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On the uniform boundedness of the solutions of systems of reaction-diffusion equations

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ABSTRACT. We consider a system of reaction-diffusion equations for which the uniform boundedness of the solutions can not be derived by existing methods. The system may represent, in particular, an epidemic model describing the spread of an infection disease within a population. We present an L^p argument allowing to establish the global existence and the uniform boundedness of the solutions of the considered system.

1. INTRODUCTION

In this work we consider the following class of reaction diffusion systems

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = c - f(u,v) - \alpha u & \text{in } \mathbb{R}^+ \times \Omega \\ \frac{\partial v}{\partial t} - d_2 \Delta v = g(u,v) - \sigma v & \text{in } \mathbb{R}^+ \times \Omega \end{cases} \tag{1}$$

with Neuman boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \mathbb{R}^+ \times \Gamma \tag{2}$$

where Ω is an open, bounded domain in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ of class C^1, $d_1,\ d_2, \alpha, \sigma$ are positive constants, $c \geq 0$ and f, g are nonnegative functions of class $C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ satisfying the following assumptions

$$f(0,.) = 0,\ g(.,0) \geq 0, \tag{A1}$$

$$\xi \geq 0, \eta \geq 0 \Longrightarrow 0 \leq f(\xi,\eta) \leq (1+\eta)^\beta \varphi(\xi) \tag{A2}$$

and

$$(\forall \xi, \eta \geq 0)\ g(\xi,\eta) \leq \psi(\eta) f(\xi,\eta) \text{ with } \lim_{\eta \to +\infty} \frac{\psi(\eta)}{\eta} = 0 \tag{A3}$$

where $\beta \geq 1$ and φ, ψ are nonnegative functions of class $C(\mathbb{R}^+)$.

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We are interested in the problem of the existence of global solutions to the system (1)-(2) in the class $C(\bar{\Omega})$ with initial data as well as their uniform boundedness

$$u(0,x) = u_0,\ v(0,x) = v_0 \tag{3}$$

with $u_0, v_0 \in L^\infty(\Omega)$ and $u_0 \geq 0, v_0 \geq 0$.

We observe that the system (1)-(3) generalizes that when $g = f$ since the assumption (A3) obviously holds in this case. In practice, when $g = f$, the system (1)-(3) may represent an epidemic model describing the spread of an infection disease within a population in which case c is the recruitment ratio into the susceptible class, α is the natural death rate and $\frac{1}{\sigma}$ denotes the average infectious period (for details see for example [**12**], [**3**], [**6**],[**8**]). When $c = 0$ and $g(u,v) = f(u,v) + \alpha u + \sigma v$ we get a system of reaction-diffusion equations with balance law initially posed by R. H. Martin for which the problem of the existence of global solutions and their uniform boundedness was studied by many authors. The question was positively answered by Alikakos [**1**] and Masuda [**10**] under the assumption (A2) and later on by Haraux and Youkana [**4**], Youkana [**13**], Barabanova [**2**] and Kanel and Kirane [**9**] when the vector field may be a function with exponential growth.

When $f = g$ and $c > 0$, it is clear that the problem of the uniform boundedness of the solutions of (1)-(3) is not as immediate as the global existence. In [**7**], Hollis, Martin and Pierre used duality arguments on L^p techniques allowing under certain conditions to derive the uniform boundedness of solutions of (1) with mixed boundary conditions from uniform estimates. However we do not see how the method in [**7**] may be applied to the problem (1)-(3). In order to establish the uniform boundedness of the solutions of (1)-(3) in this case (that is when $g = f$ and $c > 0$) one may propose the method developed by Morgan in [**11**]. Indeed the method in [**11**] assures the uniform boundedness of the solutions of (1)-(3) once it is verified that

(i) The vector field of (1) (i.e. the right hand side in (1)) is polynomial in v)

(ii) $c - f(u,v) - \alpha u$,

$$\lambda(c - f(u,v) - \alpha u) + \mu(g(u,v) - \sigma v) \tag{4}$$

for some positive constants $\lambda, \mu > 0$ are uniformly upper bounded and

(iii) $\int_\Omega (u+v) \leq C$ independent of t.

Using standard manipulations, one can check that concerning the system (1)-(3) considered here, the conditions (i)-(iii) are satisfied provided that ψ is bounded. Unfortunately, when ψ is not bounded, the condition (ii) is not satisfied.

In this paper our main concern is to establish the global existence and the uniform boundedness of the solutions of the system (1)-(3) under the assumptions (A1)-(A3). To this end we make use of the Lyapunov function techniques and present an approach similar to that developed by Haraux and Youkana [**4**] and Barabanova [**2**].

2. STATEMENT AND PROOF OF THE MAIN RESULT

It is classical that for nonnegative $u_0, v_0 \in L^\infty(\Omega)$ there exists a unique local nonnegative solution (u, v) of class $C(\bar{\Omega})$ of (1)-(3) on $]0, T^*[$, where T^* is the eventual blowing-up time in $L^\infty(\Omega)$ (see [**5**]). By the comparison principle one may also show that

$$0 \leq u(t,x) \leq \max(||u_0||_\infty, \frac{c}{\alpha}) =: K. \tag{5}$$

As a consequence our problem amounts to establish the uniform boundedness of v. To do so, we will make use of the result established in [**5**] from which the uniform boundedness of v is derived once

$$||g(u,v) - \sigma v||_p \leq C$$

(where C is a nonnegative constant independent of t) for some $p > \frac{n}{2}$. Because of the assumptions (A2) and (A3), we are led to establish the uniform boundedness of the $||v||_p$ on $]0, T^*[$ in order to get that of $||v||_\infty$ on $]0, T^*[$.

For $p \geq 2$, we let

$$\gamma = \frac{(d_1 - d_2)^2}{4 d_1 d_2}, \gamma(p) = \frac{p\gamma + 1}{p - 1},\ M_p = K + \frac{c}{\gamma(p)\sigma}.$$

The main result of this paper is stated in what follows:

PROPOSITION 1. *Assume $p \geq 2$ and let*

$$G_b(t) = \int_\Omega [bu + \exp(-\frac{p-1}{p\gamma+1}\ln(\gamma(p)[M_p - u]))v^p]dt \tag{6}$$

where (u,v) is the solution of (1)-(3) on $]0,T^*[$. Then under the assumptions (A1)-(A3) there exist two positive constants $a > 0$ and $b > 0$ such that

$$\frac{d}{dt}G_b \leq -(p-1)\sigma G_b + a \ . \tag{7}$$

Before proving this theorem we first need the following two technical lemmas:

LEMMA 1. *Let (u,v) be a solution of (1)-(3). Then*

$$\frac{d}{dt}\int_\Omega u dx + \int_\Omega f(u,v)dx + \alpha \int_\Omega u dx = c|\Omega|. \tag{8}$$

PROOF. We integrate the both sides of the equation

$$\frac{du}{dt} - d_1 \Delta u = c - f(u,v) - \alpha u$$

satisfied by u on Ω. We find

$$\int_\Omega f(u,v)dx = c|\Omega| - \alpha \int_\Omega u - \frac{d}{dt}\int_\Omega u(t,x)dx.$$

∎

LEMMA 2. *Assume that $p \geq 2$. Then under the assumptions (A1)-(A3) there exists $b_1 > 0$ such that*

$$[p\eta^{p-1}g(\xi,\eta) - \frac{1}{\gamma(p)M_p}\eta^p f(\xi,\eta)] \leq b_1 f(\xi,\eta) \tag{9}$$

for all $0 \leq \xi \leq K$ and $\eta \geq 0$.

PROOF. According to the assumption (A3), we have

$$[p\eta^{p-1}g(\xi,\eta) - \frac{1}{\gamma(p)M_p}\eta^p f(\xi,\eta)] \leq [p\frac{\psi(\eta)}{\eta} - \frac{1}{\gamma(p)M_p}]\eta^p f(\xi,\eta).$$

Since $\frac{\psi(\eta)}{\eta}$ goes to 0 as $\eta \to +\infty$, there exists $\eta_0 > 0$ such that

$$(0 \leq \xi \leq K, \eta \geq \eta_0) \Longrightarrow [p\frac{\psi(\eta)}{\eta} - \frac{1}{\gamma(p)M_p}]\eta^p f(\xi,\eta) \leq 0$$

On the other hand

$$(\xi,\eta)\longmapsto p\eta^{p-1}\psi(\eta)-\frac{1}{\gamma(p)M_p}\eta^p$$

being continuous is bounded on the compact interval $[0,\eta_0]$ so that (9) immediately follows. ∎

We proceed now to the proof of Proposition 1

PROOF OF PROPOSITION 1. Let

$$h(u)=-\frac{p-1}{p\gamma+1}\ln(\gamma(p)(M_p-u))$$

so that

$$G_b(t)=b\int_\Omega udx+L(t)$$

where

$$L(t)=\int_\Omega e^{h(u)}v^p dx.$$

Differentiating $L(t)$ with respect to t and using the Green formula one obtains

$$\frac{d}{dt}L=H+S$$

where

$$\begin{aligned} H &= -d_1\int_\Omega (h'^2(u)+h''(u))e^{h(u)}v^p(\nabla u)^2dx \\ &\quad -p(d_1+d_2)\int_\Omega h'(u)e^{h(u)}v^{p-1}\nabla u\nabla v dx \\ &\quad -d_2\int_\Omega p(p-1)e^{h(u)}v^{p-2}(\nabla v)^2dx \end{aligned}$$

and

$$\begin{aligned} S &= c\int_\Omega h'(u)e^{h(u)}v^p dx+\int_\Omega [pv^{p-1}g(u,v)-h'(u)v^pf(u,v)]e^{h(u)}dx \\ &\quad -\alpha\int_\Omega h'(u)ue^{h(u)}v^p dx-\sigma\int_\Omega pe^{h(u)}v^p dx. \end{aligned}$$

We observe that H involves a quadratic form with respect to ∇u and ∇v

$$Q=d_1(h^{'2}(u)+h^{''}(u))v^p(\nabla u)^2+p(d_1+d_2)h'(u)v^{p-1}\nabla u\nabla v+d_2p(p-1)v^{p-2}(\nabla v)^2$$

which is nonnegative if

$$(10) \qquad [p(d_1+d_2)h'(u)v^{p-1}]^2 - 4d_1d_2p(p-1)(h'^2(u)+h''(u))v^{2p-2} \le 0.$$

We have chosen $h(u)$ in such a way that

$$h'(u) = \frac{1}{\gamma(p)(M_p-u)}, h''(u) = \frac{\gamma(p)}{[\gamma(p)(M_p-u)]^2}\ .$$

The left hand side of (10) can be written

$$\begin{aligned} & v^{2p-2}\{p^2[(d_1-d_2)^2h'^2(u) - 4d_1d_2h''(u)] + 4d_1d_2p(h'^2(u)+h''(u))\} \\ = \ & 4d_1d_2pv^{2p-2}\{p[\gamma\frac{1}{[\gamma(p)(M_p-u)]^2} - \frac{\gamma(p)}{[\gamma(p)(M_p-u)]^2}] \\ & + \frac{1+\gamma(p)}{[\gamma(p)(M_p-u)]^2}\} \\ = \ & 0 \end{aligned}$$

since $p\gamma - p\gamma(p) + 1 + \gamma(p) = 0$. Therefore (10) holds, $Q \ge 0$ and consequently

$$H = -\int_\Omega Qe^{h(u)}dx \le 0.$$

Concerning the second term S we observe that

$$\begin{aligned} S &\le \int_\Omega (ch'(u) - \sigma p)e^{h(u)}v^p dx + \int_\Omega [pv^{p-1}g(u,v) - h'(u)v^p f(u,v)]e^{h(u)}dx \\ &\le -(p-1)\sigma\int_\Omega e^{h(u)}v^p dx + \int_\Omega [pv^{p-1}g(u,v) - h'(u)v^p f(u,v)]e^{h(u)}dx \end{aligned}$$

since

$$h'(u) = \frac{1}{\gamma(p)(M_p-u)} \le \frac{1}{\gamma(p)(M_p-K)} = \frac{\sigma}{c}$$

On the other hand

$$\begin{aligned} -h'(u) &= -\frac{1}{\gamma(p)(M_p-u)} \le -\frac{1}{\gamma(p)M_p} \\ h(u) &\le -\frac{1}{\gamma(p)}\ln\frac{c}{\sigma}. \end{aligned}$$

Therefore by virtue of Lemma 2 (in particular (9)) and the fact that $v \geq 0$

$$\begin{aligned} pv^{p-1}g(u,v) - h'(u)v^p f(u,v) &\leq pv^{p-1}g(u,v) - \frac{1}{\gamma(p)M_p}v^p f(u,v) \\ &\leq b_1 f(u,v). \end{aligned}$$

As a consequence

$$\begin{aligned} S &\leq -(p-1)\sigma L + b_1 \int_\Omega e^{h(u)} f(u,v)dx \\ &\leq -(p-1)\sigma L + b_1 e^{-\frac{1}{\gamma(p)}\ln\frac{c}{\sigma}} \int_\Omega f(u,v)dx. \end{aligned}$$

Let

$$b = b_1 e^{-\frac{1}{\gamma(p)}\ln\frac{c}{\sigma}}$$

where $b_1 > 0$ is a positive constant satisfying (9). Using (8) one obtains

$$\begin{aligned} S &\leq -(p-1)\sigma L + b(c|\Omega| - \frac{d}{dt}\int_\Omega u(t,x)dx) \\ &\leq -(p-1)\sigma L + bc|\Omega| - b\frac{d}{dt}\int_\Omega u(t,x)dx. \end{aligned}$$

Since $G_b(t) = b\int_\Omega udx + L(t)$, it follows that

$$\begin{aligned} S &\leq -(p-1)\sigma G_b + (p-1)\sigma b\int_\Omega udx + bc|\Omega| - b\frac{d}{dt}\int_\Omega u(t,x)dx \\ &\leq -(p-1)\sigma G_b + b((p-1)\sigma K + c)|\Omega| - b\frac{d}{dt}\int_\Omega u(t,x)dx \end{aligned}$$

from which we conclude that

$$\frac{d}{dt}G_b(t) \leq -(p-1)\sigma G_b(t) + a$$

with $a = b((p-1)\sigma K + c)|\Omega|$. ∎

We are now ready to establish the global existence and uniform boundedness of the solutions of (1)-(3)

THEOREM 1. *Under the assumptions (A1)-(A3), the solutions of (1)-(3) are global and uniformly bounded on* $[0, +\infty[$.

PROOF. By a multiplication of the inequality (7) by $e^{(p-1)\sigma t}$ and then integrating, we deduce that there exists a positive constant $C_1 > 0$ independent of t such that:

$$G_b(t) \leq C_1.$$

Since

$$e^{h(u)} \geq e^{-\frac{1}{\gamma(p)} \ln \gamma(p) M_p}$$

it follows that for all $p \geq 2$

$$\begin{aligned} \int_\Omega v^p dx &\leq e^{\frac{1}{\gamma(p)} \ln[K\gamma(p)+\frac{c}{\sigma}]} G_b(t) \\ &\leq C_1.[K\gamma(p) + \frac{c}{\sigma}]^{\frac{1}{\gamma(p)}} =: C(p). \end{aligned}$$

Select now $p > \frac{n}{2}$ and proceed to bound $||g(u,v) - \sigma v||_p$. Beforehand let

$$A = \max_{0 \leq \eta \leq \eta_0} \psi(\eta), \;\; B = \max_{0 \leq \xi \leq K} \varphi(\xi)$$

where η_0 is such that

$$\eta \geq \eta_0 \Longrightarrow \psi(\eta) \leq \eta$$

According to (A2)-(A3), we have

$$\begin{aligned} g(u,v) &\leq \psi(v) f(u,v) \\ &\leq \psi(v)\varphi(u)(1+v)^\beta \\ &\leq B\psi(v)(1+v)^\beta \end{aligned}$$

since $0 \leq u \leq K$. Therefore

$$\begin{aligned} \int_\Omega g(u,v)^p &\leq B^p \int_\Omega \psi(v)^p (1+v)^{\beta p} \\ &= B^p \Big(\int_{v \leq \eta_0} \psi(v)^p (1+v)^{\beta p} + \int_{v \geq \eta_0} \psi(v)^p (1+v)^{\beta p}\Big) \end{aligned}$$

as a consequence

$$\int_\Omega g(u,v)^p \leq (AB(1+\eta_0)^\beta)^p |\Omega| + B^p \int_\Omega v^p (1+v)^{\beta p}.$$

Using the fact that for $x, y \geq 0$ and $r \geq 1$

$$(x+y)^r \leq 2^{r-1}(x^r + y^r)$$

we obtain

$$\begin{aligned}\int_\Omega g(u,v)^p &\leq (AB(1+\eta_0)^\beta)^p|\Omega| + 2^{\beta p-1}B^p(C(p)+C((\beta+1)p)) \\ &= : R^p.\end{aligned}$$

Finally

$$\begin{aligned}||g(u,v)-\sigma v||_p &\leq ||g(u,v)||_p + \sigma||v||_p \\ &\leq R + \sigma\sqrt[p]{C(p)}.\end{aligned}$$

Using the result of [**5**, p.39, Thm 1.6.1] we conclude that the solutions of (1)-(3) are indeed global and uniformly bounded on $]0,+\infty[\times\Omega$. ∎

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