Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **47**, 1-18; **http://www.math.u-szeged.hu/ejqtde/**

ON A HIGHER-ORDER SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. Here we study the following system of difference equations

$$x_n = f^{-1}\left(\frac{c_n f(x_{n-2k})}{a_n + b_n \prod_{i=1}^{k} g(y_{n-(2i-1)}) f(x_{n-2i})}\right),$$
$$y_n = g^{-1}\left(\frac{\gamma_n g(y_{n-2k})}{\alpha_n + \beta_n \prod_{i=1}^{k} f(x_{n-(2i-1)}) g(y_{n-2i})}\right),$$

$n \in \mathbb{N}_0$, where f and g are increasing real functions such that $f(0) = g(0) = 0$, and coefficients a_n, b_n, c_n, α_n, β_n, γ_n, $n \in \mathbb{N}_0$, and initial values x_{-i}, y_{-i}, $i \in \{1, 2, \ldots, 2k\}$ are real numbers. We show that the system is solvable in closed form, and study asymptotic behavior of its solutions.

1. Introduction

Difference equations and systems of difference equations attract lots of attention (see, e.g. [1]–[49] and references therein). Among numerous topics in this area of mathematics, studying systems of difference equations is one of some recent interest [7, 9, 11, 15, 16, 17, 18, 19, 21, 23, 35, 36, 39, 40, 41, 42, 44, 45, 46, 47, 48], while solving difference equations and applying them in other areas of sciences re-attracted some attention quite recently (see, for example, [1, 2, 6, 7, 22, 28, 29, 32, 33, 35, 36, 37, 39, 40, 42, 43, 44, 45, 46, 47, 48]). Among others, the attention was trigged off by note [28] where an equation is solved in an elegant way. Some old methods for solving difference equations can be found, e.g., in [14].

In [44], S. Stević studied the following system of difference equations

$$x_n = \frac{c_n x_{n-4}}{a_n + b_n y_{n-1} x_{n-2} y_{n-3} x_{n-4}}, \quad y_n = \frac{\gamma_n y_{n-4}}{\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3} y_{n-4}}, \quad n \in \mathbb{N}_0, \quad (1)$$

with real coefficients a_n, b_n, c_n, α_n, β_n, γ_n, $n \in \mathbb{N}_0$, and initial values x_{-i}, y_{-i}, $i \in \{1, 2, 3, 4\}$, such that $c_n \neq 0$, $\gamma_n \neq 0$, $n \in \mathbb{N}_0$. He showed that system (1) is solvable in closed form, and described behavior of all well-defined solutions of the system for constant coefficients a_n, b_n, c_n, α_n, β_n and γ_n. Paper [44] is a natural continuation of his previous investigations in [7, 28, 35, 36, 37, 39, 40, 43, 45, 46, 47, 48], where related difference equations and systems of difference equations were considered.

2010 *Mathematics Subject Classification.* Primary 39A10.
Key words and phrases. Solvable system, system of difference equations, asymptotic behavior.
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Motivated by this line of investigations, here we study the following system of difference equations

$$\begin{aligned} x_n =& f^{-1}\left(\frac{c_n f(x_{n-2k})}{a_n + b_n \prod_{i=1}^{k} g(y_{n-(2i-1)}) f(x_{n-2i})}\right), \\ y_n =& g^{-1}\left(\frac{\gamma_n g(y_{n-2k})}{\alpha_n + \beta_n \prod_{i=1}^{k} f(x_{n-(2i-1)}) g(y_{n-2i})}\right), \quad n \in \mathbb{N}_0, \end{aligned} \tag{2}$$

where f and g are increasing real functions, such that

$$f(0) = g(0) = 0, \tag{3}$$

and coefficients a_n, b_n, c_n, α_n, β_n, γ_n, $n \in \mathbb{N}_0$, and initial values x_{-i}, y_{-i}, $i \in \{1, 2, \dots, 2k\}$ are real numbers.

We show that system (2) is also solvable in closed form, and study the behavior of well-defined solutions of the system when the sequences a_n, b_n, c_n, α_n, β_n and γ_n are constant.

Recall that solution $(x_n, y_n)_{n\ge -2k}$, of system (2) is periodic with period p, if

$$x_{n+p} = x_n \quad \text{and} \quad y_{n+p} = y_n, \quad n \ge -2k.$$

For some results on the periodicity or asymptotic periodicity see, e.g., [4, 5, 10, 11, 12, 13, 14, 23, 24, 26, 27, 31, 34, 38, 41, 49].

2. Solvability of system (2) in closed form

Assume that $x_{-i} \neq 0$, $y_{-i} \neq 0$, $i \in \{1, 2, \dots, 2k\}$. Then (2), the monotonicity of f and g and conditions $f(0) = g(0) = 0$, imply that $x_n \neq 0$ and $y_n \neq 0$, for every $n \in \mathbb{N}_0$. Then in this case the following change of variables along with the invertibility of functions f and g

$$u_n = \frac{1}{\prod_{i=0}^{k-1} f(x_{n-2i}) g(y_{n-2i-1})}, \quad v_n = \frac{1}{\prod_{i=0}^{k-1} g(y_{n-2i}) f(x_{n-2i-1})}, \quad n \ge -1, \tag{4}$$

transforms system (2) into the next system of linear difference equations

$$u_n = \frac{a_n}{c_n} v_{n-1} + \frac{b_n}{c_n}, \quad v_n = \frac{\alpha_n}{\gamma_n} u_{n-1} + \frac{\beta_n}{\gamma_n}, \quad n \in \mathbb{N}_0. \tag{5}$$

From (5) we have that

$$\begin{aligned} u_n &= \frac{a_n \alpha_{n-1}}{c_n \gamma_{n-1}} u_{n-2} + \frac{a_n \beta_{n-1}}{c_n \gamma_{n-1}} + \frac{b_n}{c_n}, \\ v_n &= \frac{\alpha_n a_{n-1}}{\gamma_n c_{n-1}} v_{n-2} + \frac{\alpha_n b_{n-1}}{\gamma_n c_{n-1}} + \frac{\beta_n}{\gamma_n}, \quad n \in \mathbb{N}, \end{aligned}$$

from which we get (for details see [44])

$$u_{2n}=u_0\prod_{j=1}^{n}\frac{a_{2j}\alpha_{2j-1}}{c_{2j}\gamma_{2j-1}}+\sum_{i=1}^{n}\left(\frac{a_{2i}\beta_{2i-1}}{c_{2i}\gamma_{2i-1}}+\frac{b_{2i}}{c_{2i}}\right)\prod_{s=i+1}^{n}\frac{a_{2s}\alpha_{2s-1}}{c_{2s}\gamma_{2s-1}}, \tag{6}$$

$$u_{2n-1}=u_{-1}\prod_{j=1}^{n}\frac{a_{2j-1}\alpha_{2j-2}}{c_{2j-1}\gamma_{2j-2}}+\sum_{i=1}^{n}\left(\frac{a_{2i-1}\beta_{2i-2}}{c_{2i-1}\gamma_{2i-2}}+\frac{b_{2i-1}}{c_{2i-1}}\right)\prod_{s=i+1}^{n}\frac{a_{2s-1}\alpha_{2s-2}}{c_{2s-1}\gamma_{2s-2}}, \tag{7}$$

$$v_{2n}=v_0\prod_{j=1}^{n}\frac{\alpha_{2j}a_{2j-1}}{\gamma_{2j}c_{2j-1}}+\sum_{i=1}^{n}\left(\frac{\alpha_{2i}b_{2i-1}}{\gamma_{2i}c_{2i-1}}+\frac{\beta_{2i}}{\gamma_{2i}}\right)\prod_{s=i+1}^{n}\frac{\alpha_{2s}a_{2s-1}}{\gamma_{2s}c_{2s-1}}, \tag{8}$$

$$v_{2n-1}=v_{-1}\prod_{j=1}^{n}\frac{\alpha_{2j-1}a_{2j-2}}{\gamma_{2j-1}c_{2j-2}}+\sum_{i=1}^{n}\left(\frac{\alpha_{2i-1}b_{2i-2}}{\gamma_{2i-1}c_{2i-2}}+\frac{\beta_{2i-1}}{\gamma_{2i-1}}\right)\prod_{s=i+1}^{n}\frac{\alpha_{2s-1}a_{2s-2}}{\gamma_{2s-1}c_{2s-2}}. \tag{9}$$

From (4) we have that

$$f(x_{2km+i})=\frac{v_{2km+i-1}}{u_{2km+i}}f(x_{2k(m-1)+i}),\quad i\in\{0,\dots,2k-1\},$$

$m\in\mathbb{N}_0$, and

$$g(y_{2km+i})=\frac{u_{2km+i-1}}{v_{2km+i}}g(y_{2k(m-1)+i}),\quad i\in\{0,\dots,2k-1\},$$

for $2km+i\ge 0$, from which along with the invertibility of functions f and g it follows that for every $m\in\mathbb{N}_0$ and each $i\in\{0,\dots,2k-1\}$

$$x_{2km+i}=f^{-1}\left(f(x_i)\prod_{j=1}^{m}\frac{v_{2kj+i-1}}{u_{2kj+i}}\right), \tag{10}$$

$$y_{2km+i}=g^{-1}\left(g(y_i)\prod_{j=1}^{m}\frac{u_{2kj+i-1}}{v_{2kj+i}}\right). \tag{11}$$

Using (6)–(9) in (10) and (11) we get solutions of system (2) in closed form.

3. System (2) with constant coefficients

Let

$$a_n=\hat a,\quad b_n=\hat b,\quad c_n=\hat c,\quad \alpha_n=\hat\alpha,\quad \beta_n=\hat\beta\quad\text{and}\quad\gamma_n=\hat\gamma,\quad n\in\mathbb{N}_0,$$

then we have

$$\begin{aligned}x_n&=f^{-1}\left(\frac{\hat c f(x_{n-2k})}{\hat a+\hat b\prod_{i=1}^{k}g(y_{n-(2i-1)})f(x_{n-2i})}\right),\\ y_n&=g^{-1}\left(\frac{\hat\gamma g(y_{n-2k})}{\hat\alpha+\hat\beta\prod_{i=1}^{k}f(x_{n-(2i-1)})g(y_{n-2i})}\right),\quad n\in\mathbb{N}_0.\end{aligned} \tag{12}$$

If $\hat c=0$, then since $f(0)=f^{-1}(0)=0$ we have that $x_n=0$, $n\in\mathbb{N}_0$, so that $g(y_n)=\frac{\hat\gamma}{\hat\alpha}g(y_{n-2k})$ for $n\in\mathbb{N}$ and consequently

$$y_{2km-i}=g^{-1}\left(\left(\frac{\hat\gamma}{\hat\alpha}\right)^{m}g(y_{-i})\right),$$

for every $m \in \mathbb{N}_0$ and $i \in \{0, 1, \ldots, 2k-1\}$.

If $\hat{\gamma} = 0$, then since $g(0) = g^{-1}(0) = 0$ we have that $y_n = 0$, $n \in \mathbb{N}_0$, implying $f(x_n) = \frac{\hat{c}}{\hat{a}} f(x_{n-2k})$ for $n \in \mathbb{N}$ and consequently

$$x_{2km-i} = f^{-1}\left(\left(\frac{\hat{c}}{\hat{a}}\right)^m f(x_{-i})\right),$$

for every $m \in \mathbb{N}_0$ and $i \in \{0, 1, \ldots, 2k-1\}$.

From now on we will assume that $\hat{c} \neq 0$ and $\hat{\gamma} \neq 0$. Note that in this case, system (12) can be written in the following form

$$\begin{aligned} x_n &= f^{-1}\left(\frac{f(x_{n-2k})}{a + b\prod_{i=1}^{k} g(y_{n-(2i-1)}) f(x_{n-2i})}\right), \\ y_n &= g^{-1}\left(\frac{g(y_{n-2k})}{\alpha + \beta\prod_{i=1}^{k} f(x_{n-(2i-1)}) g(y_{n-2i})}\right), \quad n \in \mathbb{N}_0, \end{aligned} \tag{13}$$

where $a = \hat{a}/\hat{c}$, $b = \hat{b}/\hat{c}$, $\alpha = \hat{\alpha}/\hat{\gamma}$ and $\beta = \hat{\beta}/\hat{\gamma}$. Therefore we will study system (13) instead of system (12).

Assume $x_{-i} \neq 0$ and $y_{-i} \neq 0$ for every $i \in \{1, 2, \ldots, 2k\}$. System (5) becomes

$$u_n = a v_{n-1} + b, \quad v_n = \alpha u_{n-1} + \beta, \quad n \in \mathbb{N}_0, \tag{14}$$

which implies that

$$u_n = a\alpha u_{n-2} + a\beta + b, \tag{15}$$

$$v_n = a\alpha v_{n-2} + \alpha b + \beta, \quad n \in \mathbb{N}. \tag{16}$$

From (15) and (16) (or (6)–(9)) we obtain

$$\begin{aligned} u_{2n-l} &= u_{-l}(a\alpha)^n + (a\beta + b)\frac{1 - (a\alpha)^n}{1 - a\alpha} \\ &= \frac{a\beta + b + (a\alpha)^n (u_{-l}(1 - a\alpha) - a\beta - b)}{1 - a\alpha}, \end{aligned} \tag{17}$$

$n \in \mathbb{N}_0$, $l \in \{0, 1\}$, if $a\alpha \neq 1$, or

$$u_{2n-l} = u_{-l} + (a\beta + b)n, \quad n \in \mathbb{N}_0,\ l \in \{0, 1\}, \tag{18}$$

if $a\alpha = 1$, and

$$\begin{aligned} v_{2n-l} &= v_{-l}(a\alpha)^n + (\alpha b + \beta)\frac{1 - (a\alpha)^n}{1 - a\alpha} \\ &= \frac{\alpha b + \beta + (a\alpha)^n (v_{-l}(1 - a\alpha) - \alpha b - \beta)}{1 - a\alpha}, \end{aligned} \tag{19}$$

$n \in \mathbb{N}_0$, $l \in \{0, 1\}$, if $a\alpha \neq 1$, or

$$v_{2n-l} = v_{-l} + (\alpha b + \beta)n, \quad n \in \mathbb{N}_0,\ l \in \{0, 1\}, \tag{20}$$

if $a\alpha = 1$.

From relations (17)–(20) we easily obtain the following formulae for solutions of system (13).

Case $a\alpha = 1$. In this case we have that

$$x_{2km+2s} = f^{-1}\left(f(x_{2s})\prod_{j=1}^{m}\frac{v_{2kj+2s-1}}{u_{2kj+2s}}\right)$$
$$= f^{-1}\left(f(x_{2s})\prod_{j=1}^{m}\frac{v_{-1}+(\alpha b+\beta)(kj+s)}{u_0+(a\beta+b)(kj+s)}\right), \quad (21)$$

$$x_{2km+2s+1} = f^{-1}\left(f(x_{2s+1})\prod_{j=1}^{m}\frac{v_{2kj+2s}}{u_{2kj+2s+1}}\right)$$
$$= f^{-1}\left(f(x_{2s+1})\prod_{j=1}^{m}\frac{v_0+(\alpha b+\beta)(kj+s)}{u_{-1}+(a\beta+b)(kj+s+1)}\right), \quad (22)$$

$$y_{2km+2s} = g^{-1}\left(g(y_{2s})\prod_{j=1}^{m}\frac{u_{2kj+2s-1}}{v_{2kj+2s}}\right)$$
$$= g^{-1}\left(g(y_{2s})\prod_{j=1}^{m}\frac{u_{-1}+(a\beta+b)(kj+s)}{v_0+(\alpha b+\beta)(kj+s)}\right), \quad (23)$$

$$y_{2km+2s+1} = g^{-1}\left(g(y_{2s+1})\prod_{j=1}^{m}\frac{u_{2kj+2s}}{v_{2kj+2s+1}}\right)$$
$$= g^{-1}\left(g(y_{2s+1})\prod_{j=1}^{m}\frac{u_0+(a\beta+b)(kj+s)}{v_{-1}+(\alpha b+\beta)(kj+s+1)}\right), \quad (24)$$

for every $m \in \mathbb{N}_0$ and $s \in \{0, 1, \ldots, k-1\}$.

Case $a\alpha \neq 1$. We have

$$x_{2km+2s} = f^{-1}\left(f(x_{2s})\prod_{j=1}^{m}\frac{v_{2kj+2s-1}}{u_{2kj+2s}}\right)$$
$$= f^{-1}\left(f(x_{2s})\prod_{j=1}^{m}\frac{(\alpha b+\beta+(a\alpha)^{kj+s}(v_{-1}(1-a\alpha)-\alpha b-\beta))}{(a\beta+b+(a\alpha)^{kj+s}(u_0(1-a\alpha)-a\beta-b))}\right), \quad (25)$$

$$x_{2km+2s+1} = f^{-1}\left(f(x_{2s+1})\prod_{j=1}^{m}\frac{v_{2kj+2s}}{u_{2kj+2s+1}}\right)$$
$$= f^{-1}\left(f(x_{2s+1})\prod_{j=1}^{m}\frac{(\alpha b+\beta+(a\alpha)^{kj+s}(v_0(1-a\alpha)-\alpha b-\beta))}{(a\beta+b+(a\alpha)^{kj+s+1}(u_{-1}(1-a\alpha)-a\beta-b))}\right), \quad (26)$$

$$y_{2km+2s} = g^{-1}\left(g(y_{2s})\prod_{j=1}^{m}\frac{u_{2kj+2s-1}}{v_{2kj+2s}}\right)$$
$$= g^{-1}\left(g(y_{2s})\prod_{j=1}^{m}\frac{(a\beta+b+(a\alpha)^{kj+s}(u_{-1}(1-a\alpha)-a\beta-b))}{(\alpha b+\beta+(a\alpha)^{kj+s}(v_0(1-a\alpha)-\alpha b-\beta))}\right), \quad (27)$$
$$y_{2km+2s+1} = g^{-1}\left(g(y_{2s+1})\prod_{j=1}^{m}\frac{u_{2kj+2s}}{v_{2kj+2s+1}}\right)$$
$$= g^{-1}\left(g(y_{2s+1})\prod_{j=1}^{m}\frac{(a\beta+b+(a\alpha)^{kj+s}(u_0(1-a\alpha)-a\beta-b))}{(\alpha b+\beta+(a\alpha)^{kj+s+1}(v_{-1}(1-a\alpha)-\alpha b-\beta))}\right), \quad (28)$$

for every $m \in \mathbb{N}_0$ and $s \in \{0, 1, \ldots, k-1\}$.

4. Behavior of solutions of system (13)

Prior to proving the main results on behavior of solutions of system (13) we present the following extension of Lemma 1 in [44] which guarantees the existence of $2k$ and $4k$ periodic solutions of system (13).

Lemma 1. *Assume that $a\alpha \neq 1$, $f, g : \mathbb{R} \to \mathbb{R}$ are increasing functions satisfying the conditions in* (3)*. Then the following statements are true.*

(a) If $\alpha b + \beta = a\beta + b$, then system (13) *has $2k$-periodic solutions.*

(b) If $\alpha b + \beta = -(a\beta + b)$, and f and g are odd, then system (13) *has $4k$-periodic solutions.*

Proof. It is easy to see that system (14) has a unique equilibrium solution
$$u_n = \bar{u} = \frac{a\beta + b}{1 - a\alpha} \neq 0, \quad v_n = \bar{v} = \frac{\alpha b + \beta}{1 - a\alpha} \neq 0, \quad n \geq -1.$$

This along with (4) implies that
$$f(x_n) = \frac{1 - a\alpha}{(a\beta + b)g(y_{n-2k+1})\prod_{j=1}^{k-1} g(y_{n-2j+1})f(x_{n-2j})}$$
$$= \frac{1 - a\alpha}{a\beta + b}v_{n-1}f(x_{n-2k}) = \frac{\alpha b + \beta}{a\beta + b}f(x_{n-2k}), \qquad n \in \mathbb{N}_0, \quad (29)$$

and
$$g(y_n) = \frac{1 - a\alpha}{(\alpha b + \beta)f(x_{n-2k+1})\prod_{j=1}^{k-1} f(x_{n-2j+1})g(y_{n-2j})}$$
$$= \frac{1 - a\alpha}{\alpha b + \beta}u_{n-1}g(y_{n-2k}) = \frac{a\beta + b}{\alpha b + \beta}g(y_{n-2k}), \qquad n \in \mathbb{N}_0. \quad (30)$$

(a) Since $\alpha b + \beta = a\beta + b$, from (29) and (30) we get $f(x_n) = f(x_{n-2k})$ and $g(y_n) = g(y_{n-2k})$, from which it follows that $x_n = x_{n-2k}$ and $y_n = y_{n-2k}$ that is, there is a $2k$-periodic solution of system (13).

(b) Since $\alpha b+\beta = -(a\beta+b)$, from (29), (30), and since f and g are odd functions, we get $f(x_n) = -f(x_{n-2k}) = f(-x_{n-2k})$ and $g(y_n) = -g(y_{n-2k}) = g(-y_{n-2k})$

which implies that $x_n = -x_{n-2k}$ and $y_n = -y_{n-2k}$, and consequently $x_n = x_{n-4k}$ and $y_n = y_{n-4k}$, that is, there is a $4k$-periodic solution of system (13). □

Theorem 1. *Assume that $a\alpha = 1$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, odd, increasing functions satisfying the conditions in (3), and $(x_n, y_n)_{n\geq -2k}$ is a well-defined solution of system (13) such that $x_{-i} \neq 0 \neq y_{-i}$, $i = 1, \ldots, 2k$. Then the following statements are true.*

(a) If $|\alpha b + \beta| < |a\beta + b|$, then $x_n \to 0$ and $|y_n| \to g^{-1}(+\infty)$, as $n \to \infty$.
(b) If $|\alpha b + \beta| > |a\beta + b|$, then $y_n \to 0$ and $|x_n| \to f^{-1}(+\infty)$, as $n \to \infty$.
(c) If $\alpha b + \beta = a\beta + b \neq 0$ and $\frac{v_{-1}-u_0}{\alpha b+\beta} > 0$, then $|x_{2km+2s}| \to f^{-1}(+\infty)$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(d) If $\alpha b+\beta = a\beta+b \neq 0$ and $\frac{v_{-1}-u_0}{\alpha b+\beta} < 0$, then $|x_{2km+2s}| \to 0$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(e) If $\alpha b + \beta = a\beta + b \neq 0$ and $v_{-1} = u_0$, then the sequences x_{2km+2s}, $s \in \{0, 1, \ldots, k-1\}$, are convergent.
(f) If $\alpha b + \beta = a\beta + b \neq 0$ and $\frac{v_0-u_{-1}}{\alpha b+\beta} > 1$, then $|x_{2km+2s+1}| \to f^{-1}(+\infty)$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(g) If $\alpha b+\beta = a\beta+b \neq 0$ and $\frac{v_0-u_{-1}}{\alpha b+\beta} < 1$, then $|x_{2km+2s+1}| \to 0$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(h) If $\alpha b + \beta = a\beta + b \neq 0$ and $v_0 = u_{-1} + \alpha b + \beta$, then the sequences $x_{2km+2s+1}$, $s \in \{0, 1, \ldots, k-1\}$, are convergent.
(i) If $\alpha b + \beta = a\beta + b \neq 0$ and $\frac{u_{-1}-v_0}{\alpha b+\beta} > 0$, then $|y_{2km+2s}| \to g^{-1}(+\infty)$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(j) If $\alpha b+\beta = a\beta+b \neq 0$ and $\frac{u_{-1}-v_0}{\alpha b+\beta} < 0$, then $y_{2km+2s} \to 0$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(k) If $\alpha b + \beta = a\beta + b \neq 0$ and $u_{-1} = v_0$, then the sequences y_{2km+2s}, $s \in \{0, 1, \ldots, k-1\}$, are convergent.
(l) If $\alpha b + \beta = a\beta + b \neq 0$ and $\frac{u_0-v_{-1}}{\alpha b+\beta} > 1$, then $|y_{2km+2s+1}| \to g^{-1}(+\infty)$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(m) If $\alpha b + \beta = a\beta + b \neq 0$ and $\frac{u_0-v_{-1}}{\alpha b+\beta} < 1$, then $y_{2km+2s+1} \to 0$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(n) If $\alpha b + \beta = a\beta + b \neq 0$ and $u_0 = v_{-1} + \alpha b + \beta$, then the sequences $y_{2km+2s+1}$, $s \in \{0, 1, \ldots, k-1\}$, are convergent.
(o) If $\alpha b + \beta = -(a\beta + b) \neq 0$ and $\frac{v_{-1}+u_0}{\alpha b+\beta} > 0$, then $|x_{2km+2s}| \to f^{-1}(+\infty)$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(p) If $\alpha b+\beta = -(a\beta+b) \neq 0$ and $\frac{v_{-1}+u_0}{\alpha b+\beta} < 0$, then $|x_{2km+2s}| \to 0$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(q) If $\alpha b + \beta = -(a\beta + b) \neq 0$ and $v_{-1} = -u_0$, then the sequences x_{4km+2s} and $x_{4km+2k+2s}$, $s \in \{0, 1, \ldots, k-1\}$, are convergent.
(r) If $\alpha b + \beta = -(a\beta + b) \neq 0$ and $\frac{v_0+u_{-1}}{\alpha b+\beta} > 1$, then $|x_{2km+2s+1}| \to f^{-1}(+\infty)$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.
(s) If $\alpha b + \beta = -(a\beta + b) \neq 0$ and $\frac{v_0+u_{-1}}{\alpha b+\beta} < 1$, then $|x_{2km+2s+1}| \to 0$, $s \in \{0, 1, \ldots, k-1\}$, as $m \to \infty$.

(t) *If* $\alpha b+\beta=-(a\beta+b)\neq 0$ *and* $v_0+u_{-1}=\alpha b+\beta$*, then the sequences* $x_{4km+2s+1}$ *and* $x_{4km+2k+2s+1}$*,* $s\in\{0,1,\ldots,k-1\}$*, are convergent.*

(u) *If* $\alpha b+\beta=-(a\beta+b)\neq 0$ *and* $\frac{u_{-1}+v_0}{\alpha b+\beta}<0$*, then* $|y_{2km+2s}|\to g^{-1}(+\infty)$*,* $s\in\{0,1,\ldots,k-1\}$*, as* $m\to\infty$*.*

(v) *If* $\alpha b+\beta=-(a\beta+b)\neq 0$ *and* $\frac{u_{-1}+v_0}{\alpha b+\beta}>0$*, then* $y_{2km+2s}\to 0$*,* $s\in\{0,1,\ldots,k-1\}$*, as* $m\to\infty$*.*

(w) *If* $\alpha b+\beta=-(a\beta+b)\neq 0$ *and* $u_{-1}=-v_0$*, then the sequences* y_{4km+2s} *and* $y_{4km+2k+2s}$*,* $s\in\{0,1,\ldots,k-1\}$*, are convergent.*

(x) *If* $\alpha b+\beta=-(a\beta+b)\neq 0$ *and* $\frac{u_0+v_{-1}+\alpha b+\beta}{\alpha b+\beta}<0$*, then* $|y_{2km+2s+1}|\to g^{-1}(+\infty)$*,* $s\in\{0,1,\ldots,k-1\}$*, as* $m\to\infty$*.*

(y) *If* $\alpha b+\beta=-(a\beta+b)\neq 0$ *and* $\frac{u_0+v_{-1}+\alpha b+\beta}{\alpha b+\beta}>0$*, then* $y_{2km+2s+1}\to 0$*,* $s\in\{0,1,\ldots,k-1\}$*, as* $m\to\infty$*.*

(z) *If* $\alpha b+\beta=-(a\beta+b)\neq 0$ *and* $u_0+v_{-1}+\alpha b+\beta=0$*, then the sequences* $y_{4km+2s+1}$ *and* $y_{4km+2k+2s+1}$*,* $s\in\{0,1,\ldots,k-1\}$*, are convergent.*

Proof. (a), (b) We have

$$\lim_{m\to\infty}\frac{v_{-1}+(\alpha b+\beta)(km+s)}{u_0+(a\beta+b)(km+s)}=\lim_{m\to\infty}\frac{v_0+(\alpha b+\beta)(km+s)}{u_{-1}+(a\beta+b)(km+s+1)}=\frac{\alpha b+\beta}{a\beta+b},$$

$$\lim_{m\to\infty}\frac{u_{-1}+(a\beta+b)(km+s)}{v_0+(\alpha b+\beta)(km+s)}=\lim_{m\to\infty}\frac{u_0+(a\beta+b)(km+s)}{v_{-1}+(\alpha b+\beta)(km+s+1)}=\frac{a\beta+b}{\alpha b+\beta}.$$

From these limits, formulae (21)–(24) and the continuity of functions f and g these two statements follow.

(c)–(n) By some calculations, and using the next known formulas

$$\ln(1+x)=x-x^2/2+O(x^3)\quad\text{and}\quad(1+x)^{-1}=1-x+O(x^2),\ x\to 0\tag{31}$$

(which we may assume that hold for all the terms in products (21)–(24)), we get

$$\begin{aligned}x_{2km+2s}&=f^{-1}\left(f(x_{2s})\prod_{j=1}^{m}\frac{\left(1+\frac{(\alpha b+\beta)s+v_{-1}}{kj(\alpha b+\beta)}\right)}{\left(1+\frac{u_0+(a\beta+b)s}{kj(a\beta+b)}\right)}\right)\\&=f^{-1}\left(f(x_{2s})\prod_{j=1}^{m}\left(1+\frac{v_{-1}-u_0}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\\&=f^{-1}\left(f(x_{2s})\exp\left(\sum_{j=1}^{m}\left(\frac{v_{-1}-u_0}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\right),\end{aligned}\tag{32}$$

$$
\begin{aligned}
f(x_{2km+2s+1}) &= f^{-1}\left(f(x_{2s+1})\prod_{j=1}^{m}\frac{\left(1+\frac{v_0+(\alpha b+\beta)s}{kj(\alpha b+\beta)}\right)}{\left(1+\frac{u_{-1}+(a\beta+b)(s+1)}{kj(a\beta+b)}\right)}\right)\\
&= f^{-1}\left(f(x_{2s+1})\prod_{j=1}^{m}\left(1+\frac{v_0-u_{-1}-(\alpha b+\beta)}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\\
&= f^{-1}\left(f(x_{2s+1})\exp\left(\sum_{j=1}^{m}\left(\frac{v_0-u_{-1}-(\alpha b+\beta)}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\right),
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
y_{2km+2s} &= g^{-1}\left(g(y_{2s})\prod_{j=1}^{m}\frac{\left(1+\frac{u_{-1}+(a\beta+b)s}{kj(a\beta+b)}\right)}{\left(1+\frac{v_0+(\alpha b+\beta)s}{kj(\alpha b+\beta)}\right)}\right)\\
&= g^{-1}\left(g(y_{2s})\prod_{j=1}^{m}\left(1+\frac{u_{-1}-v_0}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\\
&= g^{-1}\left(g(y_{2s})\exp\left(\sum_{j=1}^{m}\left(\frac{u_{-1}-v_0}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\right),
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
y_{2km+2s+1} &= g^{-1}\left(g(y_{2s+1})\prod_{j=1}^{m}\frac{\left(1+\frac{u_0+(a\beta+b)s}{kj(a\beta+b)}\right)}{\left(1+\frac{v_{-1}+(\alpha b+\beta)(s+1)}{kj(\alpha b+\beta)}\right)}\right)\\
&= g^{-1}\left(g(y_{2s+1})\prod_{j=1}^{m}\left(1+\frac{u_0-v_{-1}-(\alpha b+\beta)}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\\
&= g^{-1}\left(g(y_{2s+1})\exp\left(\sum_{j=1}^{m}\left(\frac{u_0-v_{-1}-(\alpha b+\beta)}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\right),
\end{aligned}
\tag{35}
$$

for every $s \in \{0,1,2,\dots,k-1\}$.

Using (32)–(35), the relations

$$
\sum_{j=1}^{\infty}\frac{1}{j}=+\infty \quad \text{and} \quad \sum_{j=1}^{+\infty}\left|O\left(\frac{1}{j^2}\right)\right|<+\infty, \tag{36}
$$

and the continuity of the functions f and g, these results easily follow.

(o)–(z) By some calculations and (31) (which we may also assume that hold for all the terms in products (21)–(24)), we get

$$x_{2km+2s} = f^{-1}\left(f(x_{2s})(-1)^m \prod_{j=1}^{m} \frac{\left(1+\frac{(\alpha b+\beta)s+v_{-1}}{kj(\alpha b+\beta)}\right)}{\left(1+\frac{(\alpha b+\beta)s-u_0}{kj(\alpha\beta+b)}\right)}\right)$$

$$= f^{-1}\left(f(x_{2s})(-1)^m \prod_{j=1}^{m}\left(1+\frac{v_{-1}+u_0}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)$$

$$= (-1)^m f^{-1}\left(f(x_{2s})\exp\left(\sum_{j=1}^{m}\left(\frac{v_{-1}+u_0}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\right), \quad (37)$$

$$x_{2km+2s+1} = f^{-1}\left(f(x_{2s+1})(-1)^m \prod_{j=1}^{m} \frac{\left(1+\frac{v_0+(\alpha b+\beta)s}{kj(\alpha b+\beta)}\right)}{\left(1+\frac{(\alpha b+\beta)(s+1)-u_{-1}}{kj(\alpha b+\beta)}\right)}\right)$$

$$= f^{-1}\left(f(x_{2s+1})(-1)^m \prod_{j=1}^{m}\left(1+\frac{v_0+u_{-1}-(\alpha b+\beta)}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)$$

$$= (-1)^m f^{-1}\left(f(x_{2s+1})\exp\left(\sum_{j=1}^{m}\left(\frac{v_0+u_{-1}-(\alpha b+\beta)}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\right), \quad (38)$$

$$y_{2km+2s} = g^{-1}\left(g(y_{2s})(-1)^m \prod_{j=1}^{m} \frac{\left(1+\frac{(\alpha b+\beta)s-u_{-1}}{kj(\alpha b+\beta)}\right)}{\left(1+\frac{v_0+(\alpha b+\beta)s}{kj(\alpha b+\beta)}\right)}\right)$$

$$= g^{-1}\left(g(y_{2s})(-1)^m \prod_{j=1}^{m}\left(1-\frac{u_{-1}+v_0}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)$$

$$= (-1)^m g^{-1}\left(g(y_{2s})\exp\left(-\sum_{j=1}^{m}\left(\frac{u_{-1}+v_0}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\right), \quad (39)$$

$$y_{2km+2s+1} = g^{-1}\left(g(y_{2s+1})(-1)^m \prod_{j=1}^{m} \frac{\left(1+\frac{(\alpha b+\beta)s-u_0}{kj(\alpha b+\beta)}\right)}{\left(1+\frac{v_{-1}+(\alpha b+\beta)(s+1)}{kj(\alpha b+\beta)}\right)}\right)$$

$$= g^{-1}\left(g(y_{2s+1})(-1)^m \prod_{j=1}^{m}\left(1-\frac{u_0+v_{-1}+\alpha b+\beta}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)$$

$$= (-1)^m g^{-1}\left(g(y_{2s+1})\exp\left(-\sum_{j=1}^{m}\left(\frac{u_0+v_{-1}+\alpha b+\beta}{kj(\alpha b+\beta)}+O\left(\frac{1}{j^2}\right)\right)\right)\right), \quad (40)$$

for every $s \in \{0, 1, 2, \ldots, k-1\}$.

Using (37)–(40), relations (36) and the continuity of the functions f and g, the results easily follow. □

Theorem 2. *Assume that $a\alpha \neq 1$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, odd, increasing functions satisfying the conditions in* (3), *and $(x_n, y_n)_{n\geq -2k}$ is a well-defined solution of system* (13) *such that $x_{-i} \neq 0 \neq y_{-i}$, $i = 1, \dots, 2k$. Then the following statements are true.*

(a) If $|a\alpha| > 1$, $|v_{-1}(1 - a\alpha) - \alpha b - \beta| < |u_0(1 - a\alpha) - a\beta - b|$, then $x_{2km+2s} \to 0$, $s \in \{0, 1, \dots, k-1\}$ as $m \to \infty$.

(b) If $|a\alpha| > 1$, $|v_{-1}(1 - a\alpha) - \alpha b - \beta| > |u_0(1 - a\alpha) - a\beta - b|$, then $|x_{2km+2s}| \to f^{-1}(+\infty)$, $s \in \{0, 1, \dots, k-1\}$ as $m \to \infty$.

(c) If $|a\alpha| > 1$, $v_{-1}(1-a\alpha)-\alpha b-\beta = u_0(1-a\alpha)-a\beta-b \neq 0$, then the sequences x_{2km+2s}, $s \in \{0, 1, \dots, k-1\}$ are convergent.

(d) If $|a\alpha| > 1$, $v_{-1}(1 - a\alpha) - \alpha b - \beta = -(u_0(1 - a\alpha) - a\beta - b) \neq 0$, then the sequences x_{4km+2s} and $x_{4km+2k+2s}$, $s \in \{0, 1, \dots, k-1\}$ are convergent.

(e) If $|a\alpha| < 1$ and $|\alpha b + \beta| < |a\beta + b|$, then $x_{2km+2s} \to 0$, $s \in \{0, 1, \dots, k-1\}$ as $m \to \infty$.

(f) If $|a\alpha| < 1$ and $|\alpha b+\beta| > |a\beta+b|$, then $|x_{2km+2s}| \to f^{-1}(+\infty)$, $s \in \{0, 1, \dots, k-1\}$ as $m \to \infty$.

(g) If $|a\alpha| < 1$ and $\alpha b+\beta = a\beta+b$, then the sequences x_{2km+2s}, $s \in \{0, 1, \dots, k-1\}$ are convergent.

(h) If $|a\alpha| < 1$ and $\alpha b+\beta = -(a\beta+b)$, then the sequences x_{4km+2s} and $x_{4km+2k+2s}$, $s \in \{0, 1, \dots, k-1\}$ are convergent.

(i) If $a\alpha = -1$, then

$$x_{2km+2s} = f^{-1}\left(f(x_{2s})\prod_{j=1}^{m}\left(\frac{\alpha b + \beta + (-1)^{kj+s}(2v_{-1} - \alpha b - \beta)}{a\beta + b + (-1)^{kj+s}(2u_0 - a\beta - b)}\right)\right). \tag{41}$$

Proof. Let

$$p_m^s := \frac{\alpha b + \beta + (a\alpha)^{km+s}(v_{-1}(1 - a\alpha) - \alpha b - \beta)}{a\beta + b + (a\alpha)^{km+s}(u_0(1 - a\alpha) - a\beta - b)}, \quad m \in \mathbb{N}_0,\ s \in \{0, 1, \dots, k-1\}.$$

(a) Note that in this case

$$\lim_{m\to\infty} |p_m^s| = \frac{|v_{-1}(1 - a\alpha) - \alpha b - \beta|}{|u_0(1 - a\alpha) - a\beta - b|} < 1,$$

which along with formula (25), and the continuity of function f, easily implies the result.

(b) In this case

$$\lim_{m\to\infty} |p_m^s| = \frac{|v_{-1}(1 - a\alpha) - \alpha b - \beta|}{|u_0(1 - a\alpha) - a\beta - b|} > 1,$$

from which, (25) and the continuity of function f, the result easily follows.

(c) Using (31) we have that for sufficiently large m

$$p_m^s = \frac{1+\frac{\alpha b+\beta}{(a\alpha)^{km+s}(v_{-1}(1-a\alpha)-\alpha b-\beta))}}{1+\frac{a\beta+b}{(a\alpha)^{km+s}(v_{-1}(1-a\alpha)-\alpha b-\beta))}}$$
$$= 1+\frac{\alpha b+\beta-a\beta-b}{(a\alpha)^{km+s}(v_{-1}(1-a\alpha)-\alpha b-\beta))}+\left(\frac{1}{(a\alpha)^{km}}\right). \tag{42}$$

Employing (42) in (25), then using (31), the condition $|a\alpha| > 1$, and the continuity of function f, the statement easily follows.

(d) Using (31) we have that for sufficiently large m

$$p_m^s = -\frac{1+\frac{\alpha b+\beta}{(a\alpha)^{km+s}(v_{-1}(1-a\alpha)-\alpha b-\beta))}}{1-\frac{a\beta+b}{(a\alpha)^{km+s}(v_{-1}(1-a\alpha)-\alpha b-\beta))}}$$
$$= -\left(1+\frac{\alpha b+\beta+a\beta+b}{(a\alpha)^{km+s}(v_{-1}(1-a\alpha)-\alpha b-\beta)}+\left(\frac{1}{(a\alpha)^{km}}\right)\right). \tag{43}$$

Using (43) in (25), then (31), the condition $|a\alpha| > 1$, and the continuity of function f, the statement easily follows.

(e) In this case

$$\lim_{m\to\infty} |p_m^s| = \frac{|\alpha b+\beta|}{|a\beta+b|} < 1,$$

which along with (25) and the continuity of function f, the result follows.

(f) In this case

$$\lim_{m\to\infty} |p_m^s| = \frac{|\alpha b+\beta|}{|a\beta+b|} > 1,$$

which along with (25) and the continuity of function f, the result follows.

(g) Using (31) we have that for sufficiently large m

$$p_m^s = \frac{\left(1+\frac{(a\alpha)^{km+s}(v_{-1}(1-a\alpha)-\alpha b-\beta))}{\alpha b+\beta}\right)}{\left(1+\frac{(a\alpha)^{km+s}(u_0(1-a\alpha)-\alpha b-\beta)}{\alpha b+\beta}\right)}$$
$$= 1+\frac{(a\alpha)^{km+s}(v_{-1}-u_0)(1-a\alpha)}{\alpha b+\beta}+((a\alpha)^{km}). \tag{44}$$

Employing (44) in (25), then using (31), the condition $|a\alpha| < 1$ and the continuity of function f, the statement follows.

(h) Using (31) we have that for sufficiently large m

$$p_m^s = -\frac{\left(1+\frac{(a\alpha)^{km+s}(v_{-1}(1-a\alpha)-\alpha b-\beta))}{\alpha b+\beta}\right)}{\left(1-\frac{(a\alpha)^{km+s}(u_0(1-a\alpha)+\alpha b+\beta)}{\alpha b+\beta}\right)}$$
$$= -\left(1+\frac{(a\alpha)^{km+s}(v_{-1}+u_0)(1-a\alpha)}{\alpha b+\beta}+((a\alpha)^{km})\right). \tag{45}$$

Employing (45) in (25), then using (31), the condition $|a\alpha| < 1$, the continuity and oddness of function f, the statement follows.

(i) By using the condition $a\alpha = -1$ in (25), formula (41) directly follows. □

Theorem 3. *Assume that $a\alpha \neq 1$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, odd, increasing functions satisfying the conditions in* (3)*, and that $(x_n, y_n)_{n\geq -2k}$ is a well-defined solution of system* (13) *such that $x_{-i} \neq 0 \neq y_{-i}$, $i = 1, \ldots, 2k$. Then the following statements are true.*

(a) If $|a\alpha| > 1$, $|v_0(1-a\alpha)-\alpha b-\beta| < |a\alpha||u_{-1}(1-a\alpha)-a\beta-b|$, then $x_{2km+2s+1} \to 0$, $s \in \{0, 1, \ldots, k-1\}$ as $m \to \infty$.

(b) If $|a\alpha| > 1$, $|v_0(1-a\alpha)-\alpha b-\beta| > |a\alpha||u_{-1}(1-a\alpha)-a\beta-b|$, then $|x_{2km+2s+1}| \to f^{-1}(+\infty)$, $s \in \{0, 1, \ldots, k-1\}$ as $m \to \infty$.

(c) If $|a\alpha| > 1$, $v_0(1-a\alpha)-\alpha b-\beta = a\alpha(u_{-1}(1-a\alpha)-a\beta-b)$, then the sequences $x_{2km+2s+1}$, $s \in \{0, 1, \ldots, k-1\}$ converge.

(d) If $|a\alpha| > 1$, $v_0(1-a\alpha)-\alpha b-\beta = -a\alpha(u_{-1}(1-a\alpha)-a\beta-b)$, then the sequences $x_{4km+2s+1}$ and $x_{4km+2k+2s+1}$, $s \in \{0, 1, \ldots, k-1\}$ converge.

(e) If $|a\alpha| < 1$ and $|\alpha b+\beta| < |a\beta+b|$, then $x_{2km+2s+1} \to 0$, $s \in \{0, 1, \ldots, k-1\}$ as $m \to \infty$.

(f) If $|a\alpha| < 1$ and $|\alpha b+\beta| > |a\beta+b|$, then $|x_{2km+2s+1}| \to f^{-1}(+\infty)$, $s \in \{0, 1, \ldots, k-1\}$ as $m \to \infty$.

(g) If $|a\alpha| < 1$ and $\alpha b+\beta = a\beta+b$, then the sequences $x_{2km+2s+1}$, $s \in \{0, 1, \ldots, k-1\}$ are convergent.

(h) If $|a\alpha| < 1$ and $\alpha b+\beta = -(a\beta+b)$, then the sequences $x_{4km+2s+1}$ and $x_{4km+2k+2s+1}$, $s \in \{0, 1, \ldots, k-1\}$ are convergent.

(i) If $a\alpha = -1$, then

$$x_{2km+2s+1} = f^{-1}\left(f(x_{2s+1})\prod_{j=1}^{m}\left(\frac{\alpha b+\beta+(-1)^{kj+s}(2v_0-\alpha b-\beta)}{a\beta+b+(-1)^{kj+s+1}(2u_{-1}-a\beta-b)}\right)\right). \quad (46)$$

Proof. Let

$$r_m^s := \frac{\alpha b+\beta+(a\alpha)^{km+s}(v_0(1-a\alpha)-\alpha b-\beta)}{a\beta+b+(a\alpha)^{km+s+1}(u_{-1}(1-a\alpha)-a\beta-b)}, \quad m \in \mathbb{N}_0,\ s \in \{0, 1, \ldots, k-1\}.$$

(a) Note that in this case

$$\lim_{m\to\infty} |r_m^s| = \frac{|v_0(1-a\alpha)-\alpha b-\beta|}{|u_{-1}(1-a\alpha)-a\beta-b||a\alpha|} < 1,$$

which along with formula (26) and the continuity of function f, easily implies the result.

(b) In this case

$$\lim_{m\to\infty} |r_m^s| = \frac{|v_0(1-a\alpha)-\alpha b-\beta|}{|u_{-1}(1-a\alpha)-a\beta-b||a\alpha|} > 1,$$

from which along with (26) and the continuity of function f, the result follows.

(c) Using (31) we have that for sufficiently large m

$$\begin{aligned} r_m^s &= \frac{1+\frac{\alpha b+\beta}{(a\alpha)^{km+s}(v_0(1-a\alpha)-\alpha b-\beta)}}{1+\frac{a\beta+b}{(a\alpha)^{km+s+1}(u_{-1}(1-a\alpha)-a\beta-b)}} \\ &= 1+\frac{\alpha b+\beta-a\beta-b}{(a\alpha)^{km+s}(v_0(1-a\alpha)-\alpha b-\beta))}+\left(\frac{1}{(a\alpha)^{km}}\right). \end{aligned} \quad (47)$$

Employing (47) in (26), then using (31), the condition $|a\alpha| > 1$ and the continuity of function f, the statement easily follows.

(d) Using (31) we have that for sufficiently large m

$$\begin{aligned} r_m^s &= -\frac{1 + \frac{\alpha b+\beta}{(a\alpha)^{km+s}(v_0(1-a\alpha)-\alpha b-\beta)}}{1 - \frac{a\beta+b}{(a\alpha)^{km+s}(v_0(1-a\alpha)-\alpha b-\beta)}} \\ &= -\left(1 + \frac{\alpha b + \beta + a\beta + b}{(a\alpha)^{km+s}(v_0(1-a\alpha) - \alpha b - \beta))} + \left(\frac{1}{(a\alpha)^{km}}\right)\right). \end{aligned} \tag{48}$$

Employing (48) in (26), then using (31), the condition $|a\alpha| > 1$ and the continuity of function f, the statement easily follows.

(e) In this case

$$\lim_{m\to\infty} |r_m^s| = \frac{|\alpha b + \beta|}{|a\beta + b|} < 1,$$

from which along with (26) and the continuity of function f, the result follows.

(f) In this case

$$\lim_{m\to\infty} |r_m^s| = \frac{|\alpha b + \beta|}{|a\beta + b|} > 1,$$

from which along with (26) and the continuity of function f, the result follows.

(g) Using (31) we have that for sufficiently large m

$$\begin{aligned} r_m^s &= \frac{1 + \frac{(a\alpha)^{km+s}(v_0(1-a\alpha)-\alpha b-\beta)}{\alpha b+\beta}}{1 + \frac{(a\alpha)^{km+s+1}(u_{-1}(1-a\alpha)-\alpha b-\beta)}{\alpha b+\beta}} \\ &= 1 + \frac{(a\alpha)^{km+s}(v_0 - a\alpha u_{-1} - \alpha b - \beta)(1 - a\alpha)}{\alpha b + \beta} + ((a\alpha)^{km}). \end{aligned} \tag{49}$$

Employing (49) in (26), then using (31), the condition $|a\alpha| < 1$ and the continuity of function f, the statement follows.

(h) Using (31) we have that for sufficiently large m

$$\begin{aligned} r_m^s &= -\frac{1 + \frac{(a\alpha)^{km+s}(v_0(1-a\alpha)-\alpha b-\beta)}{\alpha b+\beta}}{1 - \frac{(a\alpha)^{km+s+1}(u_{-1}(1-a\alpha)+\alpha b+\beta)}{\alpha b+\beta}} \\ &= -\left(1 + \frac{(a\alpha)^{km+s}(v_0 + \alpha a u_{-1} - \alpha b - \beta)(1 - a\alpha)}{\alpha b + \beta} + ((a\alpha)^{km})\right). \end{aligned} \tag{50}$$

Employing (50) in (26), then using (31), the condition $|a\alpha| < 1$, the continuity and oddness of function f, the statement follows.

(i) By using the condition $a\alpha = -1$ in (26) formula (46) easily follows. □

The proofs of the next two theorems use formulas (27) and (28), and are similar to those ones of Theorems 2 and 3, so they are omitted.

Theorem 4. *Assume that $a\alpha \neq 1$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, odd, increasing functions satisfying the conditions in* (3)*, and that $(x_n, y_n)_{n\geq -2k}$ is a well-defined solution of system* (13) *such that $x_{-i} \neq 0 \neq y_{-i}$, $i = 1, \dots, 2k$. Then the following statements are true.*

(a) *If* $|a\alpha| > 1$, $|v_0(1 - a\alpha) - \alpha b - \beta| > |u_{-1}(1 - a\alpha) - a\beta - b|$, *then* $y_{2km+2s} \to 0$, $s \in \{0, 1, \dots, k-1\}$ *as* $m \to \infty$.
(b) *If* $|a\alpha| > 1$, $|v_0(1 - a\alpha) - \alpha b - \beta| < |u_{-1}(1 - a\alpha) - a\beta - b|$, *then* $|y_{2km+2s}| \to g^{-1}(+\infty)$, $s \in \{0, 1, \dots, k-1\}$ *as* $m \to \infty$.
(c) *If* $|a\alpha| > 1$, $v_0(1 - a\alpha) - \alpha b - \beta = u_{-1}(1 - a\alpha) - a\beta - b$, *then the sequences* y_{2km+2s}, $s \in \{0, 1, \dots, k-1\}$ *converge.*
(d) *If* $|a\alpha| > 1$, $v_0(1 - a\alpha) - \alpha b - \beta = -(u_{-1}(1 - a\alpha) - a\beta - b)$, *then the sequences* y_{4km+2s} *and* $y_{4km+2k+2s}$, $s \in \{0, 1, \dots, k-1\}$ *converge.*
(e) *If* $|a\alpha| < 1$ *and* $|\alpha b + \beta| > |a\beta + b|$, *then* $y_{2km+2s} \to 0$, $s \in \{0, 1, \dots, k-1\}$ *as* $m \to \infty$.
(f) *If* $|a\alpha| < 1$ *and* $|\alpha b+\beta| < |a\beta+b|$, *then* $|y_{2km+2s}| \to g^{-1}(+\infty)$, $s \in \{0, 1, \dots, k-1\}$ *as* $m \to \infty$.
(g) *If* $|a\alpha| < 1$ *and* $\alpha b+\beta = a\beta+b$, *then the sequences* y_{2km+2s}, $s \in \{0, 1, \dots, k-1\}$ *are convergent.*
(h) *If* $|a\alpha| < 1$ *and* $\alpha b+\beta = -(a\beta+b)$, *then the sequences* y_{4km+2s} *and* $y_{4km+2k+2s}$, $s \in \{0, 1, \dots, k-1\}$ *are convergent.*
(i) *If* $a\alpha = -1$, *then*

$$y_{2km+2s} = g^{-1}\left(g(y_{2s}) \prod_{j=1}^{m} \left(\frac{a\beta + b + (-1)^{kj+s}(2u_{-1} - a\beta - b)}{\alpha b + \beta + (-1)^{kj+s}(2v_0 - \alpha b - \beta)} \right)^m \right).$$

Theorem 5. *Assume that* $a\alpha \neq 1$, $f, g : \mathbb{R} \to \mathbb{R}$ *are continuous, odd, increasing functions satisfying the conditions in* (3), *and that* $(x_n, y_n)_{n \geq -2k}$ *is a well-defined solution of system* (13) *such that* $x_{-i} \neq 0 \neq y_{-i}$, $i = 1, \dots, 2k$. *Then the following statements are true.*

(a) *If* $|a\alpha| > 1$, $|a\alpha||v_{-1}(1-a\alpha)-\alpha b-\beta| > |u_0(1-a\alpha)-a\beta-b|$, *then* $y_{2km+2s+1} \to 0$, $s \in \{0, 1, \dots, k-1\}$ *as* $m \to \infty$.
(b) *If* $|a\alpha| > 1$, $|a\alpha||v_{-1}(1-a\alpha)-\alpha b-\beta| < |u_0(1-a\alpha)-a\beta-b|$, *then* $|y_{2km+2s+1}| \to g^{-1}(+\infty)$, $s \in \{0, 1, \dots, k-1\}$ *as* $m \to \infty$.
(c) *If* $|a\alpha| > 1$, $a\alpha(v_{-1}(1 - a\alpha) - \alpha b - \beta) = u_0(1 - a\alpha) - a\beta - b \neq 0$, *then the sequences* $y_{2km+2s+1}$, $s \in \{0, 1, \dots, k-1\}$ *are convergent.*
(d) *If* $|a\alpha| > 1$, $a\alpha(v_{-1}(1 - a\alpha) - \alpha b - \beta) = -(u_0(1 - a\alpha) - a\beta - b) \neq 0$, *then the sequences* $y_{4km+2s+1}$ *and* $y_{4km+2k+2s+1}$, $s \in \{0, 1, \dots, k-1\}$ *are convergent.*
(e) *If* $|a\alpha| < 1$ *and* $|\alpha b + \beta| > |a\beta + b|$, *then* $y_{2km+2s+1} \to 0$, $s \in \{0, 1, \dots, k-1\}$ *as* $m \to \infty$.
(f) *If* $|a\alpha| < 1$ *and* $|\alpha b + \beta| < |a\beta + b|$, *then* $|y_{2km+2s+1}| \to g^{-1}(+\infty)$, $s \in \{0, 1, \dots, k-1\}$ *as* $m \to \infty$.
(g) *If* $|a\alpha| < 1$ *and* $\alpha b+\beta = a\beta+b$, *then the sequences* $y_{2km+2s+1}$, $s \in \{0, 1, \dots, k-1\}$ *are convergent.*
(h) *If* $|a\alpha| < 1$ *and* $\alpha b + \beta = -(a\beta + b)$, *then the sequences* $y_{4km+2s+1}$ *and* $y_{4km+2k+2s+1}$, $s \in \{0, 1, \dots, k-1\}$ *are convergent.*
(i) *If* $a\alpha = -1$, *then*

$$y_{2km+2s+1} = g^{-1}\left(g(y_{2s+1}) \prod_{j=1}^{m} \left(\frac{a\beta + b + (-1)^{kj+s}(2u_0 - a\beta - b)}{\alpha b + \beta + (-1)^{kj+s+1}(2v_{-1} - \alpha b - \beta)} \right)^m \right).$$

Theorems 2–5 and Lemma 1 yield the next corollary.

Corollary 1. *Assume that* $|a\alpha| < 1$, $f, g : \mathbb{R} \to \mathbb{R}$ *are continuous, odd, increasing functions satisfying the conditions in* (3), *and* $(x_n, y_n)_{n\geq -2k}$ *is a well-defined solution of system* (13) *such that* $x_{-i} \neq 0 \neq y_{-i}$, $i = 1, \ldots, 2k$. *Then the following statements are true.*

(a) If $\alpha b+\beta = a\beta+b$, *then the solution* $(x_n, y_n)_{n\geq -2k}$ *converges to a, not necessarily prime,* $2k$*-periodic solution of system* (13).

(b) If $\alpha b + \beta = -(a\beta + b)$, *then the solution* $(x_n, y_n)_{n\geq -2k}$ *converges to a, not necessarily prime,* $4k$*-periodic solution of system* (13).

Acknowledgements

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under grant No. (11-130/1433 HiCi). The authors, therefore, acknowledge technical and financial support of KAU.

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(Received March 6, 2013)

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