (\lambda_n > 0)\ n \ge 1. \tag{2.2}$$

Each of these two families determine on C_c a structure of Fréchet space (i.e. a linear, metrisable, and complete space), its topology being the one of the uniform convergence on compact subsets of $\mathbb{R}_+$, for every sequence λ_n. We also mention that a family $\mathcal{A} \subset C_c$ is relatively compact if and only if for each $n \ge 1$, the restrictions to $[0,n]$ of all functions from $\mathcal{A}$ form an equicontinuous and uniformly bounded set.

3. Main result

In this section we will admit the following hypotheses:

(k) there exist continuous functions $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$|K(t,s,x) - K(t,s,y)| \le \alpha(t)\,\beta(s)\,|x-y|,$$

for all $(t,s) \in \Delta$ and all $x, y \in \mathbb{R}^d$;

(g) there exist continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$, with $\int_0^\infty b(t)\,dt < \infty$, such that

$$|G(t,s,x)| \le a(t)\,b(s),$$

for all $(t,s) \in \Delta$ and all $x \in \mathbb{R}^d$.

Lemma 3.1 *Let $z : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function, satisfying the condition*

$$z(t) \le \alpha(t) \int_0^t \beta(s)\,z(s)\,ds + \gamma(t),\ t \in \mathbb{R}_+, \tag{3.1}$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous function. Then, there exists a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$z(t) \le h(t),\ \forall t \in \mathbb{R}_+.$$

Proof. Let us denote

$$w(t) := \int_0^t \beta(s)\,z(s)\,ds. \tag{3.2}$$

Then (3.1) becomes

$$z(t) \le \alpha(t)\,w(t) + \gamma(t)$$

and, since (3.2), we obtain

$$w(0) = 0,\ w'(t) = \beta(t)\,z(t) \le \alpha(t)\,\beta(t)\,w(t) + \beta(t)\,\gamma(t),\ \forall t \in \mathbb{R}_+. \tag{3.3}$$

By (3.3), classical estimates lead us to conclude that

$$z(t) \leq \alpha(t) e^{\int_0^t \alpha(s)\beta(s)ds} \int_0^t \beta(s)\gamma(s) e^{-\int_0^s \alpha(u)\beta(u)du} ds + \gamma(t) =: h(t), \ \forall t \in \mathbb{R}_+. \tag{3.4}$$

$\square$

Definition 3.1 *The operator $H : C_c \to C_c$ is called* **contraction** *if there is a sequence $L_n \in [0,1)$, such that*

$$|Hx - Hy|_{\lambda_n} \leq L_n |x - y|_{\lambda_n}, \ \forall x, y \in C_c, \ \forall n \geq 1. \tag{3.5}$$

Proposition 3.1 *(Banach) Every contraction admits a unique fixed point.*

The proof is classical and follows the proof of the known Banach's Contraction Principle. We remark that the result still holds if (3.5) is fulfilled only on a closed set M, for which $H(M) \subset M$. Finally, notice that Proposition 3.1 is a particular case of a more general result due to Cain & Nashed (see [8]).

Proposition 3.2 *([9]) Let $A, B : C_c \to C_c$ be two operators fulfilling the following hypotheses:*
(i) A is contraction;
(ii) B is compact operator;
(iii) the set $\left\{y = \lambda A\left(\frac{y}{\lambda}\right) + \lambda By, \ y \in C_c, \ \lambda \in (0,1)\right\}$ is bounded.
Then there exists $x \in S$, such that $x = Ax + Bx$.

The result contained in Proposition 3.2 has been obtained in the case of a normed space by Burton & Kirk (see [5]) and it represents the generalization of a known theorem of Krasnoselskii. The result of Burton & Kirk has been extended in [1] in the case of a Fréchet space.

Lemma 3.2 *Admit that hypothesis (k) is fulfilled. Then the equation*

$$x(t) = q(t) + \int_0^t K(t, s, x(s)) ds, \ t \in \mathbb{R}_+ \tag{3.6}$$

admits a unique solution in C_c.

Proof. We define the operator $H : C_c \to C_c$ through

$$(Hx)(t) := q(t) + \int_0^t K(t, s, x(s)) ds, \ x \in C_c, \ t \in \mathbb{R}_+.$$

Let $n \geq 1$ be fixed. Obviously, for $t \in [0, n]$,

$$\begin{aligned} |(Hx)(t) - (Hy)(t)| &\leq \alpha(t) \int_0^t \beta(s) |x(s) - y(s)| ds \\ &\leq L_n e^{\lambda_n t} |x - y|_{\lambda_n}, \end{aligned}$$

where $L_n = (1/\lambda_n) \sup_{(t,s) \in \Delta_n} \{\alpha(t)\beta(s)\}$, $\Delta_n = \{(t,s) \in [0,n] \times [0,n], \ s \leq t\}$, and so

$$|Hx - Hy|_{\lambda_n} \leq L_n |x - y|_{\lambda_n}.$$

If we choose $\lambda_n > \sup_{(t,s) \in \Delta_n} \{\alpha(t)\beta(s)\}$, it follows by Proposition 3.1 that Eq. (3.6) has a unique fixed point. $\square$

In what follows we will denote by $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ the unique solution to (3.6).

The main result of this paper is contained in the following theorem.

Theorem 3.1 *Admit that hypotheses (k) and (g) are fulfilled. Then, Eq.* (1.1) *admits solutions in the set*

$$\mathcal{U} := \{x \in C_c, \ \ |x(t) - \xi(t)| \le h(t), \ \forall t \in \mathbb{R}_+\},$$

where $h(t)$ *is given by* (3.4), *with* $\gamma(t) = a(t)\int_0^\infty b(s)\,ds$.

If, in addition, $\lim_{t\to\infty} h(t) = 0$, *then every solution* $x \in \mathcal{U}$ *to* (1.1) *is asymptotically stable and moreover, for every solution* $x \in \mathcal{U}$ *to* (1.1) *we have*

$$\lim_{t\to\infty} |x(t) - \xi(t)| = 0. \tag{3.7}$$

Proof. For the proof, we will apply Proposition 3.2. To this aim, let us set in (1.1) $x = y + \xi(t)$. Then we write Eq. (1.1) as

$$y(t) = (Ay)(t) + (By)(t), \tag{3.8}$$

where

$$\begin{aligned} (Ay)(t) \ &: = q(t) + \int_0^t K(t,s,y(s)+\xi(s))\,ds - \xi(t), \\ (By)(t) \ &: = \int_0^\infty G(t,s,y(s)+\xi(s))\,ds. \end{aligned}$$

(i) As in the proof of Lemma 3.2 it follows that A is contraction.

(ii) We prove that B is compact operator.

First, since hypothesis (g), the convergence of the integral $\int_0^\infty G(t,s,y(s)+\xi(s))\,ds$ is uniform with respect to t on each compact subset of $\mathbb{R}_+$, and so $(By)(t)$ is a continuous function of t.

Let us consider $\{y_m\}_m \subset C_c$, $y_m \to y$ in C_c, that is, $\forall \varepsilon > 0$, $\forall n \ge 1$, $\exists N = N(\varepsilon, n)$, $\forall m \ge N$, $|y_m - y|_n < \varepsilon$.

Let us fix $n \ge 1$. From the convergence of $\{y_m\}_m$ and the continuity of ξ, there is $r \ge 0$ such that $|y_m + \xi|_n \le r$, $|y + \xi|_n \le r$, $\forall m$.

Consider $\varepsilon > 0$. By hypothesis (g), there is $T > 0$, such that

$$\int_T^\infty b(s)\,ds < \frac{\varepsilon}{3a_n}, \tag{3.9}$$

where $a_n := \sup_{t\in[0,n]}\{a(t)\}$. Since G is uniformly continuous on the set $[0,n]\times[0,T]\times\overline{B(r)}$, $\overline{B(r)} := \left\{x \in \mathbb{R}^d, \ \ |x| \le r\right\}$, it follows that for all $t \in [0,n]$, $s \in [0,T]$, and $m \ge N$,

$$|G(t,s,y_m(s)+\xi(s)) - G(t,s,y(s)+\xi(s))| < \frac{\varepsilon}{3T}.$$

Therefore, for every $t \in [0,n]$ and $m \ge N$, we have

$$\begin{aligned} |(By_m)(t) - (By)(t)| \ &\le \ \int_0^T |G(t,s,y_m(s)+\xi(s)) - G(t,s,y(s)+\xi(s))|\,ds \\ &+2a(t)\int_T^\infty b(s)\,ds < \varepsilon. \end{aligned}$$

Hence,

$$|By_m - By|_n \le \varepsilon, \ \forall m \ge N,$$

and the continuity of B is proved.

Let $\mathcal{S} \subset C_c$ be bounded and $n \ge 1$ be fixed. Then, $\exists p_n > 0$, $\forall x \in \mathcal{S}$, $|x|_n \le p_n$.

Clearly, for all $t \in [0,n]$ and $y \in \mathcal{S}$, we have

$$|(By)(t)| \le a_n \int_0^\infty b(s)\,ds.$$

So, $\left\{ By\mid_{[0,n]},\ y\in\mathcal{S}\right\}$ is uniformly bounded.

Let $\varepsilon>0$ be arbitrarily fixed and $T>0$ given by (3.9).

By hypothesis (g), it follows that $G(t,s,x)$ is uniformly continuous on $[0,n]\times[0,T]\times\overline{B(\rho)}$, where $\rho:=p_{[T]+1}+\xi_n$ and $\xi_n:=\sup_{t\in[0,n]}\{|\xi(t)|\}$.

Hence, there is a $\delta>0$ such that for all $t_1,t_2\in[0,n]$ with $|t_1-t_2|<\delta$ and all $y\in\mathcal{S}$,

$$|G(t_1,s,y(s)+\xi(s))-G(t_2,s,y(s)+\xi(s))|<\varepsilon/(3T).$$

Then it follows that for all $t_1,t_2\in[0,n]$ with $|t_1-t_2|<\delta$ and all $y\in\mathcal{S}$,

$$\begin{aligned}|(By)(t_1)-(By)(t_2)| &\leq \int_0^T |G(t_1,s,y(s)+\xi(s))-G(t_2,s,y(s)+\xi(s))|\,ds\\ &\quad +a(t_1)\int_T^\infty b(s)\,ds+a(t_2)\int_T^\infty b(s)\,ds<\varepsilon.\end{aligned}$$

Hence the set $\left\{ By\mid_{[0,n]},\ y\in\mathcal{S}\right\}$ is equicontinuous.

(iii) Let $y\in C_c$, $y=\lambda A\left(\frac{y}{\lambda}\right)+\lambda By$, $\lambda\in(0,1)$. Due to hypothesis (k),

$$|(Ay)(t)|\leq\int_0^t|K(t,s,y(s)+\xi(s))-K(t,s,\xi(s))|\,ds\leq\alpha(t)\int_0^t\beta(s)\,|y(s)|\,ds.$$

Hence, for all $t\in\mathbb{R}_+$,

$$\begin{aligned}|y(t)| &\leq \lambda\,|(Ay)(t)/\lambda|+\lambda\,|(By)(t)|\\ &\leq \alpha(t)\int_0^t\beta(s)\,|y(s)|\,ds+a(t)\int_0^\infty b(s)\,ds.\end{aligned}$$

By applying Lemma 3.1 with $\gamma(t)=a(t)\int_0^\infty b(s)\,ds$, it follows that $|y(t)|\leq h(t)$, $\forall t\in\mathbb{R}_+$. Hence, $|y|_n\leq|h|_n$, $\forall n\geq1$ and so the set $\left\{y=\lambda A\left(\frac{y}{\lambda}\right)+\lambda By,\ y\in C_c,\ \lambda\in(0,1)\right\}$ is bounded.

Therefore, by applying Proposition 3.2, Eq. (3.8) admits solutions. If y is such a solution, then $y+\xi$ is a solution to (1.1).

Let us suppose that $\lim_{t\to\infty}h(t)=0$. Then for every solution y to (3.8) one has $\lim_{t\to\infty}y(t)=0$ and so for every solution x to (1.1) we have $\lim_{t\to\infty}|x(t)-\xi(t)|=0$.

Now, let $x_1,x_2\in\mathcal{U}$ two solutions to (1.1). It follows immediately that

$$|x_1(t)-x_2(t)|\leq|x_1(t)-\xi(t)|+|x_2(t)-\xi(t)|\leq2h(t),\ \forall t\in\mathbb{R}_+.$$

But, obviously, $\forall\varepsilon>0$, $\exists T=T(\varepsilon)>0$, such that $\forall t>T$, $h(t)<\varepsilon/2$ and the proof of Theorem 3.1 is complete. □

Taking into account that

$$h(t)=\alpha(t)\,e^{\int_0^t\alpha(s)\beta(s)ds}\int_0^t\beta(s)\,\gamma(s)\,e^{-\int_0^s\alpha(u)\beta(u)du}ds+\gamma(t),\ \forall t\geq0,$$

with $\gamma(t)=a(t)\int_0^\infty b(s)\,ds$, then $\lim_{t\to\infty}h(t)=0$, if the following conditions are satisfied:

(α) $\lim_{t\to\infty}\alpha(t)=0$;

(β) $\int_0^\infty\alpha(t)\,\beta(t)\,dt<\infty$;

(a_1) $\lim_{t\to\infty}a(t)=0$;

(a_2) $\int_0^\infty a(s)\,\beta(s)\,ds<\infty$.

Hence we obtain the following corollary.

Corollary 3.1 *If hypotheses (k), (g), (α), (β), (a_1), (a_2) are fulfilled, then every solution $x\in\mathcal{U}$ of Eq. (1.1) is asymptotically stable.*

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(Received November 3, 2005)