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Existence and multiplicity of solutions for the nonlocal $p(x)$-Laplacian equations in R^N

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Abstract

This work deals with the nonlocal $p(x)$-Laplacian equations in R^N with non-variational form

$$\begin{cases} A(u)\big(-\Delta_{p(x)}u+|u|^{p(x)-2}u\big)=B(u)f(x,u) \quad \text{in } R^N, \\ u\in W^{1,p(x)}(R^N), \end{cases}$$

and with the variational form

$$\begin{cases} a\Big(\displaystyle\int_{R^N}\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx\Big)(-\Delta_{p(x)}u+|u|^{p(x)-2}u) \\ =B\Big(\displaystyle\int_{R^N}F(x,u)dx\Big)f(x,u) \quad \text{in } R^N, \\ u\in W^{1,p(x)}(R^N), \end{cases}$$

where $F(x,t)=\int_0^t f(x,s)ds$, and a is allowed to be singular at zero. Using (S_+) mapping theory and the variational method, some results on existence and multiplicity for the problems in R^N are obtained.

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1 Introduction

The study on the problems of the nonlocal $p(x)$-Laplacian has attracted more and more interest in the recent years(e.g., see [1, 2, 3]), they mainly concerned the problems of the bounded domain, however the study on the existence of solutions for problems of nonlocal $p(x)$-Laplacian in R^N is rare. We know that in the study of p-Laplacian equations in R^N, a main difficulty arises from the lack of compactness. In this paper, we study the nonlocal $p(x)$-Laplacian equations in R^N with non-variational form

$$\begin{cases} A(u)\big(-\Delta_{p(x)}u+|u|^{p(x)-2}u\big)=B(u)f(x,u) \quad \text{in } R^N, \\ u\in W^{1,p(x)}(R^N), \end{cases} \tag{1.1}$$

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and with the variational form

$$
\begin{cases} a\Big(\int_{R^N}\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx\Big)(-\Delta_{p(x)}u+|u|^{p(x)-2}u) \\ = B\Big(\int_{R^N}F(x,u)dx\Big)f(x,u) \qquad\qquad\qquad\qquad \text{in } R^N, \\ u\in W^{1,p(x)}(R^N), \end{cases} \tag{1.2}
$$

where A and B are two functionals defined on $W^{1,p(x)}(R^N)$, $F(x,t)=\int_0^t f(x,s)ds$, and a is allowed to be singular at zero. To deal with the problems (1.1) and (1.2), we will overcome the difficulty caused by the absence of compactness through the method of weight function.

The variable exponent problems have been studied by many authors. We refer to [4, 5] for applied background, to [6, 7, 8] for the variable exponent Lebesgue-Sobolev spaces and to [9, 10, 11, 12] for the $p(x)$-Laplacian equations without nonlocal coefficient.

This paper is organized as follows: in Section 2, we deal with the problem with non-variational form; in Section 3, we deal with the problem with variational form.

2 The non-variational form

Let $\Omega\subset R^N(N\geq 2)$ be an open subset of R^N, set

$$L_+^\infty(\Omega)=\{p\in L^\infty(\Omega): ess\inf_{\Omega}\ p(x)\geq 1\}.$$

For $p\in L_+^\infty(\Omega)$, let

$$p^-(\Omega)=ess\inf_{x\in\Omega}p(x),\qquad p^+(\Omega)=ess\sup_{x\in\Omega}p(x).$$

Denote by $S(\Omega)$ the set of all measurable real functions defined on Ω. Two measurable functions are considered as the same element of $S(\Omega)$ when they are equal almost everywhere. For $p\in L_+^\infty(\Omega)$, define

$$L^{p(x)}(\Omega)=\{u\in S(\Omega):\int_\Omega |u|^{p(x)}dx<\infty\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)}=|u|_{p(x)}=\inf\{\lambda>0:\int_\Omega |\frac{u}{\lambda}|^{p(x)}dx\leq 1\}$$

and

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):|\nabla u|\in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)}=|u|_{L^{p(x)}(\Omega)}+|\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

For some basic properties of the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ we may refer to [6, 7, 8].

Proposition 2.1([6], [8])**.** *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are all separable and reflexive Banach spaces if $p^- > 1$.*

Proposition 2.2([6], [8])**.** *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}dx$. If $u, u_k \in L^{p(x)}(\Omega)$, we have*
(1) *For $u \neq 0$, $\quad |u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$.*
(2) *$|u|_{p(x)} < 1(= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$.*
(3) *If $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$.*
(4) *If $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$.*
(5) *$\lim_{k\to\infty} |u_k|_{p(x)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0$.*
(6) *$\lim_{k\to\infty} |u_k|_{p(x)} = \infty \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = \infty$.*

In this section we consider problem (1.1), the nonlocal $p(x)$-Laplacian equation in R^N without the variational structure.

$u \in W^{1,p(x)}(R^N)$ is said to be a (weak) solution of (1.1) if

$$A(u)\int_{R^N}(|\nabla u|^{p(x)-2}\nabla u\nabla v + |u|^{p(x)-2}uv)dx = B(u)\int_{R^N} f(x,u)vdx,$$

for every $v \in W^{1,p(x)}(R^N)$.

In what follows, for simplicity, we write $X = W^{1,p(x)}(R^N)$ and c_i, C, C_i are positive constants.

Define the mapping T, G, $L_{p(\cdot)}$ and $N_f : X \to X^*$ respectively by

$$\begin{aligned}
T(u)v &= A(u)\int_{R^N}(|\nabla u|^{p(x)-2}\nabla u\nabla v + |u|^{p(x)-2}uv)dx,\ \forall u, v \in X,\\
G(u)v &= B(u)\int_{R^N} f(x,u)vdx,\ \forall u, v \in X,\\
L_{p(\cdot)}(u)v &= \int_{R^N}(|\nabla u|^{p(x)-2}\nabla u\nabla v + |u|^{p(x)-2}uv)dx,\ \forall u, v \in X,\\
N_f(u)v &= \int_{R^N} f(x,u)vdx,\ \forall u, v \in X.
\end{aligned}$$

Then $T(u) = A(u)L_{p(\cdot)}(u)$ and $G(u) = B(u)N_f(u)$ for $u \in X$. It is clear that $u \in X$ is a solution of (1.1) if and only if $T(u) - G(u) = 0$.

Proposition 2.3. *Suppose that A satisfies the following condition:*
(A_1) *$A : X \to [0, +\infty)$ is continuous and bounded on any bounded subset of X, $A(u) > 0$ for all $u \in X \setminus \{0\}$, and for any bounded sequence $\{u_n\} \subset X$ for which $A(u_n) \to 0$, u_n must converge strongly to 0 in X.*

Then the mapping $T : X \to X^*$ *is continuous and bounded, and is of type* (S_+).

The proof is similar to [3], so omit it.

Proposition 2.4. *Suppose that the following conditions are satisfied:*
(f_1)

$$|f(x,t)| \le b(x)|t|^{q(x)-1}, \qquad \forall (x,t) \in R^N \times R,$$

where $b(x) \ge 0$, $b \in L^{r(x)}(R^N) \bigcap L^\infty(R^N)$, $r,\ q \in L^\infty_+(R^N)$, $q(x) \ll p^*(x)$, *and there is* $s \in L^\infty(R^N)$ *such that*

$$p(x) \le s(x) \le p^*(x), \qquad \frac{1}{r(x)} + \frac{q(x)}{s(x)} = 1.$$

(B_1) *The functional* $B : X \to R$ *is continuous and bounded on any bounded subset of* X. *Then the mapping* $G : X \to X^*$ *is completely continuous.*

Proof. Under the condition (f_1), the mapping $N_f : X \to X^*$ is sequentially weakly-strongly continuous (see [9]). The continuity of G is obvious. Assume $\{u_n\}$ is bounded, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $N_f(u_{n_k})$ and $B(u_{n_k})$ are strongly convergent, so is $G(u_{n_k})$. This shows that $G : X \to X^*$ is completely continuous. $\square$

We know that the sum of an (S_+) type mapping and a completely continuous mapping is of type (S_+), so from Propositions 2.3 and 2.4 we have the following:

Corollary 2.1. *Let* (A_1), (B_1) *and* (f_1) *hold. Then the mapping* $T - G : X \to X^*$ *is continuous and bounded, and is of type* (S_+).

Theorem 2.1. *Let* (A_1), (B_1) *and* (f_1) *hold. Suppose that the following conditions are satisfied:*
(A_2) *There are constants* $\alpha \in R$, $M > 0$ *and* $c_1 > 0$ *such that*

$$A(u) \ge c_1 \|u\|^\alpha \qquad for\, u \in X \quad with\ \|u\| \ge M.$$

(B_2) *There are constants* $\beta \in R$, $M > 0$ *and* $c_2 > 0$ *such that*

$$|b(u)| \le c_2 \|u\|^\beta \qquad for\, u \in X \quad with\ \|u\| \ge M.$$

Then problem (1.1) has at least one solution. If, in addition, $\alpha + p_- > 1$, *then the mapping* $T - G : X \to X^*$ *is subjective, and consequently for any* $h \in X^*$ *the operator equation* $T(u) - G(u) = h$ *has at least one solution.*

Proof. Under the hypotheses of Theorem 2.1, by Corollary 2.1, the mapping $T - G : X \to X^*$ is continuous and bounded, and is of type (S_+). For sufficiently large $\|u\|$, we have that

$$\begin{aligned}\Big(T(u) - G(u)\Big)u &= A(u)\int_{R^N}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - B(u)\int_{R^N} f(x,u)u dx \\ &\ge c_1\|u\|^\alpha \|u\|^{p^-} - c_2 c_3 \|u\|^\beta |u|^{q^+}_{q(\cdot)}\end{aligned}$$

$$
\begin{aligned}
&\geq c_1\|u\|^{\alpha+p^-} - c_2c_4\|u\|^{\beta}\|u\|^{q^+}\\
&= c_1\|u\|^{\alpha+p^-} - c_5\|u\|^{\beta+q^+}\\
&\geq c_6\|u\|^{\alpha+p^-} > 0.
\end{aligned}
$$

So, by the degree theory for (S_+) type mappings(see [13]), for $R > 0$ large enough, we have $deg(T-G, B(0,R), 0) = 1$, and consequently, there exists $u \in B(0,R)$ such that $T(u) - G(u) = 0$, that is, (1.1) has at least one solution $u \in B(0,R)$. If in addition, $\alpha + p^- > 1$, then

$$
\lim_{\|u\|\to+\infty} \frac{(T(u)-G(u))u}{\|u\|} \geq \lim_{\|u\|\to+\infty} c_6\|u\|^{\alpha+p^--1} = +\infty,
$$

that is, the mapping $T-G$ is coercive, and consequently, by the surjection theorem for the pseudomonotone mappings(see [14]), the mapping $T-G$ is surjective. □

Remark 2.1. *In Theorem 2.1, α and β are allowed to be negative.*

3 The variational form

In this section we consider problem (1.2) with variational form, where f satisfies condition (f_1), k and g are two real functions satisfying the following conditions.
(k_1) $k : (0,+\infty) \to (0,+\infty)$ *is continuous and* $k \in L^1(0,t)$ *for any* $t > 0$.
(g_1) $g : R \to R$ *is continuous.*

Note that the function k satisfying (k_1) may be singular at $t = 0$.

Define

$$
\begin{aligned}
&\hat{k}(t) = \int_0^t k(s)ds, \quad \forall t \geq 0; \quad \hat{g}(t) = \int_0^t g(s)ds, \quad \forall t \in R,\\
&I_1(u) = \int_{R^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)}dx, \quad I_2(u) = \int_{R^N} F(x,u)dx, \forall u \in X,\\
&J(u) = \hat{k}(I_1(u)) = \hat{k}(\int_{R^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)}dx), \forall u \in X,\\
&\Phi(u) = \hat{g}(I_2(u)) = \hat{g}(\int_{R^N} F(x,u)dx), \forall u \in X,\\
&E(u) = J(u) - \Phi(u), \forall u \in X.
\end{aligned}
$$

Proposition 3.1. *Let (f_1), (k_1), (g_1) hold. Then the following statements hold:*
(1) $\hat{k} \in C^0([0,+\infty)) \bigcap C^1\big((0,+\infty)\big)$, $\hat{k}(0) = 0$, $\hat{k}'(t) = k(t) > 0$, *for any* $t > 0$, $\hat{k}$ *is increasing on* $[0,+\infty)$; $\hat{g} \in C^1(R)$, $\hat{g}(0) = 0$.
(2) J, Φ, $E \in C^0(X)$, $J(0) = \Phi(0) = E(0) = 0$. $J \in C^1(X \setminus \{0\})$. *For every* $u \in X \setminus \{0\}$ *and* $v \in X$, *it holds that*

$$
E'(u)v = k\big(I_1(u)\big) \int_{R^N} (|\nabla u|^{p(x)-2}\nabla u\nabla v + |u|^{p(x)-2}uv)dx
$$

$$- \quad g\big(I_2(u)\big)\int_{R^N} f(x,u)vdx.$$

Thus $u \in X \setminus \{0\}$ is a (weak) solution of (1.2) if and only if u is a nontrivial critical point of E.
(3) *The functional $J : X \to R$ is sequentially weakly lower semi-continuous, $\Phi : X \to R$ is sequentially weakly continuous, and thus E is sequentially weakly lower semi-continuous.*
(4) *The mapping $\Phi' : X \to X^*$ is sequentially weakly-strongly continuous. For any open set $D \subset X \setminus \{0\}$ with $\overline{D} \in X \setminus \{0\}$, the mapping J' and $E' : \overline{D} \to X^*$ are bounded, and are of type (S_+) .*

Proof. The proof of statements (1) and (2) is obvious. Since the function $\hat{k}(t)$ is increasing and the functional I_1 is sequentially weakly lower semi-continuous, we can see that the functional $J : X \to R$ is sequentially weakly lower semi-continuous. Moreover, under the condition (f_1), Φ and Φ' are sequentially weakly-strongly continuous. Note let $\overline{D} \in X \setminus \{0\}$, it is clear that the mapping J' and $E' : \overline{D} \to X^*$ are bounded. In order to prove that $J' : \overline{D} \to X^*$ is of type (S_+), assuming that $\{u_n\} \subset \overline{D}$, $u_n \rightharpoonup u$ in X and $\overline{lim}_{n\to\infty} J'(u_n)(u_n - u) \leq 0$, then there exist positive constants c_1 and c_2 such that $c_1 \leq \int_{R^N} \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}dx \leq c_2$ and so there exist positive constants c_3 and c_4 such that $c_3 \leq k(\int_{R^N} \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}dx) \leq c_4$. Noting that $J'(u_n) = k(\int_{R^N} \frac{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}}{p(x)}dx)L_{p(\cdot)}(u_n)$, it follows from $\overline{lim}_{n\to\infty} J'(u_n)(u_n - u) \leq 0$ that $\overline{lim}_{n\to\infty} L_{p(\cdot)}(u_n)(u_n - u) \leq 0$. Since $L_{p(\cdot)}$ is of type (S_+), we obtain $u_{n_k} \to u$ in X. This shows that the mapping $J' : \overline{D} \to X^*$ is of type (S_+). Moreover, since Φ' is sequentially weakly-strongly continuous, the mapping $E' : \overline{D} \to X^*$ is of type (S_+). □

Remark 3.1. *To verify that E satisfies $(P.S)$ condition on E, it is enough to verify that any $(P.S)$ sequence is bounded.*

Theorem 3.1. *Let (f_1), (k_1), (g_1) and the following conditions hold.*
(k_2) There are positive constants α_1, M and C such that $\hat{k}(t) \geq Ct^{\alpha_1}$ for $t \geq M$.
(g_2) There are positive constants β_1 and C_1 such that $|\hat{g}(t)| \leq C_1(1 + |t|^{\beta_1})$ for $t \in R$.
(E_2) $\beta_1 q_+ < \alpha_1 p_-$.
Then the functional E is coercive and obtain its infimum in X at some $u_0 \in X$. Thus u_0 is a solution of (1.2) if E is differentiable at u_0, and in particular, if $u_0 \neq 0$.

Proof. For $\|u\|$ large enough, by (f_1), (k_2), (g_2) and (E_2), we have that

$$\begin{aligned}
&J(u) = \hat{k}(\int_{R^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)}dx) \geq \hat{k}(C\|u\|^{p_-}) \geq C_2\|u\|^{\alpha_1 p_-},\\
&\Big|\int_{R^N} F(x,u)dx\Big| \leq C_3|b|_{r(x)}||u|^{q(x)}|_{\frac{s(x)}{q(x)}} \leq C_4\|u\|^{q_+},\\
&|\Phi(u)| = \hat{g}(\int_{R^N} F(x,u)dx) \leq C_5\|u\|^{\beta_1 q_+},\\
&E(u) = J(u) - \Phi(u) \geq C_2\|u\|^{\alpha_1 p_-} - C_5\|u\|^{\beta_1 q_+} \geq C_6\|u\|^{\alpha_1 p_-},
\end{aligned}$$

hence E is coercive. Since E is sequentially weakly lower semi-continuous and X is reflexive, E attains its infimum in X at some $u_0 \in X$. In the case where E is differential at

u_0, u_0 is a solution of (1.2). The proof is complete. $\square$

As X is a separable and reflexive Banach space, there exist (see [15, Section 17]) $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that

$$f_n(e_m) = \begin{cases} 1 & \text{if}\, n = m, \\ 0 & \text{if}\, n \neq m. \end{cases}$$

$$X = \overline{span}\{e_n : n = 1, 2, \cdots, \}, \quad X^* = \overline{span}^{W^*}\{e_n : n = 1, 2, \cdots, \}.$$

For $k = 1, 2, \cdots$, denote

$$X_k = span\{e_k\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{3.1}$$

Proposition 3.2. *Assume that $\Phi : X \to R$ is weakly-strongly continuous and $\Phi(0) = 0$, $\gamma > 0$ is a given positive constant. Set*

$$\beta_k = \sup_{u \in Z_k, \|u\| \geq \gamma} |\Phi(u)|,$$

then $\beta_k \to 0$ as $k \to \infty$.

The proof of the Proposition 3.2 is similar to [9], here we omit it.

Theorem 3.2. *Let (f_1), (k_1), (k_2), (g_1), (g_2), (E_2) and the following conditions hold.*
(k_3) There exists $\alpha_2 > 0$ such that $\overline{lim}_{t \to 0^+} \frac{\hat{k}(t)}{t^{\alpha_2}} < +\infty$.
(g_3) There exists $\beta_2 > 0$ such that $\underline{lim}_{t \to 0^+} \frac{\hat{g}(t)}{t^{\beta_2}} > 0$.
(f_2) $f(x, -t) = -f(x, t)$ for $x \in \Omega$ and $t \in R$.
(f_3) $\exists \delta > 0$,

$$f(x, t) \geq b_0(x) t^{q_0(x)-1} \quad for\, x \in R^N \, and\, 0 < t \leq \delta,$$

where $b_0 > 0$, $b_0 \in C(R^N, R)$, $b_0 \not\equiv 0$, $q_0 \in L_+^{\infty}(R^N)$.
(E_3) $q_0^+ \beta_2 < \alpha_2 p^-$.
Then problem (1.2) has a sequence of solutions $\{\pm u_k : k = 1, 2, \cdots, \}$ such that $E(\pm u_k) < 0$ and $E(\pm u_k) \to 0$ as $k \to \infty$.

Proof. As E is coercive, by Remark 3.1 we know that E satisfies $(P.S)$ condition. By (f_2), E is an even functional. Denote by $\gamma(A)$ the genus of A(see [16]). Set

$$\begin{aligned}
&\Sigma = \{A \subset X \setminus \{0\} : A\, is\, compact\, and\, A = -A\}, \\
&\Sigma_k = \{A \in \Sigma : \gamma(A) \geq k\}, \quad k = 1, 2, \cdots, \\
&c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} \varphi(u), \quad k = 1, 2, \cdots,
\end{aligned}$$

we have

$$-\infty < c_1 \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdots.$$

Now let us prove that $c_k < 0$ for every k.

As $b_0 \not\equiv 0$ and $b_0 \geq 0$, we can find a bounded open set $\Omega \subset R^N$, such that $b_0(x) > 0$ for $x \in \Omega$. The space $W_0^{1,p(x)}(\Omega)$ is a subspace of X. For any k, we can choose a k dimensional linear subspace E_k of $W_0^{1,p(x)}(\Omega)$ such that $E_k \subset C_0^\infty(\Omega)$. As the norms on E_k are equivalent each other, there exists $\rho_k \in (0,1)$ such that $u \in E_k$ with $\|u\| \leq \rho_k$ implies $|u|_{L^\infty} \leq \delta$. Set

$$S_{\rho_k}^{(k)} = \{u \in E_K : \|u\| = \rho_k\},$$

the compactness of $S_{\rho_k}^{(k)}$ with condition (f_3) concludes the existence of a constant d_k such that

$$\int_\Omega \frac{b_0(x)|u|^{q(x)}}{q_0(x)} dx \geq d_k, \quad \forall u \in S_{\rho_k}^{(k)}.$$

For $u \in S_{\rho(k)}^{(k)}$ and sufficiently small $\lambda > 0$ we have

$$\begin{aligned} E(\lambda u) &= \hat{k}\Big(\int_{R^N} \frac{\lambda^{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})}{p(x)} dx\Big) - \hat{g}(\int_{R^N} F(x, \lambda u) dx) \\ &\leq C_7\Big(\int_{R^N} \frac{\lambda^{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})}{p(x)} dx\Big)^{\alpha_2} - C_8(\int_\Omega \frac{b_0 \lambda^{q_0(x)}|u|^{q_0(x)}}{q_0(x)} dx)^{\beta_2} \\ &\leq C_9 \lambda^{\alpha_2 p^-} \rho_k^{\alpha_2 p^-} - C_{10} \lambda^{\beta_2 q_0^+} d_k^{\beta_2}. \end{aligned}$$

As $q_0^+ \beta_2 < \alpha_2 p^-$, we can find $\lambda_k \in (0,1)$ and $\epsilon_k > 0$ such that

$$E(\lambda_k u) \leq -\epsilon_k < 0, \quad \forall u \in S_{\rho_k}^{(k)},$$

that is

$$E(u) \leq -\epsilon_k < 0, \quad \forall u \in S_{\lambda_k \rho_k}^{(k)}.$$

We know that $\gamma(S_{\lambda_k \rho_k}^{(k)}) = k$, so $c_k \leq -\epsilon_k < 0$.
By the genus theory (see [16]), each c_k is a critical value of E, hence there is a sequence of solutions $\{\pm u_k : k = 1, 2, \cdots,\}$ of (1.2) such that $E(\pm u_k) = c_k < 0$.
It remains to prove $c_k \to 0$ as $k \to \infty$.
By the coerciveness of E, there exists a constant $\gamma > 0$ such that $E(u) > 0$ when $\|u\| \geq \gamma$. Taking arbitrarily $A \in \Sigma_k$, then $\gamma(A) \geq k$. Let Y_k and Z_k be the subspaces of X as mentioned in (3.1), according to the properties of genus we know that $A \cap Z_k \neq \varnothing$. Let

$$\beta_k = \sup_{u \in Z_k, \|u\| \geq \gamma} |\Phi(u)|,$$

by Proposition 3.2 we have $\beta_k \to 0$ as $k \to \infty$. When $u \in Z_k$ and $\|u\| \geq \gamma$, we have

$$E(u) = J(u) - \Phi(u) \geq -\Phi(u) \geq -\beta_k,$$

hence

$$\sup_{u \in A} E(u) \geq -\beta_k,$$

and then $c_k \geq -\beta_k$, this concludes $c_k \to 0$ as $k \to \infty$. $\square$

Proposition 3.3. *Let (f_1), (k_1), $(g_1)$and the following conditions be satisfied:*
(k_2') (k_2) with $\alpha_1 p^- > 1$ hold.
(k_4) There exists $\lambda > 0$ such that $\lambda\hat{k}(t) \geq tk(t)$ for $t > 0$.
(g_4) There exists $\nu > 0$ such that $\nu\hat{g}(t) \leq g(t)t$ for $t > 0$.
(f_4) There exists $\mu > 0$, such that $0 < \mu F(x,t) \leq f(x,t)t$, $t \neq 0$ and $\forall x \in R^N$.
(E_4) $\lambda p^+ < \nu\mu$.
Then E satisfies condition $(P.S)_c$ for any $c \neq 0$.

Proof. By (k_4) for $\|u\|$ large enough,

$$\begin{aligned}\lambda p^+ J(u) &= \lambda p^+ \hat{k}\Big(\int_{R^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx\Big)\\ &\geq p^+ k\Big(\int_{R^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx\Big)\int_{R^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx\\ &\geq k\Big(\int_{R^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx\Big)\int_{R^N} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx\\ &= J'(u)u. \end{aligned} \tag{3.2}$$

From (f_4) we may know

$$0 \leq \mu \int_{R^N} F(x,u)dx \leq \int_{R^N} f(x,u)udx, \ \forall u \in X.$$

Moreover by (g_4),

$$\begin{aligned}\nu\mu\Phi(u) &= \nu\mu\hat{g}\Big(\int_{R^N} F(x,u)dx\Big)\\ &\leq \mu g\Big(\int_{R^N} F(x,u)dx\Big)\int_{R^N} F(x,u)dx\\ &\leq g\Big(\int_{R^N} F(x,u)d\Big)\int_{R^N} f(x,u)udx = \Phi'(u)u,\end{aligned}$$

so

$$\Phi'(u)u - \nu\mu\Phi(u) \geq 0, \text{ for every } u \in X. \tag{3.3}$$

Now let $\{u_n\} \subset X \setminus \{0\}$ and $E'(u_n) \to 0$ and $E(u_n) \to c$ with $c \neq 0$. Applying (3.2) and (3.3) and (k_2'), for sufficiently large n, we have

$$\begin{aligned}\nu\mu c + 1 + \|u\| &\geq \nu\mu E(u_n) - E'(u_n)u_n\\ &\geq (\nu\mu - \lambda p^+)J(u_n) + \Big(\lambda p^+ J(u_n) - J'(u_n)u_n\Big) + \Phi'(u_n) - \nu\mu\Phi(u_n)\\ &\geq C_{11}\|u\|^{\alpha_1 p^-} - C_{12}.\end{aligned} \tag{3.4}$$

Since $\alpha_1 p^- > 1$, (3.4) implies that $\{\|u_n\|\}$ is bounded. By Proposition 3.1, E satisfies condition $(P.S)_c$ for any $c \neq 0$. □

Proposition 3.4. *Under the hypotheses of Proposition 3.3, for any $\omega \in X \setminus \{0\}$, $E(s\omega) \to -\infty$ as $s \to +\infty$.*

For the proof of Proposition 3.4, we refer to [3].

Proposition 3.5. *Let* (f_1),(k_1),(g_1) *and the following conditions be satisfied:*
(k_5) *There exists* $\alpha_3 > 0$ *such that* $\underline{\lim}_{t\to 0^+} \frac{\hat{k}(t)}{t^{\alpha_3}} > 0$.
(g_5) *There exists* $\beta_3 > 0$ *such that* $\overline{\lim}_{t\to 0} \frac{\hat{g}(t)}{t^{\beta_3}} < \infty$.
(f_5) *There exists* $r \in L^\infty_+(R^N)$ *such that* $p(x) \le r(x) \le p^*(x)$ *for* $x \in R^N$ *and* $\underline{\lim}_{t\to 0} \frac{|F(x,t)|}{|t|^{r(x)}} < +\infty$ *uniformly in* $x \in R^N$.
$(E_5)\alpha_3 p^+ < \beta_3 r^-$.
Then there exist positive constants ρ *and* δ *such that* $E(u) \ge \delta$ *for* $\|u\| = \rho$.

Proof. It follows from (k_5) that $J(u) \ge C_{13}\|u\|^{\alpha_3 p^+}$ for $\|u\|$ small enough. It follows from (g_5), (f_5) and (f_1) that $|\Phi(u)| \le C_{14}\|u\|^{\beta_3 r^-}$ for $\|u\|$ small enough. So, by (E_5), we may obtain the conclusion of Proposition 3.5. □

By the Mountain Pass lemma(see [17]), from Proposition 3.3-3.5, we have

Theorem 3.3. *Let all hypotheses of Propositions 3.3-3.5 hold. Then problem (1.2) has a nontrivial solution with positive energy.*

By the Symmetric Mountain Pass lemma(see e.g. [17]), we have

Theorem 3.4. *Under the hypotheses of Theorem 3.3, if, in addition, f satisfies* (f_2), *then problem (1.2) has a sequence of solutions* $\{\pm u_n\}$ *such that* $E(\pm u_n) \to +\infty$ *as* $n \to \infty$.

Example 3.1. *Let* $f(x,t) = b(x)|t|^{q(x)-2}t$ *for* $t \in R$, *where* $b(x)$, $q(x)$ *satisfies* (f_1). $k(t) = t^{\alpha-1}$ *for* $t > 0$, *where* $\alpha > 0$. $g(t) = |t|^{\beta-2}t$, *for* $t \in R$ *where* $\beta \ge 1$. *Suppose* $\alpha p^+ < \beta q^-$, *then all hypotheses of Theorems 3.3-3.4 are satisfied.*

Remark 3.1. *(1) In this paper,* R^N *can be replaced by an unbounded domain* Ω *with cone property, in this case the solution of problems (1.1) and (1.2) is defined in the space* $W_0^{1,p(x)}(\Omega)$.

(2) If $p(x)$ *and* $f(x,u)$ *are radially symmetric in* x, *one can find the radially symmetric solutions of problem (1.2). The corresponding problem become much easier.*

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