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New Criteria for the Existence of Periodic and Almost Periodic Solutions for Some Evolution Equations in Banach Spaces

Khalil EZZINBI[1,*,†] **and Mustapha JAZAR**[2,‡]
[1]Université Cadi Ayyad
Faculté des Sciences Semlalia
Département de Mathématiques
B.P. 2390, Marrakech, Morocco
e-mail: ezzinbi@ucam.ac.ma

[2]Université Libanaise, Faculté des Sciences
Département de Mathématiques
B.P. 155-012, Beyrouth, Liban
e-mail: mjazar@ul.edu.lb

Abstract

In this work we give a new criteria for the existence of periodic and almost periodic solutions for some differential equation in a Banach space. The linear part is nondensely defined and satisfies the Hille-Yosida condition. We prove the existence of periodic and almost periodic solutions with condition that is more general than the known exponential dichotomy. We apply the new criteria for the existence of periodic and almost periodic solutions for some partial functional differential equation whose linear part is nondensely defined.

Key Words: Hille-Yosida operator, integral solution, evolution family, discrete equation, monodromy operator, partial functional differential equations.
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†To whom all correspondence should be sent.

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1 Introduction

In this work, we are concerned with the existence of periodic and almost periodic solutions for the following differential equation:

$$\begin{cases} \dfrac{d}{dt}x(t) = (A + B(t))\, x(t) + f(t), \text{ for } t \geq s \\ x(s) = x_0 \end{cases} \tag{1}$$

where $A : D(A) \subset E \to E$ is a nondensely defined linear operator on a Banach space E and $f : \mathbb{R} \to E$ is continuous, p-periodic or almost periodic (f is not identically zero). For every $t \geq 0$, $B(t)$ is a bounded linear operator on E. Throughout this work, we suppose that A is a Hille-Yosida operator which means that there exist $M_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ such that

$$(\omega_0, +\infty) \subset \rho(A) \text{ and } |R(\lambda, A)^n| \leq \frac{M_0}{(\lambda - \omega_0)^n}, \text{ for } n \in \mathbb{N} \text{ and } \lambda > \omega_0, \tag{2}$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$.

Differential equations with nondense domain have many applications in partial differential equations. About this topic we refer to [14] where the authors studied the well-posedness of equation (1) with $B = 0$ and A is a Hille-Yosida operator. The existence of periodic and almost periodic solutions for partial functional differential equations has been extensively studied in literature, for the reader we refer to [5], [9], [10], [11], [12] and references therein. In [8], the authors established the existence of periodic and almost periodic solutions of equation (1) and they applied their results for the following partial functional differential equation:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + K(t)x_t + h(t), \text{ for } t \geq s, \\ x_s = \varphi \in C = C([-r, 0]; E), \end{cases} \tag{3}$$

where C is the space of continuous functions from $[-r, 0]$ into E endowed with the uniform norm topology, for every $t \geq 0$, the history function $x_t \in C$ is defined by

$$x_t(\theta) = x(t + \theta), \text{ for } \theta \in [-r, 0]. \tag{4}$$

$K(t)$ is a bounded linear operator from C to E and $t \to K(t)\varphi$ is continuous, for every φ, p-periodic in t and $h : \mathbb{R} \to E$ is continuous and p-periodic.

The famous Massera's Theorem [9] on two dimensional periodic ordinary differential equations explains the relationship between the boundedness of solutions and periodic solutions. In this work we use Massera's approach [9], we give sufficient conditions such that the equivalence between the existence of a $p-$periodic solution and the existence of a bounded solution holds. Note that Massera's approach holds for equation (1) if A generates a compact semigroup on E. Since in this case the Poincaré map is compact for $p > r$. In [6], the authors proved the existence of a periodic solution for nonlinear partial functional differential equations that are bounded and ultimate bounded, using Horn's fixed point theorem they proved that the Poincaré map has a fixed point which gives a periodic solution. Recently in [10], the authors obtained a new spectral criteria for the existence of bounded solutions for the following difference equation

$$x_{n+1} = Px_n + y_{n+1}, n \in \mathbb{Z}. \tag{5}$$

where $(y_n)_{n\in\mathbb{Z}} \in l^\infty(E)$ is given and P is a bounded linear operator on E and they applied the criteria to show the existence of periodic and almost periodic solutions for some partial functional differential equations with infinite delay and some differential equations in Banach spaces.

The results obtained in this paper (together with the idea) are intimately relates to those in [10]. Several results in [10] are extended to the equation (1) whose linear part is nondensely defined.

The organization of this work is as follows: in section 2, we recall some preliminary results that will be used later. In section 3, we establish a new criteria for the existence of p-periodic and almost periodic solutions of equation (1). Finally we propose an application to equation (3).

2 Preliminary results

In the following we assume that

($\mathbf{H}_1$) A is a Hille-Yosida operator.

($\mathbf{H}_2$) For every $t \geq 0$, the operator $B(t)$ is a bounded linear operator on E, p-periodic in t and $t \to B(t)x$ is continuous, for every x in E.
Let us introduce some notions which will be used in this work.

Definition 2.1 *A continuous function $x : [s,\infty) \to E$ is said to be an integral solution of equation (1) if*
i) $\int_s^t x(\tau)d\tau \in D(A)$, *for* $t \geq s$
ii) $x(t) = x(s) + A\int_s^t x(\tau)d\tau + \int_s^t B(\tau)x(\tau)d\tau + \int_s^t f(\tau)d\tau,$ *for* $t \geq s$.

Proposition 2.1 *[14] For every $s \in \mathbb{R}$ and $x_0 \in \overline{D(A)}$, equation (1) has a unique integral solution for $t \geq s$.*

Theorem 2.1 ***(Theorem 4.1.2 in [13])****Let A_0 be the part of A in $\overline{D(A)}$ which is defined by*

$$\begin{cases} D(A_0) = \left\{x \in D(A) : Ax \in \overline{D(A)}\right\} \\ A_0x = Ax. \end{cases} \tag{6}$$

Then A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.

By [8], the integral solution x of equation (1) is given by,

$$x(t) = T_0(t-s)x_0 + \lim_{\lambda\to\infty} \int_s^t T_0(t-\tau)B_\lambda \left(B(\tau)x(\tau) + f(\tau)\right) d\tau. \qquad (7)$$

where $B_\lambda = \lambda R\left(\lambda, A\right)$.
Define $U_B(t,s)_{t\geq s}$ by

$$U_B(t,s)x_0 = x(t,s,x_0), \text{ for } x_0 \in \overline{D(A)},$$

where $x(.,.,x_0)$ is the integral solution of the equation (1) with $f = 0$.

Proposition 2.2 *[8] Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. Then* $(U_B(t,s))_{t\geq s}$ *is an evolution family:*
i) $U_B(t,t) = I$, *for every* $t \in \mathbb{R}$,
ii) $U_B(t,s)U_B(s,r) = U_B(t,r)$, *for* $t \geq s \geq r$,
iii) for all $x_0 \in \overline{D(A)}$, $(t,s) \to U_B(t,s)x_0$ *is continuous.*
Moreover the integral solution of equation (1) is given by

$$u(t) = U_B(t,s)u(s) + \lim_{\lambda\to\infty} \int_s^t U_B(t,\tau)B_\lambda f(\tau)d\tau, \ t \geq s. \qquad (8)$$

Definition 2.2 $(U_B(t,s))_{t\geq s}$ *has an exponential dichotomy on* $\overline{D(A)}$ *with constant* $\beta > 1$ *and* $L \geq 1$, *if there exists a bounded strongly continuous family of projection* $(P(t))_{t\in\mathbb{R}}$ *on* $\overline{D(A)}$ *such that for* $t \geq s$ *one has*
i) $P(t)U_B(t,s) = U_B(t,s)P(s)$.
ii) the map $U_B(t,s) : (Id - P(s))\overline{D(A)} \to (Id - P(t))\overline{D(A)}$ *is invertible.*
iii) $|U_B(t,s)z| \leq Le^{-\beta(t-s)}|z|$, *for* $z \in P(s)\overline{D(A)}$.
iv) $\left|U_B(t,s)^{-1}z\right| \leq Le^{-\beta(t-s)}|z|$, *for* $z \in (Id - P(t))\overline{D(A)}$.

For the sequel, $C_b(\mathbb{R},E)$ denotes the space of bounded continuous functions on $\mathbb{R}$ with values in E.

Theorem 2.2 *[8] Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. Then the following propositions are equivalent:*
i) $U_B(t,s)_{t\geq s}$ *has an exponential dichotomy,*
ii) for any f in $C_b(\mathbb{R},E)$, *equation (1) has a unique integral solution in* $C_b(\mathbb{R},E)$.

If we suppose that $B(t+p) = B(t)$, for all t in $\mathbb{R}$, then the evolution family $U_B(t,s)_{t\geq s}$ is p-periodic:

$$U_B(t+p,s+p) = U_B(t,s), \text{ for } t \geq s.$$

Definition 2.3 *The Carleman spectrum* $sp(u)$ *of a function* u *in* $C_b(\mathbb{R},E)$, *is consisting of* $\xi \in \mathbb{R}$ *such that the Fourier-Carleman transform*

$$\widehat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t}u(t)dt, \ Re\lambda > 0 \\ -\int_0^\infty e^{\lambda t}u(-t)dt, \ Re\lambda < 0 \end{cases}$$

has no holomorphic extension to a neighborhood of $i\xi$.

Recall that a function $v \in C_b(\mathbb{R}, E)$, is said to be almost periodic if the set $\{v_\tau : \tau \in \mathbb{R}\}$ is relatively compact in $C_b(\mathbb{R}, E)$, where v_τ is defined by

$$v_\tau(s) = v(\tau + s), \text{ for } s \in \mathbb{R}.$$

Proposition 2.3 *[7] Let $u \in C_b(\mathbb{R}, E)$. Then the following statements hold true:*
i) $sp(u)$ is closed.
ii) If u^n is a sequence in $C_b(\mathbb{R}, E)$ converging to u uniformly and $sp(u^n) \subset \Lambda$, for any $n \geq 0$, then $sp(u) \subset \overline{\Lambda}$.
iii) $sp(\alpha u) \subset sp(u)$.
iv) If u is uniformly continuous, $sp(u)$ is countable and E doesn't contain a copy of c_0, then u is almost periodic.

The spectrum $\sigma(u)$ of a bounded continuous function u is defined by: $\sigma(u) = \overline{e^{isp(u)}}$. Then a criteria for the existence of a $p-$periodic solution of equation (1) is obtained in [8].

Theorem 2.3 *(**Theorem 3.7** and **Corollary 3.8** in [8]) Let f be in $C_b(\mathbb{R}, E)$ such that*

$$\sigma(U_B(p, 0) \cap \overline{\{e^{i\eta p} : \eta \in sp(f)\}} = \emptyset. \tag{9}$$

Then equation (1) has at most one solution in $C_b(\mathbb{R}, E)$. Moreover if f is almost periodic, then equation (1) has a unique almost periodic solution.

Remark:$U_B(p, 0)$ is called the monodromy operator. Condition (9) is more general than the exponential dichotomy condition. Indeed, if the evolution family $U_B(t, s)_{t \geq s}$ has an exponential dichotomy, then

$$\Gamma = \{z \in \mathbb{C} : |z| = 1\} \subset \rho(U_B(p, 0)),$$

where $\rho(U_B(p, 0))$ denotes the resolvent set of $U_B(p, 0)$. Moreover, it's well-known that f is p-periodic if and only if $sp(f) \subset \frac{2\pi}{p}\mathbb{Z}$. Consequently if $1 \in \rho(U_B(p, 0))$, then equation (1) has a unique p-periodic solution. In the following, we give an extension of Theorem 2.3 and we prove the existence of a p-periodic solution of equation (1) if (1) has a bounded solution on the whole line and 1 is isolated in $\sigma(U_B(p, 0)$.
Define $l^\infty(E) = \left\{ (\alpha_n)_{n \in \mathbb{Z}} \subset E : \sup_{n \in \mathbb{Z}} |\alpha_n| < \infty \right\}$. For any $\alpha = (\alpha_n)_{n \in \mathbb{Z}} \in l^\infty(E)$ and any $p \in \mathbb{Z}$, we define $S(p)\alpha$ by

$$S(p)\alpha = (\alpha_{n+p})_{n \in \mathbb{Z}}.$$

Definition 2.4 *[10] Let $(\alpha_n)_{n \in \mathbb{Z}}$ be a sequence in $l^\infty(E)$. Then the subset of all λ on the unit circle $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ at which*

$$\widehat{\alpha}(\lambda) = \begin{cases} \sum_{n=0}^\infty \lambda^{n-1} S(n)\alpha, & |\lambda| > 1 \\ -\sum_{n=1}^\infty \lambda^{n-1} S(-n)\alpha, & |\lambda| < 1 \end{cases}$$

has non holomorphic extension, is said to be the spectrum $\sigma(\alpha)$ of the sequence $(\alpha_n)_{n \in \mathbb{Z}}$.

Let $\alpha \in l^\infty(E)$. Then α is said to be almost periodic if $\{(\alpha_{n+k})_{n\in\mathbb{Z}} : k \in \mathbb{Z}\}$ is relatively compact in $l^\infty(E)$.

Lemma 2.1 *(**Corollary 2.5** in [10]) Let α be an element of $l^\infty(E)$. Then $\alpha_n = \alpha_{n+1} \neq 0$, for all $n \in \mathbb{Z}$ if and only if $\sigma(\alpha) = \{1\}$. Similarly, $\alpha_n = -\alpha_{n+1} \neq 0$ for all $n \in \mathbb{Z}$ if and only if $\sigma(\alpha) = \{-1\}$.*

Proposition 2.4 *[10] Let q be an almost periodic function. Then*

$$\sigma(q(n)_{n\in\mathbb{Z}}) \subset \sigma(q).$$

Theorem 2.4 *[10] (**Theorem 3.4 and Corollary 3.6 in** [10]). Assume that equation (5) has a bounded solution and the following condition holds*

$$\sigma_\Gamma(P)\backslash\sigma(y) \text{ is closed,}$$

where $\sigma_\Gamma(P) = \sigma(P) \cap \Gamma$. Then there exists a bounded solution x of equation (5) such that $\sigma(x) = \sigma(y)$. Moreover if $\sigma(y)$ is countable and E doesn't contain a copy of c_0, then there is an almost periodic solution of equation (5).

3 Main results

3.1 Periodic solutions

Theorem 3.1 *Assume that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold. If equation (1) has a bounded integral solution on the whole line and*

$$\sigma_\Gamma(U_B(p,0))\backslash\{1\} \text{ is closed,} \tag{10}$$

where $\sigma_\Gamma(U_B(p,0)) = \sigma(U_B(p,0)) \cap \Gamma$, then equation (1) has a $p-$periodic integral solution.

Proof. Let u be a bounded integral solution of equation (1) on the whole line. Then

$$u(t) = U_B(t, t-p)u(t-p) + \lim_{\lambda\to\infty} \int_{t-p}^{t} U_B(t,\tau)B_\lambda f(\tau)d\tau,\ t \in \mathbb{R},$$

which implies that

$$u(t) = U_B(t, t-p)u(t-p) + g(t),\ t \in \mathbb{R},$$

where g is defined by

$$g(t) = \lim_{\lambda\to\infty} \int_{t-p}^{t} U_B(t,\tau)B_\lambda f(\tau)d\tau,\ t \in \mathbb{R}.$$

Periodicity of f and $U_B(t,s)_{t\geq s}$ imply that g is $p-$periodic. Let $(x_n)_{n\in\mathbb{Z}}$ and $(g_n)_{n\in\mathbb{Z}}$ be defined by

$$x_n = u(np) \text{ and } g_n = g(np),\ \text{for } n \in \mathbb{Z}.$$

Then periodic integral solutions of equation (1) correspond to constant solutions of the following discrete equation

$$x_{n+1} = U_B(p,0)x_n + g_{n+1}, n \in \mathbb{Z}. \tag{11}$$

Since f is not identically zero, then $g_{n+1} = g_n \neq 0$ and $\sigma((g_n)_{n\in\mathbb{Z}}) = \{1\}$. By Theorem 2.4, we deduce that equation (11) has a bounded solution $(x_n)_{n\in\mathbb{Z}}$ such that $\sigma((x_n)_{n\in\mathbb{Z}}) = \sigma((g_n)_{n\in\mathbb{Z}}) = \{1\}$. By Lemma (2.1), we conclude that $x_{n+1} = x_n$ for every $n \in \mathbb{Z}$ and by uniqueness of solutions with initial data we get that the integral solution of equation (1) starting from x_0 is $p-$periodic. ■

Remark 3.1 *Condition 10 means that if* 1 *is in* $\sigma_\Gamma(U_B(p,0))$*, then* 1 *is an isolated point in* $\sigma_\Gamma(U_B(p,0))$*.*

Theorem 3.2 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *hold and* f *is anti-periodic which means that* $f(t+p) = -f(t)$*, for all* $t \in \mathbb{R}$*. If*

$$\sigma_\Gamma(U_B(p,0))\backslash\{-1\} \text{ is closed,}$$

then equation (1) has an anti $p-$*periodic integral solution if and only if it has a bounded integral solution on the whole line.*

Proof. Arguing as above, we get that equation (11) has a bounded solution $(x_n)_{n\in\mathbb{Z}}$ such that $\sigma((x_n)_{n\in\mathbb{Z}}) = \sigma((g_n)_{n\in\mathbb{Z}}) = \{-1\}$ and by Lemma (2.1), we obtain that $x_{n+1} = -x_n$, which gives an anti $p-$periodic integral solution of equation (1). ■

3.2 Almost periodic solutions

Let g be defined by

$$g(t) = \lim_{\lambda\to\infty} \int_{t-1}^{t} U_B(t,\tau)B_\lambda f(\tau)d\tau,\ t \in \mathbb{R}.$$

Theorem 3.3 *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. Furthermore we assume that* B *is* $1-$*periodic,* f *is almost periodic and*

$$\sigma_\Gamma(U_B(1,0))\backslash\sigma(g(n)_{n\in\mathbb{Z}}) \text{ is closed.} \tag{12}$$

If $\sigma(g(n)_{n\in\mathbb{Z}})$ *is countable and* E *doesn't contain a copy of* c_0*, then equation (1) has an almost periodic integral solution if and only if it has a bounded integral solution on the whole line.*

We start by the following fundamental Lemma which plays an important role in the proof of Theorem 3.3, its proof is similar to the one given in [10].

Lemma 3.1 *Assume that* $U_B(t,s)_{t\geq s}$ *is a* $1-$*periodic evolution family. Let* w *be a solution on the whole line of*

$$w(t) = U_B(t,s)w(s) + \lim_{\lambda\to\infty} \int_{s}^{t} U_B(t,\tau)B_\lambda\theta(\tau)d\tau,\ t \geq s, \tag{13}$$

where θ is an almost periodic function with values in E. Then w is almost periodic if and only if the sequence $(w(n))_{n\in\mathbb{Z}}$ is almost periodic.

Proof of Theorem 3.3. Theorem 2.4 implies that equation (11) has an almost periodic solution $(v_n)_{n\in\mathbb{Z}}$. Let v be defined by

$$v(t) = U_B(t,n)v_n + \lim_{\lambda\to\infty}\int_n^t U_B(t,\tau)B_\lambda f(\tau)d\tau,\ t\in[n,n+1)\,.$$

Then v is well defined, continuous and by Lemma 3.1 v is an almost periodic integral solution of equation (1). ■

In the sequel we give some sufficient conditions for which condition (12) is satisfied.

Corollary 3.1 *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. Furthermore we assume that B is 1−periodic and f is almost periodic. If $T_0(t)$ is compact for $t>0$, $\sigma(f)$ is countable and E doesn't contain a copy of c_0, then equation (1) has an almost periodic integral solution if and only if it has a bounded integral solution on the whole line.*

Proof. We claim that $U_B(1,0)$ is compact if $T_0(t)$ is compact for $t>0$. In fact, let D be a bounded set in $\overline{D(A)}$, by formula (7), we have for any $x_0\in D$,

$$U_B(1,0)x_0 = T_0(1)x_0 + \lim_{\lambda\to\infty}\int_0^1 T_0(1-\tau)B_\lambda\left(B(\tau)U_B(\tau,0)x_0\right)d\tau.$$

Let $\varepsilon>0$ such that $1-\varepsilon>0$. Then

$\lim\limits_{\lambda\to\infty}\int_0^1 T_0(1-\tau)B_\lambda\left(B(\tau)U_B(\tau,0)x_0\right)d\tau =$
$T_0(\varepsilon)\lim\limits_{\lambda\to\infty}\int_0^{1-\varepsilon} T_0(1-\varepsilon-\tau)B_\lambda\left(B(\tau)U_B(\tau,0)x_0\right)d\tau + \lim\limits_{\lambda\to\infty}\int_{1-\varepsilon}^1 T_0(1-\tau)B_\lambda\left(B(\tau)U_B(\tau,0)x_0\right)d\tau.$

Let χ denote the measure of noncompactness of sets which is defined for any bounded set B in E by
$\chi(B)=\inf\{d>0 : B \text{ has a finite cover of diameter } < d\}$. Then it's well known that
$\chi(B)=0$ if and only if B is relatively compact,
$\chi(B_1+B_2)\le\chi(B_1)+\chi(B_2)$, for any bounded sets B_1 and B_2.
Using the above properties, we can that

$$\chi\left\{U_B(1,0)x_0 : x_0\in D\right\}\le a\varepsilon,\ \text{for some } a>0.$$

Letting $\varepsilon\to 0$, we get $\chi\left\{U_B(1,0)x_0 : x_0\in D\right\}=0$ and $U_B(1,0)$ is compact. Consequently condition (12) is satisfied. To end the proof, we will show that $\sigma(g(n)_{n\in\mathbb{Z}})$ is countable.

Lemma 3.2 *[8] Let $\xi:\mathbb{R}\to E$ be uniformly continuous such that its range is relatively compact in E. Then for any $s>0$, the limit*

$$\lim_{\lambda\to\infty}\int_{t-s}^t U_B(t,\tau)B_\lambda\xi(\tau)d\tau$$

exists uniformly for t in $\mathbb{R}$.

Let g_λ be defined by

$$g_\lambda(t) = \int_{t-1}^{t} U_B(t,\tau)B_\lambda f(\tau)d\tau,\ t \in \mathbb{R}.$$

Then g_λ is almost periodic and by Lemma 3.2 g_λ converges to g uniformly in $t \in \mathbb{R}$ as $\lambda \to \infty$, consequently g is almost periodic. Since Lemma 4.6 in [8] implies that $\sigma(g_\lambda) \subset \sigma(f)$ and $\sigma(g) \subset \sigma(f)$. Moreover Lemma 2.4 gives that $\sigma(g(n)_{n\in\mathbb{Z}}) \subset \sigma(g) \subset \sigma(f)$ and $\sigma(g(n)_{n\in\mathbb{Z}})$ is also countable, by Theorem 3.3 we deduce that equation (1) has an almost periodic integral solution. ■

4 Partial functional differential equation with finite delay

In this section, we apply the previous results for equation (3).

Definition 4.1 *A continuous function $u : [s-r,\infty) \to E$ is said to be an integral solution of equation (3) if and only if*
i) $\int_s^t u(\tau)d\tau \in D(A)$, *for* $t \geq s$,
ii) $u(t) = u(s) + A\int_s^t u(\tau)d\tau + \int_s^t (K(\tau)u_\tau + h(\tau))\, d\tau$, *for* $t \geq s$,
iii) $u_s = \varphi$.

Proposition 4.1 *[2] For $t \geq s$ and $\varphi \in C$ such that $\varphi(0) \in \overline{D(A)}$, equation (3) has a unique integral solution which is defined for $t \geq s$.*

By [2], the integral solution of equation (3) is given by

$$u(t) = T_0(t-s)\varphi(0) + \lim_{\lambda\to\infty}\int_s^t T_0(t-\tau)B_\lambda\,(K(\tau)u_\tau + h(\tau))\, d\tau,\ \text{for } t \geq s.$$

Note that the phase space C_0 of equation (3) is given by

$$C_0 = \left\{\varphi \in C : \varphi(0) \in \overline{D(A)}\right\}.$$

Theorem 4.1 *[1] Let $U(t)$ be defined for every $t \geq 0$, on C_0 by*

$$(U(t)\varphi)(\theta) = \begin{cases} T_0(t+\theta)\varphi(0) & \text{if } t+\theta \geq 0 \\ \varphi(t+\theta) & \text{if } t+\theta \leq 0. \end{cases}$$

Then $(U(t)\ _{t\geq 0}$ is a strongly continuous semigroup on C_0, its generator is given by

$$\begin{cases} D(A_U) = \left\{ \begin{array}{l} \varphi \in C^1([-r,0];E) : \varphi(0) \in D(A),\ \varphi'(0) \in \overline{D(A)} \text{ and} \\ \varphi'(0) = A\varphi(0) \end{array} \right\} \\ A_U\varphi = \varphi'. \end{cases}$$

Define the space $\langle X_o \rangle$ by

$$\langle X_o \rangle = \{X_o c : c \in E\},$$

where the function $X_o c$ is defined by

$$(X_o c)(\theta) = \begin{cases} 0 \text{ if } \theta \in [-r, 0[\\ c \text{ if } \theta = 0. \end{cases}$$

The space $C \oplus \langle X_o \rangle$ is provided with the following norm

$$|\varphi + X_0 c| = |\varphi| + |c|.$$

Theorem 4.2 *[1] The continuous extension $\widetilde{A_U}$ of the operator A_U defined on $C \oplus \langle X_o \rangle$ by*

$$\begin{cases} D(\widetilde{A_U}) = \left\{\varphi \in C^1([-r, 0], E) : \varphi(0) \in D(A) \text{ and } \varphi'(0) \in \overline{D(A)}\right\} \\ \widetilde{A_U}\varphi = \varphi' + X_0(A\varphi(0) - \varphi'(0)), \end{cases}$$

is a Hille-Yosida operator. If u is an integral solution of equation (3), then $x(t) = u_t$ is an integral solution of

$$\begin{cases} \dfrac{d}{dt}x(t) = \left(\widetilde{A_U} + \widetilde{B}(t)\right)x(t) + \widetilde{h}(t), \text{ for } t \geq s \\ x(s) = \varphi \end{cases} \tag{14}$$

where $\widetilde{B}(t)(\varphi + X_0 c) = X_0 K(t)\varphi$ and $\widetilde{h}(t) = X_0 h(t)$, $\varphi \in C$, $c \in E$ and $t \in \mathbb{R}$. Conversely if x is an integral solution of equation (14), then

$$u(t) = \begin{cases} x(t)(0) \text{ if } t \geq s \\ \varphi(t) \text{ if } s - r \leq t \leq s \end{cases}$$

is an integral solution of equation (3).

Let $(V(t,s))_{t \geq s}$ be the evolution family defined on C_0 by

$$V(t,s)\varphi = x_t(., s, \varphi), \text{ for } t \geq s,$$

where $x(., s, \varphi)$ is the integral solution of equation (3) with $h = 0$. Then by Theorem 3.1, we obtain

Proposition 4.2 *Suppose that equation (3) has a bounded integral solution on the whole line. If h is $p-$periodic and 1 is an isolated point in $\sigma_\Gamma(V(p,0))$, then equation (3) has a $p-$periodic integral solution. Moreover if h is anti $p-$periodic and -1 is an isolated point in $\sigma_\Gamma(V(p,0))$, then equation (3) has an anti $p-$periodic integral solution.*

Corollary 4.1 *Assume that $T_0(t)$ is compact whenever $t > 0$ and f is p-periodic. Then the existence of a bounded integral solution of (3) on the whole line implies the existence of a p-periodic integral solution of (3).*

Proof. Recall that if $T_0(t)$ is compact whenever $t > 0$, then the monodromy operator is compact for $p > r$, for more details we refer to [1] . Consequently condition (10) is satisfied. If $p \leq r$, then equation has a mp-periodic integral solution, for some m such that $mp > r$. Let z be the mp-periodic integral solution and $K = \overline{co}\,\{z_{np} : n \in \mathbb{N}\}$, then K is a convex compact set. Define the Poincaré map P_0 on K by $P_0\phi = x_p(.,0,\phi,h)$, where $x_.(.,0,\phi,h)$ is the integral solution of equation (3). Then $P_0K \subset K$ and by Schauder's fixed point theorem, we get that P_0 has a fixed point which gives that equation (3) has a p-periodic integral solution. ■

We remark that Corollary 4.1 is extendable for a partial functional differential equation with infinite delay in which linear part is nondensely defined and the phase space is a uniform fading memory space. For more details we refer to [5, Theorem 6].

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