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Existence of positive solution for a third-order three-point BVP with sign-changing Green's function*

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Abstract

By using the Guo-Krasnoselskii fixed point theorem, we investigate the following third-order three-point boundary value problem

$$\begin{cases} u'''(t) = f(t, u(t)),\ t \in [0,1], \\ u'(0) = u(1) = 0,\ u''(\eta) + \alpha u(0) = 0, \end{cases}$$

where $\alpha \in [0, 2)$ and $\eta \in [\frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)}, 1)$. The emphasis is mainly that although the corresponding Green's function is sign-changing, the solution obtained is still positive.

Keywords: Third-order three-point boundary value problem; Sign-changing Green's function; Positive solution; Existence; Fixed point

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1 Introduction

Third-order differential equations arise from a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [3].

Recently, the existence of single or multiple positive solutions to some third-order three-point boundary value problems (BVPs for short) has received much attention from many authors, see [1, 2, 5, 12, 15, 16] and the references therein.

However, all the above-mentioned papers are achieved when the corresponding Green's functions are positive, which is a very important condition. A natural question is that whether we can obtain the existence of positive solutions to some third-order three-point BVPs when the corresponding Green's functions are sign-changing.

In 2008, Palamides and Smyrlis [11] studied the existence of at least one positive solution to the singular third-order three-point BVP with an indefinitely signed Green's function

$$\begin{cases} u'''(t) = a(t)f(t, u(t)),\ t \in (0, 1), \\ u(0) = u(1) = u''(\eta) = 0, \end{cases}$$

where $\eta \in \left(\frac{17}{24}, 1\right)$. Their technique was a combination of the Guo-Krasnoselskii fixed point theorem and properties of the corresponding vector field.

In 2012, by using the Guo-Krasnoselskii and Leggett-Williams fixed point theorems, Sun and Zhao [13, 14] discussed the third-order three-point BVP with sign-changing Green's function

$$\begin{cases} u'''(t) = f(t, u(t)),\ t \in [0, 1], \\ u'(0) = u(1) = u''(\eta) = 0, \end{cases} \tag{1.1}$$

where $\eta \in (\frac{1}{2}, 1)$. They obtained the existence of single or multiple positive solutions to the BVP (1.1) and proved that the obtained solutions were concave on $[0, \eta]$ and convex on $[\eta, 1]$.

It is worth mentioning that there are other type of works on sign-changing Green's functions which prove the existence of sign-changing solutions, positive in some cases, see Infante and Webb's papers [6–8].

In this paper we study the following third-order three-point BVP

$$\begin{cases} u'''(t) = f(t, u(t)), \ t \in [0, 1], \\ u'(0) = u(1) = 0, \ u''(\eta) + \alpha u(0) = 0. \end{cases} \tag{1.2}$$

Throughout this paper, we always assume that $\alpha \in [0, 2)$ and $\eta \in [\frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)}, 1)$. Obviously, the BVP (1.1) is a special case of the BVP (1.2). However, it is necessary to point out that this paper is not a simple extension of [13]. In fact, if we let $\alpha = 0$, then $\eta \in [\frac{1}{2}, 1)$, which is different from the restriction in [13]. On the other hand, compared with [13], we can only prove that the obtained solution is concave on $[0, \eta]$.

Our main tool is the following well-known Guo-Krasnoselskii fixed point theorem [4, 9]:

Theorem 1.1 *Let E be a Banach space and K be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that either*
(1) $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$, or
(2) $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$.
Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Preliminaries

For the BVP

$$\begin{cases} u'''(t) = 0, \ t \in [0, 1], \\ u'(0) = u(1) = 0, \ u''(\eta) + \alpha u(0) = 0, \end{cases} \tag{2.1}$$

we have the following lemma.

Lemma 2.1 *The BVP (2.1) has only trivial solution.*

Proof. It is simple to check. □

In the remainder of this paper, we always assume that Banach space $C[0, 1]$ is equipped with the norm $\|u\| = \max\limits_{t \in [0,1]} |u(t)|$.

Now, for any $y \in C[0,1]$, we consider the BVP

$$\begin{cases} u'''(t) = y(t), \ t \in [0,1], \\ u'(0) = u(1) = 0, \ u''(\eta) + \alpha u(0) = 0. \end{cases} \tag{2.2}$$

After a direct computation, one may obtain the expression of Green's function $G(t,s)$ of the BVP (2.2) as follows:

$$G(t,s) = g_1(t,s) + g_2(t,s) + g_3(\eta,t,s),$$

where

$$g_1(t,s) = -\frac{(2-\alpha t^2)(1-s)^2}{2(2-\alpha)}, \ (t,s) \in [0,1] \times [0,1],$$

$$g_2(t,s) = \begin{cases} 0, & 0 \le t \le s \le 1, \\ \frac{(t-s)^2}{2}, & 0 \le s \le t \le 1 \end{cases}$$

and

$$g_3(\eta,t,s) = \begin{cases} 0, & s \ge \eta, \\ \frac{1-t^2}{2-\alpha}, & s < \eta. \end{cases}$$

It is not difficult to verify that the $G(t,s)$ has the following properties:

$$G(t,s) \ge 0 \ \text{ for } \ 0 \le s \le \eta \ \text{ and } \ G(t,s) \le 0 \ \text{ for } \ \eta \le s \le 1.$$

Moreover, for $s \ge \eta$,

$$\max\{G(t,s) : t \in [0,1]\} = G(1,s) = 0,$$
$$\min\{G(t,s) : t \in [0,1]\} = G(0,s) = -\frac{(1-s)^2}{2-\alpha}$$

and for $s < \eta$,

$$\max\{G(t,s) : t \in [0,1]\} = G(0,s) = \frac{2s-s^2}{2-\alpha},$$
$$\min\{G(t,s) : t \in [0,1]\} = G(1,s) = 0.$$

Let

$$K_0 = \{y \in C[0,1] : y(t) \text{ is nonnegative and decreasing on } [0,1]\}.$$

Then K_0 is a cone in $C[0,1]$.

Lemma 2.2 *Let $y \in K_0$ and $u(t) = \int_0^1 G(t,s)y(s)ds$, $t \in [0,1]$. Then u is the unique solution of the BVP (2.2) and $u \in K_0$. Moreover, $u(t)$ is concave on $[0,\eta]$.*

Proof. For $0 \le t \le \eta$, we have

$$u(t) = \int_0^t \left[g_1(t,s) + \frac{(t-s)^2}{2} + \frac{1-t^2}{2-\alpha}\right] y(s)ds + \int_t^\eta \left[g_1(t,s) + \frac{1-t^2}{2-\alpha}\right] y(s)ds + \int_\eta^1 g_1(t,s)y(s)ds.$$

Since $\eta \ge \frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)}$ implies that $\eta \ge \frac{2\alpha}{3\alpha+6}$, we get

$$\begin{aligned}
u'(t) &= -\frac{\alpha t}{2-\alpha}\int_0^\eta (2s-s^2)y(s)ds - \int_0^t sy(s)ds - t\int_t^\eta y(s)ds + \frac{\alpha t}{2-\alpha}\int_\eta^1 (1-s)^2 y(s)ds \\
&\le y(\eta)\left[-\frac{\alpha t}{2-\alpha}\int_0^\eta (2s-s^2)ds - \int_0^t sds - t\int_t^\eta ds + \frac{\alpha t}{2-\alpha}\int_\eta^1 (1-s)^2 ds\right] \\
&= ty(\eta)\left[\frac{\alpha(1-3\eta)}{3(2-\alpha)} - \eta + \frac{t}{2}\right] \\
&\le ty(\eta)\left[\frac{\alpha(1-3\eta)}{3(2-\alpha)} - \frac{\eta}{2}\right] \\
&\le 0.
\end{aligned}$$

At the same time, $\eta \ge \frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)} > \frac{1}{3}$ shows that

$$\begin{aligned}
u''(t) &= -\frac{\alpha}{2-\alpha}\int_0^\eta (2s-s^2)y(s)ds - \int_t^\eta y(s)ds + \frac{\alpha}{2-\alpha}\int_\eta^1 (1-s)^2 y(s)ds \\
&\le -\frac{\alpha y(\eta)}{2-\alpha}\int_0^\eta (2s-s^2)ds - y(\eta)\int_t^\eta ds + \frac{\alpha y(\eta)}{2-\alpha}\int_\eta^1 (1-s)^2 ds \\
&\le \frac{\alpha y(\eta)(1-3\eta)}{3(2-\alpha)} \\
&\le 0.
\end{aligned}$$

For $\eta < t \le 1$, we have

$$u(t) = \int_0^\eta \left[g_1(t,s) + \frac{(t-s)^2}{2} + \frac{1-t^2}{2-\alpha}\right] y(s)ds + \int_\eta^t \left[g_1(t,s) + \frac{(t-s)^2}{2}\right] y(s)ds + \int_t^1 g_1(t,s)y(s)ds.$$

Since $\eta \geq \frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)}$ implies that $\eta \geq \frac{6-\alpha}{12}$, we get

$$\begin{aligned} u'(t) &= -\frac{\alpha t}{2-\alpha}\int_0^\eta (2s-s^2)y(s)ds + \int_\eta^t (t-s)y(s)ds - \int_0^\eta sy(s)ds + \frac{\alpha t}{2-\alpha}\int_\eta^1 (1-s)^2 y(s)ds \\ &\leq -\frac{\alpha t y(\eta)}{2-\alpha}\int_0^\eta (2s-s^2)ds + \frac{y(\eta)(\eta-t)^2}{2} - y(\eta)\int_0^\eta sds + \frac{\alpha t y(\eta)(1-\eta)^3}{3(2-\alpha)} \\ &= ty(\eta)\left[\frac{\alpha(1-3\eta)}{3(2-\alpha)} + \frac{t-2\eta}{2}\right] \\ &\leq 0. \end{aligned}$$

Obviously, $u'''(t) = y(t)$ for $t \in [0,1]$, $u'(0) = u(1) = 0$ and $u''(\eta) + \alpha u(0) = 0$. This shows that u is a solution of the BVP (2.2). The uniqueness follows immediately from Lemma 2.1. Since $u'(t) \leq 0$ for $t \in [0,1]$ and $u(1) = 0$, we have $u(t) \geq 0$ for $t \in [0,1]$. So, $u \in K_0$. In view of $u''(t) \leq 0$ for $t \in [0,\eta]$, we know that $u(t)$ is concave on $[0,\eta]$. □

Lemma 2.3 *Let $y \in K_0$. Then the unique solution u of the BVP (2.2) satisfies*

$$\min_{t\in[0,\theta]} u(t) \geq \theta^* \|u\|,$$

where $\theta \in (0, \frac{1}{3}]$ and $\theta^* = \frac{\eta-\theta}{\eta}$.

Proof. By Lemma 2.2, we know that $u(t)$ is concave on $[0,\eta]$, thus for $t \in [0,\eta]$,

$$u(t) \geq (1-\frac{t}{\eta})u(0) + \frac{t}{\eta}u(\eta). \tag{2.3}$$

In view of $u \in K_0$, we know that $\|u\| = u(0)$, which together with (2.3) implies that

$$u(t) \geq \frac{\eta - t}{\eta}\|u\|,\ 0 \leq t \leq \eta.$$

Consequently,

$$\min_{t\in[0,\theta]} u(t) = u(\theta) \geq \frac{\eta-\theta}{\eta}\|u\| = \theta^*\|u\|.$$

□

3 Main results

For convenience, we denote

$$A = \int_0^\eta G(0,s)ds \text{ and } B = \int_0^\theta G(\eta,s)ds.$$

Then it is obvious that $0 < B < A$.

Theorem 3.1 *Assume that* $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and satisfies the following conditions:*

$(H1)$ *For each* $u \in [0,+\infty)$*, the mapping* $t \mapsto f(t,u)$ *is decreasing;*

$(H2)$ *For each* $t \in [0,1]$*, the mapping* $u \mapsto f(t,u)$ *is increasing;*

$(H3)$ *There exist two positive constants* r *and* R *with* $r \neq R$ *such that*

$$f(0,r) \leq \frac{r}{A} \text{ and } f(\theta, \theta^* R) \geq \frac{R}{B}.$$

Then the BVP (1.2) has a positive and decreasing solution u *satisfying* $\min\{r,R\} \leq \|u\| \leq \max\{r,R\}$*. Moreover, the obtained solution* $u(t)$ *is concave on* $[0,\eta]$*.*

Proof. Let

$$K = \left\{ u \in K_0 : \min_{t\in[0,\theta]} u(t) \geq \theta^* \|u\| \right\}.$$

Then it is easy to see that K is a cone in $C[0,1]$. Now, we define an operator T on K by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))ds, \ t \in [0,1].$$

Obviously, if u is a fixed point of T in K, then u is a nonnegative and decreasing solution of the BVP (1.2). In what follows, we will seek a fixed point of T in K by using Theorem 1.1.

First, by Lemma 2.2 and Lemma 2.3, we know that $T : K \to K$. Furthermore, although $G(t,s)$ is not continuous, it follows from known textbook results, for example see [10], that $T : K \to K$ is completely continuous.

Next, for any $u \in K$, we claim that

$$\int_\theta^\eta G(\eta,s) f(s,u(s))ds + \int_\eta^1 G(\eta,s) f(s,u(s))ds \geq 0. \tag{3.1}$$

In fact, if $u \in K$, recall that $G(t,s) \geq 0$ for $0 \leq s \leq \eta$ and $G(t,s) \leq 0$ for $\eta \leq s \leq 1$, then it follows from $\eta \geq \frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)}$ that

$$\begin{aligned}
&\int_\theta^\eta G(\eta,s)f(s,u(s))ds + \int_\eta^1 G(\eta,s)f(s,u(s))ds \\
&\geq f(\eta,u(\eta))\left[\int_\theta^\eta G(\eta,s)ds + \int_\eta^1 G(\eta,s)ds\right] \\
&= f(\eta,u(\eta))\left[\int_\theta^\eta \left(g_1(\eta,s) + \frac{(\eta-s)^2}{2} + \frac{1-\eta^2}{2-\alpha}\right)ds + \int_\eta^1 g_1(\eta,s)ds\right] \\
&= \frac{(1-\eta)f(\eta,u(\eta))}{6(2-\alpha)}\left[(4+\alpha)\eta^2 + (4+\alpha\theta^3-3\alpha\theta^2)\eta - 6\theta^2 + \alpha\theta^3 - 2\right] \\
&\geq \frac{(1-\eta)f(\eta,u(\eta))}{6(2-\alpha)}\left[(4+\alpha)\eta^2 + \frac{10}{3}\eta - \frac{8}{3}\right] \\
&\geq 0.
\end{aligned}$$

Now, without loss of generality, we assume that $r < R$. Let

$$\Omega_1 = \{u \in C[0,1] : \|u\| < r\} \text{ and } \Omega_2 = \{u \in C[0,1] : \|u\| < R\}.$$

For any $u \in K \cap \partial\Omega_1$, we get $0 \leq u(s) \leq r$ for $s \in [0,1]$, which together with $(H3)$ implies that

$$\begin{aligned}
0 \leq (Tu)(t) &\leq \int_0^\eta \max_{t\in[0,1]} G(t,s)f(s,u(s))ds + \int_\eta^1 \max_{t\in[0,1]} G(t,s)f(s,u(s))ds \\
&= \int_0^\eta G(0,s)f(s,u(s))ds \\
&\leq \int_0^\eta G(0,s)f(0,r)ds \\
&\leq r = \|u\|, \ t \in [0,1].
\end{aligned}$$

This shows that

$$\|Tu\| \leq \|u\| \text{ for } u \in K \cap \partial\Omega_1. \tag{3.2}$$

For any $u \in K \cap \partial\Omega_2$, we get $\theta^* R \leq u(s) \leq R$ for $s \in [0,\theta]$, which together with (3.1) and

$(H3)$ implies that

$$
\begin{aligned}
Tu(\eta) &= \int_0^1 G(\eta,s)f(s,u(s))ds \\
&= \int_0^\theta G(\eta,s)f(s,u(s))ds + \int_\theta^\eta G(\eta,s)f(s,u(s))ds + \int_\eta^1 G(\eta,s)f(s,u(s))ds \\
&\geq \int_0^\theta G(\eta,s)f(s,u(s))ds \\
&\geq \int_0^\theta G(\eta,s)f(\theta,\theta^* R)ds \\
&\geq R = \|u\|,
\end{aligned}
$$

This indicates that

$$\|Tu\| \geq \|u\| \text{ for } u \in K \cap \partial\Omega_2. \tag{3.3}$$

Therefore, it follows from Theorem 1.1, (3.2) and (3.3) that the operator T has a fixed point $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is a desired positive and decreasing solution of the BVP (1.2) with $r \leq \|u\| \leq R$. Moreover, similar to the proof of Lemma 2.2, we can prove that the obtained solution $u(t)$ is concave on $[0,\eta]$. □

Example 3.2 *We consider the BVP*

$$\begin{cases} u'''(t) = \frac{u^2(t)}{4} + \frac{9(1-t^2)}{2}, \ t \in [0,1], \\ u'(0) = u(1) = 0, \ u''(\frac{1}{2}) + u(0) = 0. \end{cases} \tag{3.4}$$

Since $\alpha = 1$ and $\eta = \frac{1}{2}$, if we choose $\theta = \frac{1}{3}$, then a simple calculation shows that

$$\theta^* = \frac{1}{3}, \ A = \frac{5}{24} \text{ and } B = \frac{7}{108}.$$

Let $f(t,u) = \frac{u^2}{4} + \frac{9(1-t^2)}{2}$, $(t,u) \in [0,1]\times[0,+\infty)$. Then $(H1)$ and $(H2)$ are satisfied. Moreover, it is easy to verify that

$$f(\theta, \frac{\theta^*}{4}) \geq \frac{1}{4B}, \ f(0,1) \leq \frac{1}{A}$$

and

$$f(0,18) \leq \frac{18}{A}, \ f(\theta, 556\theta^*) \geq \frac{556}{B}.$$

Therefore, it follows from Theorem 3.1 that the BVP (3.4) has positive and decreasing solutions u_1 and u_2 satisfying

$$\frac{1}{4} \leq \|u_1\| \leq 1 < 18 \leq \|u_2\| \leq 556.$$

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