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On the minimum of certain functional related to the Schrödinger equation

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Abstract

We consider the infimum $\inf_f \max\limits_{j=1,2,3} \|f^{(j)}\|_{L^\infty(0,T_0)}$, where the infimum is taken over every function f which runs through the set $KC^3(0,T_0)$ consisting of all functions $f:[0,T_0]\to\mathbb{R}$ satisfying the boundary conditions $f^{(j)}(0)=a_j$, $f^{(j)}(T_0)=0$ for $j=0,1,2$, whose derivatives $f^{(j)}$ are continuous for $j=0,1,2$ and the third derivative $f^{(3)}$ may have a finite number of discontinuities in the interval $(0,T_0)$, and find this infimum explicitly for certain choice of boundary conditions. This problem is motivated by some conditions under which the solution of the nonlinear Schrödinger equation with periodic boundary condition blows up in a finite time.

Keywords: Nonfixed termination time, optimal control problem, periodic boundary conditions, Schrödinger equation, blow up, minimum.

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1 Introduction

Let $KC^3(A,B)$ be a class consisting of all functions $f : [A,B] \to \mathbb{R}$ satisfying the following conditions: $D^j f \in C(A,B)$, i.e. the jth derivative of f is continuous for $j = 0,1,2$, but $D^3 f$ may have a finite number of discontinuities in the interval (A,B). (Here, A can be $-\infty$ and B can be ∞.) Throughout, we denote $D = \frac{\partial}{\partial x}$ and $D^j f = f^{(j)}$ for $j \geqslant 0$, so, in particular, $D^0 f = f$. In this paper we consider the functional

$$\begin{aligned} M(f) &= \max_{j=1,2,3} \|D^j f\|_{L^\infty(0,T_0)} \\ &= \max\{\max_{0\leqslant x\leqslant T_0} |f^{(1)}(x)|, \max_{0\leqslant x\leqslant T_0} |f^{(2)}(x)|, \max_{0\leqslant x\leqslant T_0} |f^{(3)}(x)|\}, \end{aligned}$$

where the real valued function $f \in KC^3(0,T_0)$ satisfies the boundary conditions

$$D^j f(0) = a_j \quad \text{and} \quad D^j f(T_0) = 0 \quad \text{for} \quad j = 0,1,2. \tag{1}$$

We find $\inf M(f)$, where the infimum is taken over each f lying in the class $KC^3(0,T_0)$ with boundary conditions (1) for some choice of $a_0, a_1, a_2 \in \mathbb{R}$ and $T_0 > 0$.

The question concerning $\inf M(f)$ arises from certain nonlinear Schrödinger equation, where one needs to estimate the integral

$$\Big| \int_I D^j f(x)|u(x)|\mathrm{d}x \Big| \leqslant \|D^j f\|_{L^\infty(I)} \int_I |u(x)|\mathrm{d}x \leqslant M(f) \int_I |u(x)|\mathrm{d}x \tag{2}$$

in proving that the solution of the nonlinear Schrödinger equation with periodic boundary condition

$$i\frac{\partial u}{\partial t} + D^2 u = -|u|^4 u, \quad t \geqslant 0, \quad x \in I, \tag{3}$$

$$u(0,x) = u_0(x), \quad x \in I, \tag{4}$$

$$u(t,-2) = u(t,2), \quad t \geqslant 0, \tag{5}$$

blows up, i.e.

$$\|Du\|_{L^2(I)} \to \infty \quad \text{as} \quad t \to t_0$$

(see [10]). Here $I = (-2, 2)$.

One of the aims of this paper is to investigate the size of the constant $M = M(f)$ in (2). In general, our results can be applied to all mathematical problems, where the estimates (2) are used for $f(x) \in KC^3(\mathbb{R})$ such that $D^j f(x) = 0$ for $x \notin I$, $j = 0, 1, 2$. For example, in some problems of mathematical physics the estimates for derivatives of a truncated function have been used (see [10], [11] and Theorem 1 below).

The blow up problem of the solution given by (3) and (4) in the whole real line $I = \mathbb{R}$ has been studied by many authors; see, for example, [4], [9], [11], [14]. Put

$$E(u) = \|Du\|^2_{L^2(I)} - \frac{1}{3}\|u\|^6_{L^6(I)}.$$

In the case $I = \mathbb{R}$, the inequality $E(u_0) < 0$ is a sufficient condition for the solution of (3) and (4) to blow up at finite time $t_0 > 0$ (see [4]). However, in general the condition $E(u_0) < 0$ is not sufficient for the collapse of (3) – (5) (see [10]).

The nonlinear Schrödinger equation with periodic boundary condition have been considered in [2], [5], [6], [10]. Some problems related to Schrödinger equation in bounded domain have been studied in [12], [13], etc. Ogawa and Tsutsumi [10] found a sufficient condition for the blow up of the solution of (3) – (5). Before stating their result let us first give some notation. Assume that $\phi(x) = -\phi(-x)$, $D^j\phi \in L^\infty(\mathbb{R})$ for $j = 0, 1, 2, 3,$

$$\phi(x) = \begin{cases} x, & 0 \leqslant x < 1, \\ x - (x-1)^3, & 1 \leqslant x < 1 + 1/\sqrt{3}, \\ \text{arbitrary}, & 1 + 1/\sqrt{3} \leqslant x < 2, \\ 0, & 2 \leqslant x, \end{cases} \tag{6}$$

and $D\phi(x) \leqslant 0$ for $x \geqslant 1 + 1/\sqrt{3}$. Of course, although $\phi(x)$ is arbitrary in the interval $[1+1/\sqrt{3}, 2)$, the function $\phi(x)$ still must satisfy $D^j\phi \in L^\infty(\mathbb{R})$

for $j = 0, 1, 2, 3$. Set $\Phi(x) = \int\limits_0^x \phi(y)\mathrm{d}y$. The sufficient conditions of blow up solution is the following theorem in [10].

Theorem 1 *Let* $u_0 \in H^1(I)$, $u_0(-2) = u_0(2)$ *and* $E(u_0) < 0$. *In addition we assume that*

$$\eta = -2E(u_0) - 80(1+M)^2\|u_0\|_{L^2(I)}^6 - \frac{M}{2}\|u_0\|_{L^2(I)}^2 > 0, \tag{7}$$

$$\Big(\int_I \Phi(x)|u_0(x)|^2\mathrm{d}x\Big)\Big(\frac{2}{\eta}\|Du_0\|_{L^2(I)}^2 + 1\Big) \leqslant \frac{1}{16}, \tag{8}$$

where $M = \sum_{j=1}^3 \|D^j\phi\|_{L^\infty(I)}$. *Then the solution* $u(t)$ *in* $H^1(\mathbb{R})$ *blows up in a finite time.*

The theorem raises the following natural question: *how small can the constant* M *be?* In the present note we shall answer this question. Clearly, the functional $M = \sum_{j=1}^3 \|D^j\phi\|_{L^\infty(I)}$ of Theorem 1 can be replaced by the smaller functional

$$M(\phi) = \max_{j=1,2,3} \|D^j\phi\|_{L^\infty(I)} = \max_{j=1,2,3} \|D^j\phi\|_{L^\infty(\mathbb{R})},$$

because in the proof of Theorem 1 the authors used the estimate (2).

The results of the present paper make Theorem 1 applicable in practice. Take the initial function $u_0 \in H^1(I)$, $u_0(-2) = u_0(2)$. To answer the question on whether the solution of (3)-(5) blows up one needs verify conditions (7) and (8). However, we cannot verify the conditions (7) and (8), if we do not know how small M is, i.e. we cannot use this result in practice. For this one can apply Theorem 1 using the results of Theorem 2 below.

Evaluating the constant M and finding its exact value is complicated, because the function $\phi(x)$ is not defined in the interval $(1 + 1/\sqrt{3}, 2)$. Numerical calculations show that $M \leqslant 1076.007\ldots$ if we restrict the search to polynomials of degree at most 5 in the interval $(1 + 1/\sqrt{3}, 2)$ such that $\phi(x) \in KC^3(\mathbb{R})$. Below, we will show that the smallest value of M is $562.986\ldots$.

Weinstein [14] determined the best (smallest) constant $C_{\sigma,n}^{2\sigma+2}$ in the n dimensional case for the interpolation estimate

$$\|f\|_{2\sigma+2}^{2\sigma+2} \leqslant C_{\sigma,n}^{2\sigma+2} \|\nabla f\|_2^{\sigma n} \|f\|_2^{2+\sigma(2-n)}, \quad 0 < \sigma < 2/(n-2), \quad n \geqslant 2$$

and obtained a sufficient condition for the global existence of the solution of the Schrödinger equation. For this, he solved the Euler-Lagrange equation minimizing the functional

$$\frac{\|\nabla f\|_2^{\sigma n} \|f\|_2^{2+\sigma(2-n)}}{\|f\|_{2\sigma+2}^{2\sigma+2}}.$$

The case of the functional $M(f) = \max\limits_{j=1,2,3} \|D^j f\|_{L^\infty(0,T_0)}$ is more complicated, because the derivative of this functional does not exist, so we cannot solve the corresponding Euler-Lagrange equation. For minimizing the functional $M(f)$ we shall use the optimal control problem with a nonfixed termination time. The optimal control problem is one of the cases of Pontryagin's maximum principle and was considered in many papers; see, for example, [1], [7], [8] and the references in those papers.

We solve the following optimal control problem with a nonfixed termination time:

$$\begin{aligned} T \to \min, \quad D^j f(0) = a_j, \quad D^j f(T) = 0, \qquad j = 0, 1, 2,\\ |D^3 f(x)| \leqslant a, \quad f \in KC^3, \end{aligned} \tag{9}$$

i.e. we find the minimal number T for which the conditions (9) are satisfied. We find that the minimal number T is attained at the function (13), (14) or (15) below. We consider only the function (15) with $\delta = -1$ because this function is applicable to Theorem 1. The function (15) with $\delta = -1$

belongs to $KC^3[0,T]$ if the following system of equations

$$\left\{\begin{array}{rl} -at_1+a_2= & at_1+2b_2, \\ at_2+2b_2= & -a(t_2-T), \\ -at_1^2/2+a_2t_1+a_1= & at_1^2/2+2b_2t_1+b_1, \\ at_2^2/2+2b_2t_2+b_1= & -a(t_2-T)^2/2, \\ -at_1^3/6+a_2t_1^2/2+a_1t_1+a_0= & at_1^3/6+b_2t_1^2+b_1t_1+b_0, \\ at_2^3/6+b_2t_2^2+b_1t_2+b_0= & -a(t_2-T)^3/6 \end{array}\right. \tag{10}$$

is satisfied.

In the system (10) $b_0, b_1, b_2, t_1, t_2, a$ are unknowns and a_0, a_1, a_2, T are parameters. The system (10) can be solved by using resultants. For instance, one can take b_2 from the first equation, b_1 from the third and b_0 from the fifth. Then, since the resulting system with unknowns t_1, t_2, a consists of polynomial equations, we can use the elimination of the variables t_1 and t_2 with resultants. (For instance, if $P(t,x,y)$ and $Q(t,x,y)$ are two polynomials in $\mathbb{Q}[t,x,y]$ then the resultant of P and Q with respect to t is a polynomial $R(x,y)\in\mathbb{Q}[x,y]$ which is the determinant of a corresponding Sylvester matrix [3].) The elimination of t_1 and t_2 gives the following equation relating a and T:

$$\begin{aligned} &3T^4a^4+(-12T^3a_2-96a_0T-48T^2a_1)a^3 \\ &+(-6T^2a_2^2-48a_1^2+96a_0a_2)a^2+4Ta_2^3a-a_2^4=0. \end{aligned} \tag{11}$$

This was checked with Mathematica (using `Eliminate`[*eqns*, *vars*]) and with Maple. Unfortunately, the resulting equations for other variables (like t_1 and t_2) have large degree which leads to many (real and complex) solutions or to no solutions at all. The solution of (10) is applicable to our problem in case the following hypothesis holds:

Hypothesis (H) *The system* (10)*, where* $b_0, b_1, b_2, t_1, t_2, a$ *are unknowns and* a_0, a_1, a_2, T *are parameters, has a unique real solution* $(b_0, b_1, b_2, t_1, t_2, a)$ *satisfying* $0<t_1<t_2<T$ *and* $a>0$.

The main result of this paper is the following theorem.

Theorem 2 *Suppose that the hypothesis (H) holds with $T = T_0$, where $0 < T_0 \leqslant 1$, and some $a_0 > 0$, $a_1 \geqslant 0$, $a_2 \leqslant 0$. If $a > \max\{4a_1/3, -a_2\}$ then $\inf M(f)$, where f runs through the class $KC^3(0, T_0)$ with boundary conditions* (1)*, is attained at the function given in* (15) *with $\delta = -1$ and is equal to a, where a is the positive root of the equation* (11)*.*

Finally, we give some numerical calculations and an application of Theorem 2 to Theorem 1. We use these numerical calculations to show that all the conditions of Theorem 2 are satisfied and find $\inf M(f) = 562.986\ldots$ for

$$a_0 = 1 + 2/(3\sqrt{3}), \quad a_1 = 0, \quad a_2 = -2\sqrt{3}, \quad T_0 = 1 - 1/\sqrt{3}. \tag{12}$$

In particular, we show that

Corollary 3 *With the conditions of Theorem 1, the smallest possible constant $M = M(\phi) = \max_{j=1,2,3} ||D^j(\phi)||_{L^\infty(\mathbb{R})}$ for ϕ defined in* (6) *is equal to positive root of the equation* (11)*, i.e.* $\inf M = 562.986\ldots$.

The numerical value $562.986\ldots$ is obtained by inserting the values given in (12) into (11) and dividing by $3T_0^4$, where $T = T_0$. This gives the equation

$$a^4-(284+156\sqrt{3})a^3-(2472+1428\sqrt{3})a^2-(360+216\sqrt{3})a-(756+432\sqrt{3}) = 0$$

with the unique positive root $562.986\ldots$. Corollary 3 and the computations in Section 4 show that the "arbitrary" part of ϕ in (6) which minimizes the functional $M(\phi)$ must be $\phi(x) = f(x - 1 - 1/\sqrt{3})$, where $f(x)$ is the function (15) with $\delta = -1$ in the interval $[0, 1 - 1/\sqrt{3}] = [0, 0.422\ldots]$ and is given by

$$f(x) = \begin{cases} -93.831x^3 - 1.732x^2 + 1.384, & 0 \leqslant x \leqslant 0.101, \\ 93.831x^3 - 58.645x^2 + 5.753x + 1.191, & 0.101 \leqslant x \leqslant 0.315, \\ -93.831(x - 1 + 1/\sqrt{3})^3, & 0.315 \leqslant x \leqslant 0.422. \end{cases}$$

Here, three decimal digits are correct.

Remark 4 *The constant* $M = 562.986\ldots$ *is the best (smallest) possible in Theorem 1 only for the function* ϕ *defined in* (6)*. In principle, for another function the corresponding* M *can be smaller than* $562.986\ldots$*.*

2 The optimal control problem

We first solve the following optimal control problem with a nonfixed termination time (9). The simplest optimal control problem with a nonfixed termination time was solved in [1]. Some other problems with a nonfixed termination time have been considered in [7], [8]. The following lemma is a necessary condition in our optimal control problem.

Lemma 5 *Suppose that the solution* T *of the optimal control problem* (9) *is attained at a function* $f(x)$*,* $a > 0$*, and suppose that* $\delta \in \{-1, 1\}$*. Then the function* $f(x)$ *can only be one of the following:*

$$f(x) = \delta a x^3/6 + a_2 x^2/2 + a_1 x + a_0, \tag{13}$$

$$f(x) = \begin{cases} \delta a x^3/6 + a_2 x^2/2 + a_1 x + a_0, & 0 \leqslant x \leqslant t_1, \\ -\delta a (x-T)^3/6, & t_1 \leqslant x \leqslant T, \end{cases} \tag{14}$$

or

$$f(x) = \begin{cases} \delta a x^3/6 + a_2 x^2/2 + a_1 x + a_0, & 0 \leqslant x \leqslant t_1, \\ -\delta a x^3/6 + b_2 x^2 + b_1 x + b_0, & t_1 \leqslant x \leqslant t_2, \\ \delta a (x-T)^3/6, & t_2 \leqslant x \leqslant T, \end{cases} \tag{15}$$

where $0 < t_1 < t_2 < T$ *and the constants* b_j*,* $j = 0, 1, 2$*, are such that* $f(x) \in KC^3(0,T)$*.*

Proof: In this lemma, we shall use the usual notation $Df = f'$. Let us reduce our problem to the standard problem of Pontryagin's maximum principle [1], by changing the variables $f_1(x) = f(x)$, $f_2(x) = f_1'(x)$, $f_3(x) =$

$f_2'(x)$, $u = f_3'(x)$:

$$T \to \inf,\ f_2(x) = f_1'(x),\ f_3(x) = f_2'(x),\ f_3'(x) = u,\ u \in [-a, a],$$

$$f_j(0) = a_{j-1},\ f_j(T) = 0,\ j = 1, 2, 3.$$

The Lagrange function for this problem is

$$\mathcal{L} = \int_0^T (p_1(f_1' - f_2) + p_2(f_2' - f_3) + p_3(f_3' - u))\mathrm{d}x$$

$$+\lambda_0 T + \sum_{j=1}^{3} \lambda_j (f_j(0) - a_{j-1}) + \sum_{j=4}^{6} \lambda_j f_{j-3}(T).$$

If the solution exists, then there exist some constants λ_j, $j = 0, 1, \ldots, 6$, the functions $p_k(x)$, $k = 1, 2, 3$, do not vanish simultaneously and satisfy the following conditions $(a), (b), (c), (d)$ given below.

(a) The solutions of the Euler equations $-\frac{\partial}{\partial x} L_{f_k'} + L_{f_k} = 0$, $k = 1, 2, 3$, for the Lagrangian $L = p_1(f_1' - f_2) + p_2(f_2' - f_3) + p_3(f_3' - u)$ are

$$p_1' = 0,\ -p_2' - p_1 = 0,\ -p_3' - p_2 = 0. \tag{16}$$

(b) The conditions of transversality for

$$l = \lambda_0 T + \sum_{j=1}^{3} \lambda_j (f_j(0) - a_{j-1}) + \sum_{j=4}^{6} \lambda_j f_{j-3}(T)$$

are

$$p_k(0) = \lambda_k, \quad p_k(T) = -\lambda_{k+3}, \quad \text{where} \quad k = 1, 2, 3. \tag{17}$$

(c) The condition of optimality in u, namely, $\min\limits_{u \in [-a,a]} (-p_3(x)u)$ gives

$$u = a \cdot \mathrm{sign} p_3(x). \tag{18}$$

(d) Finally, the stationarity of T implies $\lambda_0 + \sum_{j=4}^{6} \lambda_j f_{j-3}(T) = 0$.

By solving the equations (16) and taking into account (17) we obtain

$$p_3(x) = \frac{\lambda_1 x^2}{2} + \lambda_2 x + \lambda_3.$$

Note that $p_3(x)$ is not identically zero, because $p_3(x) \equiv 0$ implies $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and so $p_k(x) \equiv 0$ for each $k = 1, 2, 3$. The function $p_3(x)$ changes its sign at most twice. The condition (18) leads to (13), (14) and (15) in case the function $p_3(x)$ changes its sign zero, one and two times, respectively, in the interval $(0, T)$. $\square$

The following lemma is a sufficient condition for the optimal control problem.

Lemma 6 *Suppose hypothesis (H) holds and* $a_0 > 0$. *Then the solution of the optimal control problem is attained at the function* $f(x)$ *as defined* (15) *with* $\delta = -1$, *i.e.* $D^3 f(x) = -a$ *for* $x \in (0, t_1) \cup (t_2, T)$ *and* $D^3 f(x) = a$ *for* $x \in (t_1, t_2)$.

Proof: By hypothesis (H), there exists a unique real positive number $a = a(T_0)$ such that $(b_0, b_1, b_2, t_1, t_2, a)$ is a solution of the system (10) satisfying $0 < t_1 < t_2 < T_0$. Hence there exists at least one function $f(x)$ (as in (15) with $\delta = -1$) satisfying $f \in KC^3$, $D^j f(T_0) = 0$, $j = 0, 1, 2$. Lemma 5 shows that the solution of the optimal control problem (9) with $a = a(T_0)$ is equal to T_0 if this solution exists. We shall prove that the solution of the optimal control problem is attained at this function $f(x)$ given in (15) with $\delta = -1$.

Set $f(x) = f_1(x)$. For a contradiction assume that the solution of the optimal control problem is attained at another function $f_2(x) \in KC^3(0, T_1)$, i.e. $|D^3 f_2(x)| \leqslant a$, $D^j f_2(0) = a_j$, $D^j f_2(T_1) = 0$ for $j = 0,\ 1,\ 2$, and $T_1 < T_0$ (so $f_2(x)$ is distinct from $f_1(x)$). Define $f_2(x) \equiv 0$ in the interval $(T_1, T_0]$. We next prove that

$$D^j f_1(t_1) < D^j f_2(t_1) \quad \text{for} \quad j = 0, 1, 2 \tag{19}$$

and

$$(-1)^j D^j f_1(t_2) > (-1)^j D^j f_2(t_2) \quad \text{for} \quad j = 0, 1, 2. \tag{20}$$

Indeed, using boundary conditions and integration by parts, we obtain

$$\begin{aligned} f_1(t_1) - f_2(t_1) &= \frac{1}{2}\int_0^{t_1} (t_1 - s)^2 (D^3 f_1(s) - D^3 f_2(s))\mathrm{d}s \\ &= -\frac{1}{2}\int_0^{t_1} (t_1 - s)^2 (a + D^3 f_2(s))\mathrm{d}s \leqslant 0. \end{aligned}$$

By the same argument,

$$Df_1(t_1) - Df_2(t_1) = -\int_0^{t_1} (t_1 - s)(a + D^3 f_2(s))\mathrm{d}s \leqslant 0$$

and

$$D^2 f_1(t_1) - D^2 f_2(t_1) = -\int_0^{t_1} (a + D^3 f_2(s))\mathrm{d}s \leqslant 0.$$

Similarly, by integrating over the interval (t_2, T_0), we find that

$$\begin{aligned} f_1(t_2) - f_2(t_2) &= -\frac{1}{2}\int_{t_2}^{T_0} (s - t_2)^2 (D^3 f_1(s) - D^3 f_2(s))\mathrm{d}s \\ &= -\frac{1}{2}\int_{t_2}^{T_1} (s - t_2)^2 (-a - D^3 f_2(s))\mathrm{d}s + \frac{1}{2}\int_{T_1}^{T_0} a(s - t_2)^2 \mathrm{d}s > 0 \end{aligned}$$

and in the same way $Df_1(t_2) < Df_2(t_2)$, $D^2 f_1(t_2) > D^2 f_2(t_2)$. This completes the proof of (20).

To complete the proof of (19) assume that $D^j f_1(t_1) = D^j f_2(t_1)$ for some $j \in \{0, 1, 2\}$. Let S be a finite set of points in $(0, t_1)$, where the derivative $D^3 f_2(x)$ does not exist. Then from the expression of the difference $D^j f_1(t_1) - D^j f_2(t_1)$ by a corresponding integral we see that $a + D^3 f_2(x)$ must be zero in the set $(0, t_1) \setminus S$. Thus $D^3 f_2(x) = -a = D^3 f_1(x)$ for each $x \in (0, t_1) \setminus S$. This equality and the boundary conditions $D^j f_1(0) = D^j f_2(0) = a_j$ for $j = 0, 1, 2$ give us $D^j f_1(t_1) = D^j f_2(t_1)$ for each $j = 0, 1, 2$. Now, by integrating by parts over the interval (t_1, t_2), we deduce that

$$
\begin{aligned}
Df_1(t_2) - Df_2(t_2) &= \int_{t_1}^{t_2} (t_2 - s)(D^3 f_1(s) - D^3 f_2(s))\mathrm{d}s \\
&= \int_{t_1}^{t_2} (t_2 - s)(a - D^3 f_2(s))\mathrm{d}s \geqslant 0.
\end{aligned}
$$

This contradicts to $D^2 f_1(t_2) < D^2 f_2(t_2)$ (which is already proved in (20)), and so completes the proof of (19).

We next prove that there exist τ_1 and τ_2 satisfying $t_1 < \tau_1 < \tau_2 < t_2$ for which

$$
D^2 f_2(\tau_1) < D^2 f_1(\tau_1) \quad \text{and} \quad D^2 f_2(\tau_2) > D^2 f_1(\tau_2). \tag{21}
$$

Since the function $f_1(x) - f_2(x)$ and its derivative in the interval (t_1, t_2) are continuous, the inequalities $f_1(t_1) < f_2(t_1)$ and $f_1(t_2) > f_2(t_2)$ imply that there exists a point $\theta \in (t_1,\, t_2)$ such that $f_1(\theta) = f_2(\theta)$ and $Df_1(\theta) \geqslant Df_2(\theta)$. The inequality

$$
\begin{aligned}
0 < Df_2(t_1) - Df_1(t_1) &= -\int_{t_1}^{\theta} (D^2 f_2(s) - D^2 f_1(s))\mathrm{d}s + Df_2(\theta) - Df_1(\theta) \\
&\leqslant -\int_{t_1}^{\theta} (D^2 f_2(s) - D^2 f_1(s))\mathrm{d}s
\end{aligned}
$$

yields $D^2 f_2(s) < D^2 f_1(s)$ for some point $s \in (t_1,\, \theta)$. This proves the first inequality in (21). In the same way the second inequality of (21) with $\tau_2 \in (\theta, t_2)$ follows from

$$
0 < Df_2(t_2) - Df_1(t_2) \leqslant \int_{\theta}^{t_2} (D^2 f_2(s) - D^2 f_1(s))\mathrm{d}s.
$$

Now, by (21), we find that

$$
D^2 f_2(\tau_2) - D^2 f_2(\tau_1) > D^2 f_1(\tau_2) - D^2 f_1(\tau_1) = a(\tau_2 - \tau_1). \tag{22}
$$

If the derivative $D^3 f_2$ exists in the interval $(\tau_1,\,\tau_2)$ then, by (22) and the Lagrange theorem, we conclude that there exists $\theta_2 \in (\tau_1,\,\tau_2)$ for which

$$D^3 f_2(\theta_2) > a, \tag{23}$$

which is a contradiction with $|D^3 f_2(x)| \leqslant a$ for each x, where $D^3 f_2(x)$ exists.

Suppose $D^3 f_2$ has $m \geqslant 1$ points of discontinuity, say, $\alpha_1 < \cdots < \alpha_m$ in the interval (τ_1, τ_2). Put $\alpha_0 = \tau_1$ and $\alpha_{m+1} = \tau_2$. Then (22) implies that

$$\begin{aligned} D^2 f_2(\alpha_{m+1}) - D^2 f_2(\alpha_0) &= \sum_{j=0}^{m} (D^2 f_2(\alpha_{j+1}) - D^2 f_2(\alpha_j)) \\ &> \sum_{j=0}^{m+1} a(\alpha_{j+1} - \alpha_j) = a(\alpha_{m+1} - \alpha_0). \end{aligned}$$

So we must have $D^2 f_2(\alpha_{j+1}) - D^2 f_2(\alpha_j) > a(\alpha_{j+1} - \alpha_j)$ for at least one $j \in \{0, \ldots, m\}$. As above this leads to the existence of some point $\theta_2 \in (\alpha_j, \alpha_{j+1})$ with the property (23), which is contradiction to the inequality $|f_2(x)| \leqslant a$. $\square$

Corollary 7 *Suppose hypothesis (H) holds for some* $a_0 > 0$, a_1, a_2 *and* T_0. *Then the system* (10), *where* $b_0, b_1, b_2, t_1, t_2, T$ *are unknowns and* $a_0, a_1, a_2, a = a(T_0)$ *are parameters, has a solution* $(b_0, b_1, b_2, t_1, t_2, T_0)$ *satisfying* $0 < t_1 < t_2 < T_0$ *and in no other solution (if it exists) the last coordinate can be* T_0.

Proof: Suppose for a contradiction that there exist two real numbers $T = T_0$ and $T = T_1$, $T_0 \neq T_1$, which are the last coordinates of some solutions of (10). Then there exists two functions (15), say, $f_1(x)$ and $f_2(x)$, such that $f_k(x) \in KC^3(0, T_{k-1})$, $|D^3 f_k(x)| = a$, $D^j f_k(0) = a_j$, $D^j f_k(T_{k-1}) = 0$ for $j = 0,\,1,\,2$ and $k = 1, 2$. However, the inequality (23) gives $D^3 f_2(x) > a$ for some $x \in (0,\,T_1)$ if $T_0 > T_1$ and $D^3 f_1(x) > a$ if $T_0 < T_1$. $\square$

Remark 8 *Corollary 7 does not give the uniqueness of the function* (15)*, because it does not state that the the first five coordinates* b_0*,* b_1*,* b_2*,* t_1*,* t_2 *are the same. Numerical calculations show that the function* (15) *is unique. However, we will not prove the uniqueness of the function* (15)*, because we do not need it in the proof of Theorem 2.*

3 Proof of Theorem 2

The system (10) describes the function (15) with $\delta = -1$ whose third derivative is $-a$ for $0 \leqslant x < t_1$ and $t_2 < x \leqslant T$ and a for $t_1 < x < t_2$. Of course, the six equations of (10) are obtained from the condition $f \in KC^3$ of the function (15) by evaluating $D^j f(x)$, where $j = 0, 1, 2$, at the points $x = t_1$ and $x = t_2$.

In order to estimate the derivatives of the function (15) we will use the next lemma.

Lemma 9 *If* $a_1 \geqslant 0$*,* $a_2 \leqslant 0$ *and hypothesis (H) holds then* $|b_2| < aT/2$ *and* $|a_2| < aT(\sqrt{2} - 1) + 2\sqrt{aa_1}$*.*

Proof: In the proof of this lemma we shall only use the first four equations of (10). Adding the first and the second equations we obtain

$$t_2 - t_1 = T/2 - a_2/2a \geqslant T/2 \tag{24}$$

and therefore

$$t_1 \leqslant T/2 \leqslant t_2. \tag{25}$$

Hence the second equation implies

$$|b_2| = |aT/2 - at_2| = a(t_2 - T/2) < a(T - T/2) = aT/2.$$

From the third and the first equations we find that

$$b_1 = -at_1^2 + (a_2 - 2b_2)t_1 + a_1 = -at_1^2 + 2at_1^2 + a_1 = at_1^2 + a_1. \tag{26}$$

Thus the fourth equation combined with the second yields

$$\begin{aligned} at_2^2 &= at_2T - aT^2/2 - 2b_2t_2 - b_1 = t_2(aT - 2b_2) - aT^2/2 - at_1^2 - a_1 \\ &= 2at_2^2 - aT^2/2 - at_1^2 - a_1. \end{aligned}$$

Hence $t_2^2 = T^2/2 + t_1^2 + a_1/a$, and so $t_2 = \sqrt{T^2/2 + t_1^2 + a_1/a}$. From (24) it follows that

$$\begin{aligned} -a_2/a &= 2t_2 - 2t_1 - T = \sqrt{2T^2 + 4t_1^2 + 4a_1/a} - T - 2t_1 \\ &\leqslant \sqrt{2T^2} + \sqrt{4t_1^2} + 2\sqrt{a_1/a} - T - 2t_1 = T(\sqrt{2} - 1) + 2\sqrt{a_1/a}. \end{aligned}$$

This completes the proof of the lemma in view of $a_2 \leqslant 0$. □

Proof of Theorem 2: Hypothesis (H) gives us the solution $(b_0, b_1, b_2, t_1, t_2, a)$ of (10), where a is a root of (11) with $T = T_0$. We next prove that the absolute values of the first and the second derivatives of f in the interval $(0, T_0)$ are smaller than a. Recall that

$$f(x) = \begin{cases} -ax^3/6 + a_2x^2/2 + a_1x + a_0, & 0 \leqslant x \leqslant t_1, \\ ax^3/6 + b_2x^2 + b_1x + b_0, & t_1 \leqslant x \leqslant t_2, \\ -a(x - T_0)^3/6, & t_2 \leqslant x \leqslant T_0. \end{cases} \tag{27}$$

We estimate the first derivative of (27) in the interval $[0,\, t_1]$. From (25), (27) we have

$$|Df(x)| = a_1 - ax^2/2 + a_2x \leqslant a_1 < a,$$

if $a_1 - ax^2/2 + a_2x > 0$. From (25), (27) and Lemma 9 we obtain

$$\begin{aligned} |Df(x)| &= ax^2/2 - a_2x - a_1 \leqslant a/8 + |a_2|/2 - a_1 \\ &< a/8 + (aT_0(\sqrt{2} - 1) + 2\sqrt{aa_1})/2 - a_1 \\ &\leqslant a/8 + a/8 + \sqrt{aa_1} - a_1 \leqslant a/4 + a/4 < a \end{aligned}$$

if $a_1 - ax^2/2 + a_2 x \leqslant 0$. We estimate the second derivative of (27)

$$\begin{aligned}|D^2 f(x)| &\leqslant at_1 - a_2 = at_1 + |a_2| < aT_0/2 + aT_0(\sqrt{2}-1)\\ &< aT_0/2 + aT_0/2 = aT_0 \leqslant a\end{aligned}$$

for $x \in (0, t_1]$. For $x \in [t_2, T_0)$, using (25), we obtain

$$\begin{aligned}|Df(x)| &= |-a(x-T_0)^2/2| < |-a(x-T_0)| = |D^2 f(x)|\\ &= a(T_0 - t_2) \leqslant aT_0/2 \leqslant a/2.\end{aligned}$$

This proves that $|Df(x)|, |D^2 f(x)| < a$ for each $x \in (0, t_1] \cup [t_2, T_0)$.

It remains to prove that the same holds for $x \in [t_1, t_2]$. By the above we have $|D^2 f(t_1)| < a$ and $|D^2 f(t_2)| < a$. Thus

$$|D^2 f(x)| = |ax + 2b_2| \leqslant \max(|D^2 f(t_1)|, |D^2 f(t_2)|) < a$$

for $x \in [t_1, t_2]$. The local extremum of the function $Df(x) = ax^2/2 + 2b_2 x + b_1$ is attained at the point $x = -2b_2/a$ and is equal $-2b_2^2/a + b_1$. By (25) and Lemma 9, we have $at_1^2 < aT_0^2/4$ and $2b_2^2/a < aT_0^2/2$. Hence, using (25), (26), $T_0 \leqslant 1$ and the condition $a_1 < 3a/4$ of Theorem 2, we obtain

$$\begin{aligned}|-2b_2^2/a + b_1| &= |-2b_2^2/a + at_1^2 + a_1| \leqslant \max(2b_2^2/a, at_1^2 + a_1)\\ &\leqslant \max(aT_0^2/2, aT_0^2/4 + a_1) \leqslant a.\end{aligned}$$

Consequently, the maximum of $|Df(x)| = |ax^2/2 + 2b_2 x + b_1|$ in the interval $x \in [t_1, t_2]$ is at most $\max(|D(f(t_1))|, |D(f(t_2))|, a) \leqslant a$. This completes the proof of the theorem. $\square$

4 Numerical calculations and applications

We use numerical calculations to show that all the conditions of Theorem 2 hold if the equalities (12) are satisfied. Recall that equalities (12) are obtained from the conditions

$$\phi(1+1/\sqrt{3}) = 1+2/(3\sqrt{3}), \quad D\phi(1+1/\sqrt{3}) = 0, \quad D^2\phi(1+1/\sqrt{3}) = -2\sqrt{3}.$$

The solutions of (11) are

$$562.98642\ldots, \quad -8.67110\ldots, \quad -0.05769\ldots \pm i0.55209\ldots,$$

where $i = \sqrt{-1}$, i.e. the equation (11) has a unique real positive solution. The system (10), where a_0, a_1, a_2, T_0 are defined in (12), has the following four solutions (all five decimal digits are correct).

a	562.98642	−8.67110	$-0.05769 + 0.55209\delta i$
t_1	0.10109	3.85235	$0.05597 + 1.56603\delta i$
t_2	0.31549	3.86392	$-0.05701 - 1.53729\delta i$
b_0	1.19102	166.63070	$0.67288 + 0.00217\delta i$
b_1	5.75346	−128.68439	$0.04453 - 1.36238\delta i$
b_2	−58.64534	31.67207	$-0.86421 + 0.05945\delta i$

Since there exists a unique real solution of the system (10) satisfying $0 < t_1 < t_2 < T$ and $a > 0$, hypothesis (H) holds. All other conditions of Theorem 2 hold too.

Suppose Theorem 2 holds. Let us prove Corollary 3. Set

$$\phi(x) = f(x - 1 - 1/\sqrt{3}), \quad 1 + 1/\sqrt{3} \leqslant x \leqslant 2,$$

where f is defined in (27). The corresponding smallest value of the functional

$$\max_{j=1,2,3} ||D^j\phi||_{L^\infty(1+1/\sqrt{3},2)}$$

is equal to $a = 562.986\ldots$. From $\phi(x) = -\phi(-x)$ and (6) it is easy to see that $|D\phi(x)| \leqslant 1$, $|D^2\phi(x)| \leqslant |6(x-1)| \leqslant 2\sqrt{3}$ and $|D^3\phi(x)| \leqslant 6$ for $x \notin [-2, -1-1/\sqrt{3}] \cup [1+1/\sqrt{3}, 2]$. Hence

$$\inf \max_{j=1,2,3} ||D^j(\phi)||_{L^\infty(\mathbb{R})} = \min \max_{j=1,2,3} ||D^j(\phi)||_{L^\infty(\mathbb{R})} = a = 562.986\ldots,$$

where the infimum is taken over every function ϕ of the form (6) in $KC^3(\mathbb{R})$. The Corollary 3 is proved. Note that the extremal function $\phi(x)$ in the interval $[1+1/\sqrt{3},2]$ is given by $\phi(x) = f(x-1-1/\sqrt{3})$ with f as given at the end of Section 1.

For simplicity, let us assume that $a_1 = 0$ and describe the set of numbers a_0, a_2, T_0 for which Theorem 2 holds. To do this we first establish when (14) with $\delta = 1$ gives the solution of the optimal control problem (if this solution exists). From $f(x) \in KC^3(0,T)$ we obtain the following system:

$$\begin{cases} at_1 + a_2 = & -a(t_1 - T), \\ at_1^2/2 + a_2 t_1 + a_1 = & -a(t_1-T)^2/2, \\ at_1^3/6 + a_2 t_1^2/2 + a_1 t_1 + a_0 = & -a(t_1-T)^3/6. \end{cases} \tag{28}$$

Lemma 10 *Let* $a_0 > 0$, $a_1 = 0$, $a_2 \leqslant 0$. *Suppose that the minimum in the optimal control problem is attained at the function* (14) *with* $\delta = 1$. *Then*

$$a_0 = -a_2 T^2/(6\sqrt{2}). \tag{29}$$

Proof: Let us introduce in (28) the following new variables $y_1 = t_1/T$, $y_2 = -a_2/aT$ and $y_3 = -a_0/aT^3$. Then $y_1 > 0$, $y_2 \geqslant 0$, $y_3 > 0$ and the first equality of (28) gives $y_1 - y_2 = 1 - y_1$, so $y_2 = 2y_1 - 1$. The second equality gives $y_1^2/2 - y_1 y_2 = -(1-y_1)^2/2$, and so

$$y_1^2 + (1-y_1)^2 = 2y_1 y_2 = 2y_1(2y_1 - 1).$$

Consequently, $2y_1^2 = 1$, which implies $y_1 = 1/\sqrt{2}$ and $y_2 = \sqrt{2} - 1$. With these values, the third equation of (28) gives

$$\begin{aligned} y_3 &= \frac{y_1^3}{6} - \frac{y_1^2 y_2}{2} + \frac{(y_1-1)^3}{6} = \frac{2y_1^3 - 3y_1^2 + 3y_1 - 1 - 3y_1^2 y_2}{6} \\ &= \frac{8y_1 - 5 - 3y_2}{12} = \frac{\sqrt{2} - 2}{12}. \end{aligned}$$

Thus $y_3/y_2 = -1/6\sqrt{2}$ and we deduce that

$$a_0 = -y_3 a T^3 = \frac{y_3}{y_2} a_2 T^2 = -\frac{a_2 T^2}{6\sqrt{2}},$$

which is (29). □

Numerical calculations show that for $a_1 = 0$ Theorem 2 holds when

$$a_0 > -a_2 T_0^2/(6\sqrt{2}).$$

To illustrate this with Maple, let us take $T_0 = 1$, $a_1 = 0$, $a_2 = -1$. We do not write the solutions b_0, b_1, b_2 in this table, but only t_1, t_2, a satisfying $0 < t_1 < t_2 < T_0$ and $a > 0$.

a_0	t_1	t_2	$a = \inf M(f)$
5	0.24524	0.74842	157.03169
3	0.24199	0.74737	93.05331
2	0.23785	0.74603	61.08089
1.5	0.23361	0.74469	45.10908
1	0.22485	0.74199	29.16727
0.5	0.19645	0.73388	13.35538
0.3	0.15467	0.72382	7.23066
0.2	0.10021	0.71417	4.38766
0.15	0.04874	0.70878	3.12413
0.12	0.00373	0.70711	2.45848
0.118	0.00026	0.70710	2.41726
0.11786	0.00001	0.70710	2.41439

Note that substituting $T_0 = 1$ and $a_2 = -1$ into (29) we obtain $a_0 = 1/(6\sqrt{2}) = \sqrt{2}/12 = 0.11785\ldots$. The last table shows that (14) is the "limit case" of the function (15). In fact, t_1 tends to zero if $a_0 \to \sqrt{2}/12 = 0.11785\ldots$ and the function (15) becomes the function (14) with $\delta = 1$.

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