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Continuous spectrum of a fourth order nonhomogeneous differential operators with variable exponent

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Abstract: In this article, we consider the nonlinear eigenvalue problem:

$$\begin{cases} \Delta(|\Delta u|^{p_1(x)-2}\Delta u) + \Delta(|\Delta u|^{p_2(x)-2}\Delta u) = \lambda |u|^{q(x)-2}u, \text{in } \Omega, \\ u = \Delta u = 0, \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary, λ is a positive real number, the continuous functions p_1, p_2, and q satisfy $1 < p_2(x) < q(x) < p_1(x) < \frac{N}{2}$ and $\max_{y\in\overline{\Omega}} q(y) < \frac{Np_2(x)}{N-2p_2(x)}$ for any $x \in \overline{\Omega}$. The main result of this paper establishes the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \le \lambda_1$ such that any $\lambda \in [\lambda_1, +\infty)$ is an eigenvalue, while and $\lambda \in (0, \lambda_0)$ is not an eigenvalue of the above problem.

Key words: Fourth order elliptic equation, eigenvalue problem, variable exponent, Sobolev space, critical point.

AMS Subject Classification: 35G30, 35J35, 35P30, 58E05.

§1 Introduction

The study of differential equations and variational problems with variable exponent has been a new and interesting topic. Its interest is widely justified with many physical examples, such as nonlinear elasticity theory, electrorheological fluids, etc. (see [1, 2]). It also has wide applications in different research fields, such as image processing model (see e.g. [3,4]), stationary thermorheological viscous flows (see [5]) and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium (see [6]).

In this paper, we are concerned with the study of the nonhomogeneous eigenvalue problem

$$\begin{cases} \Delta(|\Delta u|^{p_1(x)-2}\Delta u) + \Delta(|\Delta u|^{p_2(x)-2}\Delta u) = \lambda |u|^{q(x)-2}u, \text{in } \Omega, \\ u = \Delta u = 0, \text{on } \partial\Omega, \end{cases} \qquad (P)$$

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where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary, λ is a positive real number, and p_1, p_2, q are continuous functions on $\overline{\Omega}$.

The study of nonhomogeneous eigenvalue problems involving operators with variable exponents growth conditions has captured a special attention in the last few years. This is in keeping with the fact that operators which arise in such kind of problems, like the $p(x)$-Laplace operator (i.e., $\Delta(|\Delta u|^{p(x)-2}\Delta u)$, where $p(x)$ is a continuous positive function), are not homogeneous and thus, a large number of techniques which can be applied in the homogeneous case (when $p(x)$ is a positive constant) fail in this new setting. A typical example is the Lagrange multiplier theorem, which does not apply to the eigenvalue problem

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda |u|^{q(x)-2}u, \text{in } \Omega, \\ u = \Delta u = 0, \text{on } \partial\Omega, \end{cases} \tag{S}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary, λ is a positive real number, and p, q are continuous functions on $\overline{\Omega}$.

On the other hand, problems like (S) have been largely considered in the literature in the recent years. We give in what follows a concise but complete image of the actual stage of research on this topic.

In the case when $p(x) = q(x)$ on $\overline{\Omega}$, Ayoujil and Amrouss [7] established the existence of infinitely many eigenvalues for problem (S) by using an argument based on the Ljusternik-Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, they showed that $\sup \Lambda = +\infty$ and they pointed out that only under special conditions we have $\inf \Lambda = 0$. We remark that for the p-biharmonic operator (corresponding to $p(x) = p$) we always have $\inf \Lambda > 0$.

In the case when $\max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ it can be proved that the energy functional associated to problem (S) has a nontrivial minimum for any positive λ (see Theorem 3.1 in [8]).

In the case when $\min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth Ayoujil and Amrouss [8] used the Ekeland's variational principle in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin.

In the case when $\max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x) \le \max_{x\in\overline{\Omega}} q(x) < \frac{Np(x)}{N-2p(x)}$, by Theorem 3.8 in [8], for every $\lambda > 0$, the energy functional Φ_λ corresponding to (S) Mountain Pass type critical point which is nontrivial and nonnegative, and hence $\Lambda = (0, +\infty)$.

In this paper we study problem (P) under the following assumptions:

$\mathbf{H(p_1, p_2, q)}: 1 < p_2(x) < q^- \le q^+ < p_1(x) < \frac{N}{2}, \forall x \in \overline{\Omega}.$
$\quad q^+ < \frac{Np_2(x)}{N-2p_2(x)}, \forall x \in \overline{\Omega},$

where $q^- = \min_{x\in\overline{\Omega}} q(x)$ and $q^+ = \max_{x\in\overline{\Omega}} q(x)$.

Inspired by the above-mentioned papers, we study problem (P) from all the cases studied before. In this new situation we will show the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \le \lambda_1$ such that any $\lambda \ge \lambda_1$ is an eigenvalue of problem (P) while any $0 < \lambda < \lambda_0$ is not an eigenvalue of problem (P).

Need to note that our methods in this paper can be applied to the problems in the special case when $p(x)$ and $q(x)$ are all constants. Recently, we find that [9] use quite different methods from ours to consider such special problems and obtain better results.

This paper is composed of three sections. In Section 2, we recall the definition of variable exponent Lebesgue spaces, $L^p(x)(\Omega)$, as well as Sobolev spaces, $W^{1,p(x)}(\Omega)$. Moreover, some properties of these spaces will be also exhibited to be used later. In Section 3, we give the main results and their proofs.

§2 Preliminaries

Firstly, we introduce some theories of Lebesgue-Sobolev space with variable exponent. The detailed can be found in [10–12].

Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for any } x \in \overline{\Omega}\}.$$

Define

$$h^- = \min_{x\in\overline{\Omega}} h(x), \quad h^+ = \max_{x\in\overline{\Omega}} h(x) \text{ for any } h \in C_+(\overline{\Omega}).$$

For $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space:

$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real value function } \int_\Omega |u(x)|^{p(x)}dx < +\infty\}$,

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \le 1\}$,

and define the variable exponent Sobolev space

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\},$$

with the norm $\|u\|_{W^{k,p(x)}(\Omega)} = \|u\|_{k,p(x)} = \sum_{|\alpha|\le k} |D^\alpha u|_{p(x)}$.

We remember that spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are separable and reflexive Banach spaces. Denoting by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

For $p(x) \in C_+(\overline{\Omega})$, by $L^{q(x)}(\Omega)$ we denote the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then the Hölder type inequality

$$\int_\Omega |uv|dx \le (\frac{1}{p^-} + \frac{1}{q^-})|u|_{L^{p(x)}(\Omega)}|v|_{L^{q(x)}(\Omega)}, \ u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega) \quad (1)$$

holds. Furthermore, define mapping $\rho : L^{p(x)} \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega |u|^{p(x)}dx,$$

then the following relations hold

$$|u|_{p(x)} < 1(= 1, > 1) \Leftrightarrow \rho(u) < 1(= 1, > 1), \quad (2)$$

$$|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}, \quad (3)$$

$$|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}. \quad (4)$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho(u_n - u) \to 0. \quad (5)$$

Definition 2.1. Assume that spaces E, F are Banach spaces, we define the norm on the space $X := E \bigcap F$ as $\|u\|_X = \|u\|_E + \|u\|_F$.

In order to discuss problem (P), we need some theories on space $X_i := W_0^{1,p_i(x)}(\Omega) \bigcap W^{2,p_i(x)}(\Omega)(i = 1, 2)$. Since $p_1(x) > p_2(x)$ for any $x \in \overline{\Omega}$, so the space $W_0^{1,p_1(x)}(\Omega)$ is continuously embedded in $W_0^{1,p_2(x)}(\Omega)$, $W^{2,p_1(x)}(\Omega)$ is continuously embedded in $W^{2,p_2(x)}(\Omega)$, so X_1 is continuously embedded in X_2.

From the Definition 2.1, it follows that for any $u \in X_1$, $\|u\|_1 = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}$, thus $\|u\|_1 = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(x)}$.

In Zanga and Fu [13], the equivalence of the norms was proved, and it was even proved that the norm $|\Delta u|_{p(x)}$ is equivalent to the norm $\|u\|_1$ (see [13, Theorem 4.4]).

Let us choose on X_1 the norm defined by $\|u\|_1 = |\Delta u|_{p(x)}$. Note that, $(X_1, \|\cdot\|_1)$ is also a separable and reflexive Banach space. Similar to (2),(3), (4) and (5), we have the following, define mapping $\rho_1 : X_1 \to \mathbb{R}$ by

$$\rho_1(u) = \int_\Omega |\Delta u|^{p(x)} dx,$$

then the following relations hold

$$\|u\|_1 < 1 (= 1, > 1) \Leftrightarrow \rho_1(u) < 1 (= 1, > 1), \tag{6}$$

$$\|u\|_1 > 1 \Rightarrow \|u\|_1^{p^-} \leq \rho_1(u) \leq \|u\|_1^{p^+}, \tag{7}$$

$$\|u\|_1 < 1 \Rightarrow \|u\|_1^{p^+} \leq \rho_1(u) \leq \|u\|_1^{p^-}. \tag{8}$$

$$\|u_n - u\|_1 \to 0 \Leftrightarrow \rho_1(u_n - u) \to 0. \tag{9}$$

Hereafter, let

$$p_1^*(x) = \begin{cases} \dfrac{Np_1(x)}{N - 2p_1(x)}, & p_1(x) < \dfrac{N}{2}, \\ +\infty, & p_1(x) \geq \dfrac{N}{2}, \end{cases}$$

and

$$p_2^*(x) = \begin{cases} \dfrac{Np_2(x)}{N - 2p_2(x)}, & p_2(x) < \dfrac{N}{2}, \\ +\infty, & p_2(x) \geq \dfrac{N}{2}. \end{cases}$$

Remark 2.1. If $h \in C_+(\overline{\Omega})$ and $h(x) < p_i^*(x)$ $(i = 1$ or $2)$ for any $x \in \overline{\Omega}$, by Theorem 3.2 in [7], we deduce that X_i is continuously and compact embedded in $L^{h(x)}(\Omega)$.

Remark 2.2. Since $p_2(x) < p_1(x)$ for any $x \in \Omega$ it follows that $p_2^*(x) < p_1^*(x)$, using condition $\mathbf{H(p_1, p_2, q)}$ we have a compact embedding $X_i \hookrightarrow L^{q(x)}(\Omega) (i = 1, 2)$.

§3 The main results and proof of the theorem

Since $p_2(x) < p_1(x)$ for any $x \in \Omega$ it follows that $W_0^{1,p_1(x)}(\Omega)$ $(W^{2,p_1(x)}(\Omega))$ is continuously embedded in $W^{2,p_2(x)}(\Omega)$. Thus , a solution for a problem of type (P) will be sought in the variable exponent space $X_1 = W_0^{1,p_1(x)}(\Omega) \bigcap W^{2,p_1(x)}(\Omega)$.

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (P) if there exists $u \in X_1 \backslash \{0\}$ such that

$$\int_\Omega (|\Delta u|^{p_1(x)-2} + |\Delta u|^{p_2(x)-2}) \Delta u \Delta v dx - \lambda \int_\Omega |u|^{q(x)-2} uv dx = 0,$$

for all $u \in X_1$. We point out that if λ is an eigenvalue of problem (P), then the corresponding eigenfunction $u \in X_1 \backslash \{0\}$ is a weak solution of problem (P).

Define

$$\lambda_1 := \inf_{u \in X_1 \backslash \{0\}} \frac{\int_\Omega \frac{1}{p_1(x)} |\Delta u|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\Delta u|^{p_2(x)} dx}{\int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx}.$$

Our main goal is to prove the following result:

Theorem 3.1. Assume that $\mathbf{H(p_1, p_2, q)}$ holds. Then $\lambda_1 > 0$. Moreover, any $\lambda \in [\lambda_1, +\infty)$ is an eigenvalue of problem (P). Furthermore, there exists a positive constant λ_0 such that $\lambda_0 \leq \lambda_1$ and any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (P).

Proof: The proof is divided into the following four Steps.

Step 1. We will show that $\lambda_1 > 0$.

Since $p_1(x) > q^+ \geq q(x) \geq q^- > p_2(x), \forall x \in \overline{\Omega}$, we deduce that for every $u \in X_1$,

$$|\Delta u|^{q^+} + |\Delta u|^{q^-} \leq 2(|\Delta u|^{p_1(x)} + |\Delta u|^{p_2(x)})$$

and

$$|u(x)|^{q(x)} \leq |u(x)|^{q^+} + |u(x)|^{q^-}.$$

Integrating the above inequalities, for any $u \in X_1$, we have

$$\int_\Omega (|\Delta u|^{q^+} + |\Delta u|^{q^-})dx \leq 2\int_\Omega (|\Delta u|^{p_1(x)} + |\Delta u|^{p_2(x)})dx \tag{10}$$

and

$$\int_\Omega |u(x)|^{q(x)}dx \leq \int_\Omega (|u(x)|^{q^+} + |u(x)|^{q^-})dx. \tag{11}$$

Using again the fact that $q^- < \frac{Nq^-}{N-2q^-}$ and $q^+ < \frac{Nq^+}{N-2q^+}$, we deduce that $W_0^{1,q^-}(\Omega) \cap W^{2,q^-}(\Omega)$ is continuously embedded in $L^{q^-}(\Omega)$, and $W_0^{1,q^+}(\Omega) \cap W^{2,q^+}(\Omega)$ is continuously embedded in $L^{q^+}(\Omega)$, there exist two positive constants $c_1, c_2 > 0$ such that

$$\begin{aligned} \int_\Omega |u|^{q^-}dx &\leq c_1 \int_\Omega |\Delta u|^{q^-}dx, \ \forall u \in W_0^{1,q^-}(\Omega) \cap W^{2,q^-}(\Omega) \\ \int_\Omega |u|^{q^+}dx &\leq c_2 \int_\Omega |\Delta u|^{q^+}dx, \ \forall u \in W_0^{1,q^+}(\Omega) \cap W^{2,q^+}(\Omega). \end{aligned} \tag{12}$$

By $q^- \leq q^+ < p_1^*(x)$ for any $x \in \overline{\Omega}$ we deduce that X_1 is continuously embedded in $W_0^{1,q^-}(\Omega) \cap W^{2,q^-}(\Omega)$ and in $W_0^{1,q^+}(\Omega) \cap W^{2,q^+}(\Omega)$. Thus, inequalities (12) hold true for any $u \in X_1$.

Using inequalities (11) and (12) it is clear that there exists a positive constant c_3 such that

$$\int_\Omega |u(x)|^{q(x)}dx \leq c_3(\int_\Omega |\Delta u|^{q^-}dx + \int_\Omega |\Delta u|^{q^+}dx), \ \forall u \in X_1. \tag{13}$$

From (13) and (10) we get the estimate

$$\int_\Omega |u(x)|^{q(x)}dx \leq 2c_3(\int_\Omega |\Delta u|^{p_1(x)}dx + \int_\Omega |\Delta u|^{p_2(x)}dx), \ \forall u \in X_1. \tag{14}$$

Consequently, from (14) we obtain

$$
\begin{aligned}
\lambda_1 &= \inf_{u\in X_1\setminus\{0\}} \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx}\\
&\geq \inf_{u\in X_1\setminus\{0\}} \frac{\frac{1}{p_1^+}\int_\Omega |\Delta u|^{p_1(x)}dx + \frac{1}{p_2^+}\int_\Omega |\Delta u|^{p_2(x)}dx}{\frac{1}{q^-}\int_\Omega |u|^{q(x)}dx}\\
&\geq \inf_{u\in X_1\setminus\{0\}} \frac{\frac{1}{p_1^+}\int_\Omega |\Delta u|^{p_1(x)}dx + \frac{1}{p_1^+}\int_\Omega |\Delta u|^{p_2(x)}dx}{\frac{1}{q^-}\int_\Omega |u|^{q(x)}dx}\\
&= \frac{q^-}{p_1^+}\inf_{u\in X_1\setminus\{0\}} \frac{\int_\Omega |\Delta u|^{p_1(x)}dx + \int_\Omega |\Delta u|^{p_2(x)}dx}{\int_\Omega |u|^{q(x)}dx}\\
&\geq \frac{q^-}{2c_3p_1^+} > 0.
\end{aligned}
$$

Step 2. We will show that λ_1 is an eigenvalue of problem (P).

Claim 1:

$$
\lim_{\|u\|_1\to 0} \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx} = +\infty, \tag{15}
$$

$$
\lim_{\|u\|_1\to \infty} \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx} = +\infty. \tag{16}
$$

In fact, since X_2 is continuously embedded in $L^{q^\pm}(\Omega)$ it follows that there exist two positive constants c_4 and c_5 such that

$$
|u|_{q^-} \leq c_4\|u\|_2 \text{ and } |u|_{q^+} \leq c_5\|u\|_2,\ \forall u \in X_2. \tag{17}
$$

By X_1 is continuously embedded in X_2, we have

$$
\|u\|_1 \to 0 \Rightarrow \|u\|_2 \to 0.
$$

For any $u \in X_1$ with $\|u\|_1 < 1$ small enough, by (8), (11), (17) we have

$$
\begin{aligned}
&\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx}\\
&\geq \frac{\frac{1}{p_2^+}\int_\Omega |\Delta u|^{p_2(x)}dx}{\frac{|u|_{q^-}^{q^-}+|u|_{q^+}^{q^+}}{q^-}}\\
&\geq \frac{q^-}{p_2^+}\frac{\|u\|_2^{p_2^+}}{c_4^{q^-}\|u\|_2^{q^-}+c_5^{q^+}\|u\|_2^{q^+}}.
\end{aligned}
$$

Since $p_2^+ < q^- \leq q^+$, passing to the limit as $\|u\|_1 \to 0$ in the above inequality we deduce that relation (15) holds true.

On the other hand, since X_1 is continuously embedded in $L^{q^\pm}(\Omega)$ it follows that there exist two positive constants c_6 and c_7 such that

$$
|u|_{q^-} \leq c_6\|u\|_1 \text{ and } |u|_{q^+} \leq c_7\|u\|_1,\ \forall u \in X_1. \tag{18}
$$

Thus, for any $u \in X_1$ with $\|u\|_1 > 1$, by (7), (11), (18) we have

$$\begin{aligned}&\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx}\\ \geq &\frac{\frac{1}{p_1^+}\int_\Omega |\Delta u|^{p_1(x)}dx}{\frac{|u|_{q^-}^{q^-}+|u|_{q^+}^{q^+}}{q^-}}\\ \geq &\frac{q^-}{p_2^+}\frac{\|u\|_1^{p_1^-}}{c_6^{q^-}\|u\|_1^{q^-}+c_7^{q^+}\|u\|_1^{q^+}}.\end{aligned}$$

Since $p_1^- > q^+ \geq q^-$, passing to the limit as $\|u\|_1 \to +\infty$ in the above inequality we deduce that relation (16) holds true.

Claim 2: there exists $u_0 \in X_1 \backslash \{0\}$ such that

$$\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u_0|^{q(x)}dx} = \lambda_1.$$

In fact, let $\{u_n\} \subseteq X_1 \setminus \{0\}$ be a minimizing sequence for λ_1, that is,

$$\lim_{n\to\infty}\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx} = \lambda_1 > 0. \tag{19}$$

By Claim 1, we have $\{u_n\}$ is bounded in X_1. Since X_1 is reflexive it follows that there exists $u_0 \in X_1$ such that $u_n \rightharpoonup u_0$ in X_1. On the other hand, the function $\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx : X_1 \to \mathbb{R}$ is a convex function, hence it is weakly lower semi-continuous, thus

$$\begin{aligned}&\liminf_{n\to\infty}\Big(\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx\Big)\\ \geq &\Big(\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx\Big).\end{aligned} \tag{20}$$

Note that X_1 is compactly embedded in $L^{q(x)}(\Omega)$, thus,

$$u_n \to u_0 \text{ in } L^{q(x)}(\Omega). \tag{21}$$

By (5) and (21), it is easy to see that

$$\lim_{n\to\infty}\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx = \int_\Omega \frac{1}{q(x)}|u_0|^{q(x)}dx. \tag{22}$$

In view of (20) and (22), we obtain

$$\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u_0|^{q(x)}dx} = \lambda_1, \quad \text{if } u_0 \neq 0.$$

It remains to show that u_0 is nontrivial. Assume by contradiction the contrary. Then $u_n \rightharpoonup 0$ in X_1 and $u_n \to 0$ in $L^{q(x)}(\Omega)$. Thus, we have

$$\lim_{n\to\infty}\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx = 0. \tag{23}$$

Taking $\varepsilon \in (0, \lambda_1)$ be fixed. By (19) we deduce that for n large enough we have

$$\Big|\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx - \lambda_1\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx\Big| < \varepsilon\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx,$$

which deduces that

$$\begin{aligned}&(\lambda_1-\varepsilon)\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx\\ <&\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx \\ <&(\lambda_1+\varepsilon)\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx.\end{aligned} \tag{24}$$

Combining (23) and (24), we have

$$\lim_{n\to\infty}\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx=0. \tag{25}$$

By (9) and (25), we have

$$u_n\to 0 \ \text{ in } X_1, \ \text{that is } \|u_n\|_1\to 0.$$

From this information and relation (15), we get

$$\lim_{\|u_n\|_1\to 0}\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx}=+\infty$$

and this is a contradiction. Thus $u_0\neq 0$.

By Claim 2 we conclude that there exists $u_0\in X_1\setminus\{0\}$ such that

$$\begin{aligned}\lambda_1=&\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u_0|^{q(x)}dx}\\ =&\inf_{u\in X_1\setminus\{0\}}\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx}.\end{aligned}$$

Then for any $v\in X_1$ we have

$$\frac{d}{dt}\Big(\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta(u_0+tv)|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta(u_0+tv)|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u_0+tv|^{q(x)}dx}\Big)\Big|_{t=0}=0.$$

A simple computation yields

$$\begin{aligned}&\big(\int_\Omega(|\Delta u_0|^{p_1(x)-2}+|\Delta u_0|^{p_2(x)-2})\Delta u_0\Delta v dx\big)\times\int_\Omega \frac{1}{q(x)}|u_0|^{q(x)}dx\\ =&\big(\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx\big)\times\int_\Omega |u_0|^{q_2(x)-2}u_0 v dx\end{aligned} \tag{26}$$

for any $v\in X_1$.

Returning to (26) and using

$$\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u_0|^{q(x)}dx}=\lambda_1$$

and $\int_\Omega \frac{1}{q(x)}|u_0|^{q(x)}dx\neq 0$, we obtain λ_1 is an eigenvalue of problem (P). Thus, Step 2 is verified.

Step 3. We will show that any $\lambda\in(\lambda_1,+\infty)$ is an eigenvalue of problem (P).

Let $\lambda\in(\lambda_1,+\infty)$ be arbitrary but fixed. Define $T_\lambda: X_1\to\mathbb{R}$ by

$$T_\lambda(u)=\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx-\lambda\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx.$$

Clearly, $T_\lambda\in C^1(X_1,\mathbb{R})$ with

$$\langle T_\lambda'(u),v\rangle=\int_\Omega(|\Delta u|^{p_1(x)-2}+|\Delta u|^{p_2(x)-2})\Delta u\Delta v dx\big)-\lambda\int_\Omega |u|^{q(x)-2}uv dx,\forall u\in X_1.$$

Thus, λ is an eigenvalue of problem (P) if and only if there exists $u_\lambda \in X_1 \setminus \{0\}$ a critical point of T_λ.

It follows from (7) and (18) that

$$\begin{aligned} T_\lambda(u) =& \int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\ \geq& \frac{1}{p_1^+}\|u\|_1^{p_1^-} - \frac{\lambda}{q^-}(|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}) \\ \geq& \frac{1}{p_1^+}\|u\|_1^{p_1^-} - \frac{\lambda}{q^-}(c_6\|u\|_1^{q^+} + c_7\|u\|_1^{q^-}) \\ \to& \infty, \text{as } \|u\|_1 \to +\infty, \end{aligned}$$

since $p_1^- > q^+ \geq q^-$, i.e. $\lim_{\|u\|_1 \to +\infty} T_\lambda(u) = \infty$.

On the other hand, we recall that the functional T_λ is weakly lower semi-continuous (see for example Ge [14, Theorem 3.1]). So by Weierstrass theorem, there exists $u_\lambda \in X_1$ a global minimum point of T_λ and thus, a critical point of T_λ. To prove that u_λ is nontrivial.

Indeed, since

$$\lambda_1 = \inf_{u \in X_1 \setminus \{0\}} \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx}{\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx}$$

and $\lambda > \lambda_1$ it follows that there exists $v_\lambda \in X_1$ such that $T_\lambda(v_\lambda) < 0$, that is

$$\int_\Omega \frac{1}{p_1(x)}|\Delta v_\lambda|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta v_\lambda|^{p_2(x)}dx < \lambda \int_\Omega \frac{1}{q(x)}|v_\lambda|^{q(x)}dx.$$

Thus,

$$\inf_{u \in X_1} T_\lambda(u) < 0,$$

and we conclude that u_λ is a nontrivial critical point of T_λ, and then λ is an eigenvalue of problem (P).

Step 4. We will show that any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (P), where λ_0 is given by

$$\lambda_0 := \inf_{u \in X_1 \setminus \{0\}} \frac{\int_\Omega (|\Delta u|^{p_1(x)}dx + |\Delta u|^{p_2(x)})dx}{\int_\Omega |u|^{q(x)}dx}.$$

Firstly, we verify the $\lambda_0 \leq \lambda_1$. Due to Step 2, λ_1 and u_0 is an eigenvalue and is an eigenfunction corresponding to λ_1 of (P), then for every $v \in X_1$ we have

$$\int_\Omega (|\Delta u_0|^{p_1(x)-2} + |\Delta u_0|^{p_2(x)-2})\Delta u_0 \Delta v dx = \lambda_1 \int_\Omega |u|^{q_2(x)-2}u_0 v dx, \tag{27}$$

which implies that

$$\int_\Omega (|\Delta u_0|^{p_1(x)-2} + |\Delta u_0|^{p_2(x)-2})\Delta u_0 \Delta u_0 dx = \lambda_1 \int_\Omega |u_0|^{q_2(x)-2}u_0 u_0 dx,$$

that is,

$$\int_\Omega (|\Delta u_0|^{p_1(x)} + |\Delta u_0|^{p_2(x)})dx = \lambda_1 \int_\Omega |u_0|^{q_2(x)}dx.$$

Then, it follows that

$$\lambda_0 \leq \lambda_1.$$

Now we prove our assertion, arguing by contradiction: assume that there exists $\lambda \in (0, \lambda_0)$

is an eigenvalue of problem (P). Thus, there exists $u_\lambda \in X_1 \setminus \{0\}$ such that

$$\int_\Omega (|\Delta u_\lambda|^{p_1(x)-2} + |\Delta u_\lambda|^{p_2(x)-2})\Delta u_\lambda \Delta v dx = \lambda \int_\Omega |u_\lambda|^{q_2(x)-2} u_\lambda v dx \tag{28}$$

for any $v \in X_1$. Thus, for $v = u_\lambda$ we have

$$\int_\Omega (|\Delta u_\lambda|^{p_1(x)} + |\Delta u_\lambda|^{p_2(x)})dx = \lambda \int_\Omega |u_\lambda|^{q_2(x)} dx.$$

The fact that $u_\lambda \in X_1 \setminus \{0\}$ assures that $\int_\Omega |u_\lambda|^{q_2(x)} dx > 0$. Since $\lambda < \lambda_0$, the above information yields

$$\begin{aligned} &\int_\Omega (|\Delta u_\lambda|^{p_1(x)} + |\Delta u_\lambda|^{p_2(x)})dx \\ \geq &\lambda_0 \int_\Omega |u_\lambda|^{q_2(x)} dx \\ > &\lambda \int_\Omega |u_\lambda|^{q_2(x)} dx \\ = &\int_\Omega (|\Delta u_\lambda|^{p_1(x)} + |\Delta u_\lambda|^{p_2(x)})dx. \end{aligned}$$

Clearly, the above inequalities lead to a contradiction. The proof of theorem is now complete. □

Remark 3.1. In Theorem 3.1, we are not able to deduce whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. In the latter case an interesting question concerns the existence of eigenvalue of the problem (P) in the interval $[\lambda_0, \lambda_1)$. We leave it to interested readers to investigate this open problem.

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