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Multiple solutions for a perturbed mixed boundary value problem involving the one-dimensional p-Laplacian

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Abstract

The existence of three distinct weak solutions for a perturbed mixed boundary value problem involving the one-dimensional p-Laplacian operator is established under suitable assumptions on the nonlinear term. Our approach is based on recent variational methods for smooth functionals defined on reflexive Banach spaces.

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1 Introduction

Consider the following perturbed mixed boundary value problem

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + s(x)|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \;]a,b[\\ u(a) = u'(b) = 0, \end{cases} \tag{1}$$

where $p > 1$, $\lambda > 0$ and $\mu \geq 0$ are real numbers, $a, b \in \mathbb{R}$ with $a < b$, $\rho, s \in L^\infty([a,b])$ with $\rho_0 = \text{essinf}_{x\in[a,b]}\rho(x) > 0$, $s_0 = \text{essinf}_{x\in[a,b]}s(x) \geq 0$ and $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory function.

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Using two kinds of three critical points theorems obtained in [4, 8] which we recall in the next section (Theorems 2.1 and 2.2), we ensure the existence of at least three weak solutions for the problem (1); see Theorems 3.1 and 3.2. These theorems have been successfully employed to establish the existence of at least three solutions for perturbed boundary value problems in the papers [5, 6, 14, 16, 17].

Existence and multiplicity of solutions for mixed boundary value problems have been studied by several authors and, for an overview on this subject, we refer the reader to the papers [2, 3, 12, 15, 18]. We also refer the reader to the papers [7, 9, 10, 11] in which the existence of multiple solutions is ensured.

A special case of Theorem 3.1 is the following theorem.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) := \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that $F(\eta) > 0$ for some $\eta > 0$ and $F(\xi) \geq 0$ in $[0, \eta]$ and*

$$\liminf_{\xi \to 0} \frac{F(\xi)}{\xi^p} = \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p} = 0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for every L^1-Carathéodory function $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ satisfying the asymptotical condition

$$\limsup_{|t| \to \infty} \frac{\sup_{x \in [a,b]} \int_0^t g(x,s)ds}{t^p} < +\infty,$$

there exists $\delta^*_{\lambda,g} > 0$ such that, for each $\mu \in [0, \delta^*_{\lambda,g}[$, the problem

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + s(x)|u|^{p-2}u = \lambda f(u) + \mu g(x,u) & \text{in } \ ]a,b[\\ u(a) = u'(b) = 0, \end{cases}$$

admits at least three weak solutions.

Moreover, the following result is a consequence of Theorem 3.2.

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\lim_{t \to 0^+} \frac{f(t)}{t^2} = 0,$$

and

$$\int_0^1 f(\xi)d\xi < \frac{1}{222} \int_0^2 f(\xi)d\xi.$$

Then, for every $\lambda \in \left] \frac{37}{\int_0^2 f(\xi)d\xi}, \frac{1}{6\int_0^1 f(\xi)d\xi} \right[$ and for every L^1-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|t| \to \infty} \frac{\sup_{x \in [0,1]} \int_0^t g(x,s)ds}{t^3} < +\infty,$$

there exists $\delta^*_{\lambda,g} > 0$ such that, for each $\mu \in [0, \delta^*_{\lambda,g}[$, the problem

$$\begin{cases} -(|u'|u')' + |u|u = \lambda f(u) + \mu g(x,u) & \text{in }]0,1[\\ u(0) = u'(1) = 0, \end{cases}$$

admits at least three weak solutions.

The present paper is arranged as follows. In Section 2 we recall some basic definitions and preliminary results, while Section 3 is devoted to the existence of multiple weak solutions for the eigenvalue problem (1).

2 Preliminaries

Our main tools are the following three critical points theorems. In the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed.

Theorem 2.1 ([8], Theorem 2.6). *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$ such that*

(a_1) $\dfrac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \dfrac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(a_2) *for each* $\lambda \in \Lambda_r := \left] \dfrac{\Phi(\overline{x})}{\Psi(\overline{x})}, \dfrac{r}{\sup_{\Phi(x)\le r} \Psi(x)} \right[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.2 ([4], Theorem 3.3). *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

2. for each $\lambda > 0$ and for every $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \ge 0$ and $\Psi(u_2) \ge 0$, one has

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \ge 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{v} \in X$, with $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, such that

(b_1) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(b_2) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each $\lambda \in \left] \dfrac{3}{2}\dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \min\left\{ \dfrac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}, \dfrac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)} \right\} \right[$, *the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.*

In order to study the problem (1), the variational setting is the space

$$X := \left\{ u \in W^{1,p}([a,b]) : u(a) = 0 \right\}$$

endowed with the norm

$$\|u\| := \left(\int_a^b \rho(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx \right)^{1/p}.$$

We observe that the norm $\|\cdot\|$ is equivalent to the usual one.

It is well known that $(X, \|\cdot\|)$ is compactly embedded in $(C^0([a,b]), \|\cdot\|_\infty)$ and

$$\|u\|_\infty \leq \frac{(b-a)^{(p-1)/p}}{\rho_0^{1/p}} \|u\|, \tag{2}$$

for every $u \in X$.

We need the following proposition in the proof of Theorem 3.1.

Proposition 2.3. *Let $T : X \to X^*$ be the operator defined by*

$$T(u)v = \int_a^b \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dx$$

for every $u, v \in X$. Then T admits a continuous inverse on X^*.

Proof. In the proof, we use C_1, $C_2, \ldots, C_9$ to denote suitable positive constants. For any $u \in X \setminus \{0\}$,

$$\begin{aligned}
\lim_{\|u\|\to\infty} \frac{\langle T(u), u\rangle}{\|u\|} &= \lim_{\|u\|\to\infty} \frac{\int_a^b \rho(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx}{\|u\|} \\
&= \lim_{\|u\|\to\infty} \frac{\|u\|^p}{\|u\|} \\
&= \lim_{\|u\|\to\infty} \|u\|^{p-1} = \infty.
\end{aligned}$$

Thus, the map T is coercive.

Now, taking into account (2.2) in [19], we see that

$$\langle T(u) - T(v), u - v\rangle$$

$$\geq \begin{cases} C_1 \int_a^b \Big(\rho(x)|u'(x) - v'(x)|^p + s(x)|u(x) - v(x)|^p\Big) dx & \text{if } p \geq 2, \\ C_2 \int_a^b \Big(\frac{\rho(x)|u'(x) - v'(x))|^2}{(|u'(x)| + |v'(x)|)^{2-p}} + \frac{s(x)|u(x) - v(x))|^2}{(|u(x)| + |v(x)|)^{2-p}}\Big) dx & \text{if } 1 < p < 2. \end{cases} \tag{3}$$

At this point, if $p \geq 2$, then it follows that

$$\langle T(u) - T(v), u - v\rangle \geq C_1 \|u - v\|^p,$$

so T is uniformly monotone. By [20, Theorem 26.A (d)], T^{-1} exists and is continuous on X^*.

On the other hand, if $1 < p < 2$, by Hölder's inequality, we obtain

$$\begin{aligned}
\int_a^b s(x)|u(x) - v(x)|^p dx &\leq \left(\int_a^b \frac{s(x)|u(x) - v(x)|^2}{(|u(x)| + |v(x)|)^{2-p}} dx\right)^{p/2} \left(\int_a^b s(x)(|u(x)| + |v(x)|)^p dx\right)^{(2-p)/2} \\
&\leq C_3 \left(\int_a^b \frac{s(x)|u(x) - v(x)|^2}{(|u(x)| + |v(x)|)^{2-p}} dx\right)^{p/2} \left(\int_a^b s(x)(|u(x)|^p + |v(x)|^p) dx\right)^{(2-p)/2}
\end{aligned}$$

$$\leq C_4 \left(\int_a^b \frac{s(x)|u(x)-v(x)|^2}{(|u(x)|+|v(x)|)^{2-p}} dx \right)^{p/2} (\|u\|+\|v\|)^{(2-p)p/2}. \tag{4}$$

Similarly, one has

$$\int_a^b \rho(x)|u'(x)-v'(x)|^p dx \leq C_5 \left(\int_a^b \frac{\rho(x)|u'(x)-v'(x)|^2}{(|u'(x)|+|v'(x)|)^{2-p}} dx \right)^{p/2} (\|u\|+\|v\|)^{(2-p)p/2}. \tag{5}$$

Then, relation (3) together with (4) and (5), yields

$$\begin{aligned}
& \langle T(u)-T(v), u-v \rangle \\
\geq\ & \frac{C_6}{(\|u\|+\|v\|)^{2-p}} \left(\left(\int_a^b \rho(x)|u'(x)-v'(x)|^p dx \right)^{2/p} + \left(\int_a^b s(x)|u(x)-v(x)|^p dx \right)^{2/p} \right) \\
\geq\ & \frac{C_7}{(\|u\|+\|v\|)^{2-p}} \left(\int_a^b \rho(x)|u'(x)-v'(x)|^p dx + \int_a^b s(x)|u(x)-v(x)|^p dx \right)^{2/p} \\
\geq\ & C_8 \frac{\|u-v\|^2}{(\|u\|+\|v\|)^{2-p}}.
\end{aligned}$$

Thus, T is strictly monotone. By [20, Theorem 26.A (d)], T^{-1} exists and is bounded. Moreover, given $g_1, g_2 \in X^*$, by the inequality

$$\langle T(u)-T(v), u-v \rangle \geq C_8 \frac{\|u-v\|^2}{(\|u\|+\|v\|)^{2-p}},$$

choosing $u = T^{-1}(g_1)$ and $v = T^{-1}(g_2)$ we have

$$\|T^{-1}(g_1) - T^{-1}(g_2)\| \leq \frac{1}{C_9} (\|T^{-1}(g_1)\| + \|T^{-1}(g_2)\|)^{2-p} \|g_1 - g_2\|_{X^*}.$$

So T^{-1} is continuous. This completes the proof. □

We use the following notations:

$$\|\rho\|_\infty := \operatorname{ess\,sup}_{x\in[a,b]} \rho(x), \qquad \|s\|_\infty := \operatorname{ess\,sup}_{x\in[a,b]} s(x).$$

Corresponding to f and g we introduce the functions $F : [a,b]\times\mathbb{R} \to \mathbb{R}$ and $G : [a,b]\times\mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)d\xi, \quad \forall\ (x,t) \in [a,b]\times\mathbb{R}$$

and

$$G(x,t) := \int_0^t g(x,\xi)d\xi, \quad \forall\ (x,t) \in [a,b]\times\mathbb{R}.$$

Moreover, set $G^\theta := \int_{[a,b]} \max_{|t|\leq\theta} G(x,t)dt$, for every $\theta > 0$ and $G_\eta := \inf_{[a,b]\times[0,\eta]} G$, for every $\eta > 0$. If g is sign-changing, then $G^\theta \geq 0$ and $G_\eta \leq 0$.

We mean by a (weak) solution of problem (1), any function $u \in X$ such that

$$\int_a^b \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dx$$

$$-\lambda \int_a^b f(x,u(x))v(x)dx - \mu \int_a^b g(x,u(x))v(x)dx = 0,$$

for every $v \in X$.

3 Main results

Put

$$k := \frac{2(p+1)\rho_0}{2^p(p+1)\|\rho\|_\infty + (p+2)(b-a)^p\|s\|_\infty}, \tag{6}$$

Following the construction given in [6], in order to introduce our first result, fixing two positive constants θ and η such that

$$\frac{\eta^p}{k\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx} < \frac{\theta^p}{\displaystyle\int_a^b \sup_{|t|\le\theta} F(x,t)dx},$$

and taking

$$\lambda \in \Lambda := \left] \frac{\rho_0\eta^p}{pk(b-a)^{p-1}} \frac{1}{\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx},\ \frac{\rho_0\theta^p}{p(b-a)^{p-1}} \frac{1}{\displaystyle\int_a^b \sup_{|t|\le\theta} F(x,t)dx} \right[,$$

set $\delta_{\lambda,g}$ given by

$$\min\left\{ \frac{\rho_0\theta^p - \lambda p(b-a)^{p-1}\displaystyle\int_a^b \sup_{|t|\le\theta} F(x,t)dx}{p(b-a)^{p-1}G^\theta},\ \frac{\rho_0\eta^p - \lambda pk(b-a)^{p-1}\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx}{pk(b-a)^p G_\eta} \right\} \tag{7}$$

and

$$\overline{\delta}_{\lambda,g} := \min\left\{ \delta_{\lambda,g},\ \frac{1}{\max\left\{0, \dfrac{p(b-a)^p}{\rho_0} \displaystyle\limsup_{|t|\to\infty} \frac{\sup_{x\in[a,b]} G(x,t)}{t^p}\right\}} \right\}, \tag{8}$$

where we read $\rho/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in[a,b]} G(x,t)}{t^p} \le 0,$$

and $G_\eta = G^\theta = 0$.

Now, we formulate our main result.

Theorem 3.1. *Assume that there exist two positive constants θ and η with $\theta < \eta$ such that*

(A_1) $\displaystyle\int_a^{\frac{a+b}{2}} F(x,\xi)dx > 0$, *for each* $\xi \in [0,\eta]$;

(A_2) $\dfrac{\displaystyle\int_a^b \sup_{|t|\le\theta} F(x,t)dx}{\theta^p} < k\dfrac{\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx}{\eta^p}$;

(A_3) $\displaystyle\limsup_{|t|\to+\infty} \frac{\sup_{x\in[a,b]} F(x,t)}{t^p} \le 0.$

Then, for each $\lambda \in \Lambda$ and for every L^1-Carathéodory function $g : [a,b]\times\mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in[a,b]} G(x,t)}{t^p} < +\infty,$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (8) such that, for each $\mu \in [0, \overline{\delta}_{\lambda,g}[$, the problem (1) admits at least three distinct weak solutions in X.

Proof. In order to apply Theorem 2.1 to our problem, we introduce the functionals $\Phi,\ \Psi : X \to \mathbb{R}$ for each $u \in X$, as follows

$$\Phi(u) = \frac{1}{p}\|u\|^p$$

and

$$\Psi(u) = \int_a^b [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx.$$

Let us prove that the functionals Φ and Ψ satisfy the required conditions.

It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_a^b [f(x,u(x)) + \frac{\mu}{\lambda}g(x,u(x))]v(x)dx,$$

for every $v \in X$ as well as is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this end, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$, then u_n converges uniformly to u on $[a,b]$ as $n \to \infty$; see [20]. Since f, g are L^1-Carathéodory functions, f, g are continuous in $\mathbb{R}$ for every $x \in [a,b]$, so

$$f(x,u_n) + \frac{\mu}{\lambda}g(x,u_n) \to f(x,u) + \frac{\mu}{\lambda}g(x,u),$$

as $n \to \infty$. Hence $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [20].

Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_a^b r(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dx,$$

for every $v \in X$, while Proposition 2.3 gives that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Clearly, the weak solutions of the problem (1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$.

Put $r := \dfrac{\rho_0\theta^p}{p(b-a)^{p-1}}$, and

$$w(x) := \begin{cases} \dfrac{2\eta}{b-a}(x-a) & \text{if } x \in [a, \frac{a+b}{2}[\\ \eta & \text{if } x \in [\frac{a+b}{2}, b]. \end{cases} \tag{9}$$

It is easy to see that $w \in X$ and, in particular, one has

$$\|w\|^p = \frac{2^p\eta^p}{(b-a)^p}\int_a^{\frac{a+b}{2}} \rho(x)dx + \frac{2^p\eta^p}{(b-a)^p}\int_a^{\frac{a+b}{2}} (x-a)^p s(x)dx + \eta^p\int_{\frac{a+b}{2}}^b s(x)dx.$$

Taking into account $0 < \theta < \eta$, using (6), we observe that

$$0 < r < \Phi(w) < \frac{\rho_0\eta^p}{pk(b-a)^{p-1}}.$$

Bearing in mind relation (2), we see that

$$\begin{aligned} \Phi^{-1}(]-\infty, r]) &= \{u \in X;\ \Phi(u) \le r\} \\ &= \left\{u \in X;\ \frac{||u||^p}{p} \le r\right\} \\ &\subseteq \{u \in X;\ |u(x)| \le \theta \text{ for each } x \in [a,b]\}, \end{aligned}$$

and it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r])}\int_a^b [F(x,u(x))+\frac{\mu}{\lambda}G(x,u(x))]dx\\ &\leq \int_a^b \sup_{|t|\leq\theta}F(x,t)dx+\frac{\mu}{\lambda}G^\theta.\end{aligned}$$

On the other hand, by using condition (A_1), since $0\leq w(x)\leq\eta$ for each $x\in[a,b]$, we infer

$$\begin{aligned}\Psi(w) &\geq \int_{\frac{a+b}{2}}^b F(x,\eta)dx+\frac{\mu}{\lambda}\int_a^b G(x,w(x))dx\\ &\geq \int_{\frac{a+b}{2}}^b F(x,\eta)dx+(b-a)\frac{\mu}{\lambda}\inf_{[a,b]\times[0,\eta]}G\\ &= \int_{\frac{a+b}{2}}^b F(x,\eta)dx+(b-a)\frac{\mu}{\lambda}G_\eta.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\int_a^b [F(x,u(x))+\frac{\mu}{\lambda}G(x,u(x))]dx}{r}\\ &\leq \frac{\int_a^b \sup_{|t|\leq\theta}F(x,t)dx+\frac{\mu}{\lambda}G^\theta}{\frac{\rho_0\theta^p}{p(b-a)^{p-1}}}, \end{aligned}\tag{10}$$

and

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)} &\geq \frac{\int_{\frac{a+b}{2}}^b F(x,\eta)dx+\frac{\mu}{\lambda}\int_a^b G(x,w(x))dx}{\frac{\rho_0\eta^p}{pk(b-a)^{p-1}}}\\ &\geq \frac{\int_{\frac{a+b}{2}}^b F(x,\eta)dx+(b-a)\frac{\mu}{\lambda}G_\eta}{\frac{\rho_0\eta^p}{pk(b-a)^{p-1}}}.\end{aligned}\tag{11}$$

Since $\mu<\delta_{\lambda,g}$, one has

$$\mu<\frac{\rho_0\theta^p-\lambda p(b-a)^{p-1}\int_a^b\sup_{|t|\leq\theta}F(x,t)dx}{p(b-a)^{p-1}G^\theta},$$

this means

$$\frac{\int_a^b \sup_{|t|\leq\theta}F(x,t)dx+\frac{\mu}{\lambda}G^\theta}{\frac{\rho_0\theta^p}{p(b-a)^{p-1}}}<\frac{1}{\lambda}.$$

Furthermore,

$$\mu<\frac{\rho_0\eta^p-\lambda pk(b-a)^{p-1}\int_{\frac{a+b}{2}}^b F(x,\eta)dx}{pk(b-a)^pG_\eta},$$

this means

$$\frac{\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx + (b-a)\frac{\mu}{\lambda}G_\eta}{\dfrac{\rho_0\eta^p}{pk(b-a)^{p-1}}} > \frac{1}{\lambda}.$$

Then,

$$\frac{\displaystyle\int_a^b \sup_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{\dfrac{\rho_0\theta^p}{p(b-a)^{p-1}}} < \frac{1}{\lambda} < \frac{\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx + (b-a)\frac{\mu}{\lambda}G_\eta}{\dfrac{\rho_0\eta^p}{pk(b-a)^{p-1}}}. \tag{12}$$

Hence from (10)-(12), we observe that the condition (a_1) of Theorem 2.1 is satisfied.

Finally, since $\mu < \overline{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in[a,b]} G(x,t)}{t^p} < l,$$

and $\mu l < \dfrac{\rho_0}{p(b-a)^p}$.

Therefore, there exists a function $h \in L^1([a,b])$ such that

$$G(x,t) \le lt^p + h(x), \tag{13}$$

for every $x \in [a,b]$ and $t \in \mathbb{R}$.

Now, fix $0 < \epsilon < \dfrac{\rho_0}{p(b-a)^p\lambda} - \dfrac{\mu l}{\lambda}$. From (A_3) there is a function $h_\epsilon \in L^1([a,b])$ such that

$$F(x,t) \le \epsilon t^p + h_\epsilon(x), \tag{14}$$

for every $x \in [a,b]$ and $t \in \mathbb{R}$.

Taking (2) into account, it follows that, for each $u \in X$,

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \frac{1}{p}\|u\|^p - \lambda\int_a^b [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx \\
&\ge \frac{1}{p}\|u\|^p - \lambda\epsilon\int_a^b u^p(x)dx - \lambda\|h_\epsilon\|_1 - \mu l\int_a^b u^p(x)dx - \mu\|h\|_1 \\
&\ge \Big(\frac{1}{p} - \lambda\frac{(b-a)^p}{\rho_0}\epsilon - \mu\frac{(b-a)^p}{\rho_0}l\Big)\|u\|^p - \lambda\|h_\epsilon\|_1 - \mu\|h\|_1,
\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty,$$

which means the functional $\Phi - \lambda\Psi$ is coercive, and the condition (a_2) of Theorem 2.1 is verified.

By using relations (10) and (12) one also has

$$\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\right[.$$

Finally, Theorem 2.1 (with $\overline{x} = w$) ensures the conclusion. □

Now, we present a variant of Theorem 3.1 in which no asymptotic condition on the nonlinear term is requested. In such a case f and g are supposed to be nonnegative.

For our goal, let us fix positive constants θ_1, θ_2 and η such that

$$\frac{3}{2}\frac{\eta^p}{k\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx} < \min\left\{\frac{\theta_1^p}{\displaystyle\int_a^b \sup_{|t|\le\theta_1} F(x,t)dx}, \frac{\theta_2^p}{2\displaystyle\int_a^b \sup_{|t|\le\theta_2} F(x,t)dx}\right\},$$

and taking

$$\lambda \in \Lambda := \left]\frac{3}{2}\frac{\dfrac{\rho_0\eta^p}{pk(b-a)^{p-1}}}{\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx}, \frac{\rho_0}{p(b-a)^{p-1}}\min\left\{\frac{\theta_1^p}{\displaystyle\int_a^b \sup_{|t|\le\theta_1} F(x,t)dx}, \frac{\theta_2^p}{2\displaystyle\int_a^b \sup_{|t|\le\theta_2} F(x,t)dx}\right\}\right[.$$

With the above notations we have the following multiplicity result.

Theorem 3.2. *Let $f : [a,b]\times\mathbb{R}\to\mathbb{R}$ satisfies the condition $f(x,t)\ge 0$ for every $(x,t)\in[a,b]\times(\mathbb{R}^+\cup\{0\})$. Assume that there exist three positive constants θ_1, θ_2 and η with $2^{1/p}\theta_1 < \eta < \dfrac{\theta_2}{2^{1/p}}$ such that assumption (A_1) in Theorem 3.1 holds. Furthermore, suppose that*

(B_1) $$\frac{\displaystyle\int_a^b \sup_{|t|\le\theta_1} F(x,t)dx}{\theta_1^p} < \frac{2}{3}k\frac{\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx}{\eta^p};$$

(B_2) $$\frac{\displaystyle\int_a^b \sup_{|t|\le\theta_2} F(x,t)dx}{\theta_2^p} < \frac{1}{3}k\frac{\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx}{\eta^p}.$$

Then, for each $\lambda\in\Lambda$ and for every nonnegative L^1-Carathéodory function $g : [a,b]\times\mathbb{R}\to\mathbb{R}$, there exists $\delta^*_{\lambda,g} > 0$ given by

$$\min\left\{\frac{\rho_0\theta_1^p - \lambda p(b-a)^{p-1}\displaystyle\int_a^b \sup_{|t|\le\theta_1} F(x,t)dx}{p(b-a)^{p-1}G^{\theta_1}}, \frac{\rho_0\theta_2^p - \lambda p(b-a)^{p-1}\displaystyle\int_a^b \sup_{|t|\le\theta_2} F(x,t)dx}{p(b-a)^{p-1}G^{\theta_2}}\right\}.$$

such that, for each $\mu\in[0,\delta^*_{\lambda,g}[$, the problem (1) admits at least three distinct weak solutions u_i for $i=1,2,3$, such that

$$0\le u_i(x) < \theta_2, \quad \forall\, x\in[a,b],\ (i=1,2,3).$$

Proof. Fix λ, g and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.2 on Φ and Ψ are satisfied. Then, our aim is to verify (b_1) and (b_2).

To this end, put w as given in (9), as well as

$$r_1 := \frac{\rho_0\theta_1^p}{p(b-a)^{p-1}},$$

and

$$r_2 := \frac{\rho_0\theta_2^p}{p(b-a)^{p-1}}.$$

By using condition $2^{1/p}\theta_1 < \eta < \dfrac{\theta_2}{2^{1/p}}$, and bearing in mind (6), we get $2r_1 < \Phi(w) < \dfrac{r_2}{2}$. Since $\mu < \delta^*_{\lambda,g}$ and $G_\eta = 0$, one has

$$
\begin{aligned}
\frac{\displaystyle\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{r_1} &= \frac{\displaystyle\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\int_a^b [F(x,u(x))+\frac{\mu}{\lambda}G(x,u(x))]dx}{r_1} \\
&\le \frac{\displaystyle\int_a^b \sup_{|t|\le\theta_1} F(x,t)dx + \frac{\mu}{\lambda}G^{\theta_1}}{\dfrac{\rho_0\theta_1^p}{p(b-a)^{p-1}}} \\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{\displaystyle\int_{\frac{a+b}{2}}^b F(x,\eta)dx + (b-a)\frac{\mu}{\lambda}G\eta}{\dfrac{\rho_0\eta^p}{pk(b-a)^{p-1}}} \\
&\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{2\displaystyle\sup_{u\in\Phi^{-1}(]-\infty,r_2])}\Psi(u)}{r_2} &= \frac{2\displaystyle\sup_{u\in\Phi^{-1}(]-\infty,r_2])}\int_a^b [F(x,u(x))+\frac{\mu}{\lambda}G(x,u(x))]dx}{r_2} \\
&\le \frac{2\displaystyle\int_a^b \sup_{|t|\le\theta_2} F(x,t)dx + 2\frac{\mu}{\lambda}G^{\theta_2}}{\dfrac{\rho_0\theta_2^p}{p(b-a)^{p-1}}} \\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{\displaystyle\int_{\frac{a+b}{2}}^b F(x,\eta)dx + (b-a)\frac{\mu}{\lambda}G_\eta}{\dfrac{\rho_0\eta^p}{pk(b-a)^{p-1}}} \\
&\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.
\end{aligned}
$$

Therefore, (b_1) and (b_2) of Theorem 2.2 are verified.

Finally, we verify that $\Phi - \lambda\Psi$ satisfies the assumption *2.* of Theorem 2.2. Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (1). We want to prove that they are nonnegative.

Let u_0 be a weak solution of problem (1). Arguing by a contradiction, assume that the set $A = \{x \in]a,b] : u_0(x) < 0\}$ is non-empty and of positive measure. Put $\bar{v}(x) = \min\{0, u_0(x)\}$ for all $x \in [a,b]$. Clearly, $\bar{v} \in X$ and one has

$$
\int_a^b \rho(x)|u_0'(x)|^{p-2}u_0'(x)\bar{v}'(x)dx + \int_a^b s(x)|u_0(x)|^{p-2}u_0(x)\bar{v}(x)dx
$$
$$
-\lambda\int_a^b f(x,u_0(x))\bar{v}(x)dx - \mu\int_a^b g(x,u_0(x))\bar{v}(x)dx = 0,
$$

for every $v \in X$.

Thus, from our sign assumptions on the data, we have

$$
0 \le \int_A \rho(x)|u_0'(x)|^p dx + \int_A s(x)|u_0(x)|^p dx \le 0.
$$

Hence, $u_0 = 0$ in A and this is absurd. Then, we deduce $u_1(x) \geq 0$ and $u_2(x) \geq 0$ for every $x \in [a,b]$. Thus, it follows that $su_1 + (1-s)u_2 \geq 0$ for all $s \in [0,1]$, and that

$$(\lambda f + \mu g)(x, su_1 + (1-s)u_2) \geq 0,$$

and consequently, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0,1]$.

By using Theorem 2.2, for every

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)},\ \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)},\ \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are the weak solutions of the problem (1) and the desired conclusion is achieved. $\square$

Now we prove Theorems 1.1 and 1.2 in Introduction.

Proof of Theorem 1.1: Fix $\lambda > \lambda^* := \dfrac{2\rho_0\eta^p}{pk(b-a)^p F(\eta)}$ for some $\eta > 0$.

Recalling that

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^2} = 0,$$

there is a sequence $\{\theta_n\} \subset]0,+\infty[$ such that $\lim\limits_{n\to\infty}\theta_n = 0$ and

$$\lim_{n\to\infty}\frac{\sup\limits_{|\xi|\leq\theta_n} F(\xi)}{\theta_n^p} = 0.$$

Indeed, one has

$$\lim_{n\to\infty}\frac{\sup\limits_{|\xi|\leq\theta_n} F(\xi)}{\theta_n^p} = \lim_{n\to\infty}\frac{F(\xi_{\theta_n})}{\xi_{\theta_n}^p}\frac{\xi_{\theta_n}^p}{\theta_n^p} = 0,$$

where $F(\xi_{\theta_n}) = \sup\limits_{|\xi|\leq\theta_n} F(\xi)$.

Hence, there exists $\overline{\theta} > 0$ such that

$$\frac{\sup\limits_{|\xi|\leq\overline{\theta}} F(\xi)}{\overline{\theta}^p} < \min\left\{\frac{kF(\eta)}{2\eta^p};\ \frac{\rho_0}{p\lambda(b-a)^p}\right\}$$

and $\overline{\theta} < \eta$.

The conclusion follows by using Theorem 3.1. $\square$

Proof of Theorem 1.2: Our aim is to employ Theorem 3.2 by choosing $a = 0$, $b = 1$, $\rho(x) = s(x) = 1$ (for every $x \in [a,b]$) $\theta_2 = 1$ and $\eta = 2$.

Therefore, since $k = 8/37$, we see that

$$\frac{3}{2}\frac{\dfrac{\rho_0\eta^p}{pk(b-a)^{p-1}}}{\displaystyle\int_{\frac{a+b}{2}}^{b} F(x,\eta)dx} = \frac{37}{\displaystyle\int_0^2 f(\xi)d\xi}$$

and

$$\frac{\rho_0}{p(b-a)^{p-1}}\frac{\theta_2^p}{2\displaystyle\int_a^b \sup_{|t|\leq\theta_2} F(x,t)dx} = \frac{1}{6\displaystyle\int_0^1 f(\xi)d\xi}.$$

Moreover, since $\lim_{t\to 0^+}\dfrac{f(t)}{t^2}=0$, one has

$$\lim_{t\to 0^+}\frac{\displaystyle\int_0^t f(\xi)d\xi}{t^3}=0.$$

Then, there exists a positive constant $\theta_1<\sqrt[3]{4}$ such that

$$\frac{\displaystyle\int_0^{\theta_1} f(\xi)d\xi}{\theta_1^3}<\frac{\displaystyle\int_0^{2} f(\xi)d\xi}{111},$$

and

$$\frac{\theta_1^3}{\displaystyle\int_0^{\theta_1} f(\xi)d\xi}>\frac{1}{2\displaystyle\int_0^{1} f(\xi)d\xi}.$$

Finally, a simple computation shows that all assumptions of Theorem 3.2 are fulfilled. The desired conclusion follows. □

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