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# Fixed points for some non-obviously contractive operators defined in a space of continuous functions

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**Abstract**

Let $X$ be an arbitrary (real or complex) Banach space, endowed with the norm $|\cdot|$. Consider the space of the continuous functions $C([0,T],X)$ $(T>0)$, endowed with the usual topology, and let $M$ be a closed subset of it. One proves that each operator $A : M \to M$ fulfilling for all $x, y \in M$ and for all $t \in [0,T]$ the condition

$$\begin{aligned} |(Ax)(t) - (Ay)(t)| \quad &\leq \quad \beta \left| x(\nu(t)) - y(\nu(t)) \right| + \\ &+ \frac{k}{t^{\alpha}} \int_0^t \left| x(\sigma(s)) - y(\sigma(s)) \right| ds, \end{aligned}$$

(where $\alpha$, $\beta \in [0,1)$, $k \geq 0$, and $\nu$, $\sigma : [0,T] \to [0,T]$ are continuous functions such that $\nu(t) \leq t$, $\sigma(t) \leq t$, $\forall t \in [0,T]$) has exactly one fixed point in $M$. Then the result is extended in $C(\mathbb{R}_+, X)$, where $\mathbb{R}_+ := [0,\infty)$.

## 1. Introduction

A result due to Krasnoselskii (see, e.g. [1]) ensures the existence of fixed points for an operator which is the sum of two operators, one of them being compact and the other being contraction. A natural question is whether the result continues to hold if the first operator is not compact. In [2] and [3] the case when the compactity is replaced to a Lipschitz condition is considered; the result is proved only in the space of the continuous functions.

More precisely, let $X$ be a (real or complex) Banach space, endowed with the norm $|\cdot|$. Consider the space $C([0,T],X)$ of the continuous functions from $[0,T]$ into $X$ $(T>0)$, endowed with the usual topology and $M$ a closed subset of $C([0,T],X)$.

Let $A : M \to M$ be an operator with the property that there exist $\alpha, \beta \in [0,1)$, $k \geq 0$ such that for every $x, y \in M$,

$$\begin{aligned} |(Ax)(t) - (Ay)(t)| \quad &\leq \quad \beta \left| x(t) - y(t) \right| + \\ &+ \frac{k}{t^{\alpha}} \int_0^t \left| x(s) - y(s) \right| ds, \ \forall t \in [0,T]. \qquad (1.1) \end{aligned}$$

In [2] the authors resume the result contained in [3] and prove that the condition (1.1) ensures the existence in $M$ of a unique fixed point for $A$; the result is deduced through a subtle technique. Finally, by admitting that (1.1) is fulfilled for every $t \in \mathbb{R}_+$, the result is generalized to the space $BC(\mathbb{R}_+, X)$, (where $\mathbb{R}_+ := [0, \infty)$), i.e. the space of the bounded and continuous functions from $\mathbb{R}_+$ into $X$.

In the present paper we give an alternative proof of the first result contained in [2], in a more general case, by means of a new approach; more exactly, we use in $C([0,T], X)$ a special norm which is equivalent to the classical norm. Then we extend the result to the space $C(\mathbb{R}_+, X)$.

## 2. The first existence result

Consider the space $C([0,T], X)$, where $(X, |\cdot|)$ is a Banach space, $T > 0$ and let $\gamma \in (0,T)$, $\lambda > 0$.

Define for $x \in C([0,T], X)$,

$$\|x\| := \|x\|_\gamma + \|x\|_\lambda ,$$

where we denoted

$$\|x\|_\gamma := \sup_{t \in [0,\gamma]} \{|x(t)|\}, \quad \|x\|_\lambda := \sup_{t \in [\gamma, T]} \left\{ e^{-\lambda(t-\gamma)} |x(t)| \right\}.$$

It is easily seen that $\|\cdot\|$ is a norm on $C([0,T], X)$ and it defines the same topology as the norm $\|\cdot\|_\infty$, where

$$\|x\|_\infty := \sup_{t \in [0,T]} \{|x(t)|\}.$$

**Theorem 2.1** *Let $M$ be a closed subset of $C([0,T], X)$ and $A : M \to M$ be an operator. If there exist $\alpha, \beta \in [0,1)$, $k \geq 0$ such that for every $x, y \in M$ and for every $t \in [0,T]$,*

$$\begin{aligned} |(Ax)(t) - (Ay)(t)| \quad \leq \quad & \beta |x(\nu(t)) - y(\nu(t))| + \\ & + \frac{k}{t^\alpha} \int_0^t |x(\sigma(s)) - y(\sigma(s))| \, ds, \end{aligned} \tag{2.1}$$

*where $\nu$, $\sigma : [0,T] \to [0,T]$ are continuous functions such that $\nu(t) \leq t$, $\sigma(t) \leq t$, $\forall t \in [0,T]$, then $A$ has a unique fixed point in $M$.*

**Proof.** We shall apply the Banach Contraction Principle. To this aim, we show that $A$ is contraction, i.e. there exists $\delta \in [0,1)$ such that for any $x, y \in M$,

$$\|Ax - Ay\| \leq \delta \|x - y\|.$$

Let $t \in [0, \gamma]$ be arbitrary. Then we have

$$\begin{aligned}
|(Ax)(t) - (Ay)(t)| &\leq \beta |x(\nu(t)) - y(\nu(t))| + \\
&\quad + \frac{k}{t^\alpha} \int_0^t |x(\sigma(s)) - y(\sigma(s))| \, ds \leq \\
&\leq \beta \|x - y\|_\gamma + t^{1-\alpha} k \|x - y\|_\gamma \leq \\
&\leq \left(\beta + k\gamma^{1-\alpha}\right) \|x - y\|_\gamma
\end{aligned}$$

and hence

$$\|Ax - Ay\|_\gamma \leq \left(\beta + k\gamma^{1-\alpha}\right) \|x - y\|_\gamma . \tag{2.2}$$

Let $t \in [\gamma, T]$ be arbitrary. Then we get

$$\begin{aligned}
|(Ax)(t) - (Ay)(t)| &\leq \beta |x(\nu(t)) - y(\nu(t))| + \\
&\quad + \frac{k}{t^\alpha} \left( \int_0^\gamma |x(\sigma(s)) - y(\sigma(s))| \, ds + \right. \\
&\quad \left. + \int_\gamma^t |x(\sigma(s)) - y(\sigma(s))| \, e^{-\lambda((\sigma(s)) - \gamma)} e^{\lambda((\sigma(s)) - \gamma)} ds \right) \\
&\leq \beta |x(\nu(t)) - y(\nu(t))| + \frac{k}{\gamma^\alpha} \left( \gamma \|x - y\|_\gamma + \right. \\
&\quad \left. + \|x - y\|_\lambda \int_\gamma^t e^{\lambda(\sigma(s) - \gamma)} ds \right) \\
&\leq \beta |x(\nu(t)) - y(\nu(t))| + \frac{k}{\gamma^\alpha} \left( \gamma \|x - y\|_\gamma + \right. \\
&\quad \left. + \|x - y\|_\lambda \int_\gamma^t e^{\lambda(s - \gamma)} ds \right) \\
&< \beta |x(\nu(t)) - y(\nu(t))| + \frac{k}{\gamma^\alpha} \left( \gamma \|x - y\|_\gamma + \right. \\
&\quad \left. + \|x - y\|_\lambda \frac{e^{\lambda(t - \gamma)}}{\lambda} \right).
\end{aligned}$$

It follows that

$$\begin{aligned}
|(Ax)(t) - (Ay)(t)| \, e^{-\lambda(t-\gamma)} &< \beta |x(\nu(t)) - y(\nu(t))| \, e^{-\lambda(t-\gamma)} + \\
&\quad + k\gamma^{1-\alpha} \|x - y\|_\gamma + \frac{k}{\lambda} \gamma^{-\alpha} \|x - y\|_\lambda
\end{aligned}$$

and therefore

$$\begin{aligned}
\|Ax - Ay\|_\lambda &\leq \beta \sup_{t \in [\gamma, T]} \left\{ |x(\nu(t)) - y(\nu(t))| \, e^{-\lambda(t-\gamma)} \right\} + \\
&\quad + k\gamma^{1-\alpha} \|x - y\|_\gamma + \frac{k}{\lambda} \gamma^{-\alpha} \|x - y\|_\lambda \\
&\leq \beta \sup_{t \in [\gamma, T]} \left\{ |x(\nu(t)) - y(\nu(t))| \, e^{-\lambda(\nu(t)-\gamma)} \right\} +
\end{aligned} \tag{2.3}$$

$$+k\gamma^{1-\alpha}\left\|x-y\right\|_{\gamma}+\frac{k}{\lambda}\gamma^{-\alpha}\left\|x-y\right\|_{\lambda}$$
$$\leq \left(\beta+\frac{k}{\lambda}\gamma^{-\alpha}\right)\left\|x-y\right\|_{\lambda}+k\gamma^{1-\alpha}\left\|x-y\right\|_{\gamma}.$$

By (2.2) and (2.3) we obtain

$$\left\|Ax-Ay\right\|\leq\left(\beta+2k\gamma^{1-\alpha}\right)\left\|x-y\right\|_{\gamma}+\left(\beta+\frac{k}{\lambda}\gamma^{-\alpha}\right)\left\|x-y\right\|_{\lambda}. \tag{2.4}$$

Since $\beta\in[0,1)$, for $\gamma\in\left(0,\left(\frac{1-\beta}{2k}\right)^{\frac{1}{1-\alpha}}\right)$ we deduce $\beta+\frac{k}{\lambda}\gamma^{1-\alpha}<1$ and for $\lambda>\frac{k}{1-\beta}\gamma^{-\alpha}$ we deduce $\gamma+\frac{k}{\lambda}\gamma^{-\alpha}<1$. Let $\delta:=\max\left\{\beta+\frac{k}{\lambda}\gamma^{1-\alpha},\gamma+\frac{k}{\lambda}\gamma^{-\alpha}\right\}$. It follows that $\delta<1$ and, since (2.4),

$$\left\|Ax-Ay\right\|\leq\delta\left(\left\|x-y\right\|_{\gamma}+\left\|x-y\right\|_{\lambda}\right)=\delta\left\|x-y\right\|.$$

Hence, $A$ is contraction.

From the Banach Contraction Principle we conclude that $A$ has exactly one fixed point in $M$. ■

**Remark 2.1** *We remark that if* $\nu(t)=t$ *and* $\sigma(t)=t$, $\forall t\in[0,T]$, *then the conditions* (1.1) *and* (2.1) *are identical.*

## 3. The second existence result

As we mentioned in Section 1, in [2] is presented a generalization in the space $BC(\mathbb{R}_+,X)$ if (1.1) is fulfilled for every $t\in\mathbb{R}_+$. We shall prove that result under slightly more general assumptions.

Consider the space $C(\mathbb{R}_+,X)$ and for every $n\in\mathbb{N}^*$ let $\gamma_n\in(0,n)$, $\lambda_n>0$. Define the numerable family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}^*}$, where $\|x\|_n:=\|x\|_{\gamma_n}+\|x\|_{\lambda_n}$, for every $x\in C(\mathbb{R}_+,X)$, and

$$\|x\|_{\gamma_n}:=\sup_{t\in[0,\gamma_n]}\{|x(t)|\},\ \ \|x\|_{\lambda_n}:=\sup_{t\in[\gamma_n,T]}\left\{e^{-\lambda(t-\gamma_n)}|x(t)|\right\}.$$

As it is known, $C(\mathbb{R}_+,X)$ endowed with this numerable family of seminorms becomes a Fréchet space, i.e. a metrisable complete linear space. Also, the most natural metric which can be defined is

$$d(x,y):=\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{\|x-y\|_n}{1+\|x-y\|_n},\ \forall x,y\in C(\mathbb{R}_+,X).$$

Notice that a sequence $\{x_m\}_{m\in\mathbb{N}}\subset C(\mathbb{R}_+,X)$ converges to $x$ if and only if

$$\forall n\in\mathbb{N}^*,\ \lim_{m\to\infty}\|x_m-x\|_n=0.$$

In addition, a sequence $\{x_m\}_{m\in\mathbb{N}} \subset C(\mathbb{R}_+, X)$ is fundamental if and only if

$$\forall n \in \mathbb{N}^*,\ \forall \varepsilon > 0,\ \exists m_0 \in \mathbb{N}, \forall p, q \geq m_0,\ \ \|x_p - x_q\|_n < \varepsilon$$

or, more easily, if and only if

$$\forall n \in \mathbb{N}^*,\ \ \lim_{p,q\to\infty} \|x_p - x_q\|_n = 0.$$

**Theorem 3.1** *Let $M$ be a closed subset of $C(\mathbb{R}_+, X)$ and $A : M \to M$ be an operator. If for every $n \in \mathbb{N}^*$ there exist $\alpha_n, \beta_n \in [0,1)$, $k_n \geq 0$ such that for every $x, y \in M$ and for every $t \in [0, n]$,*

$$\begin{aligned} |(Ax)(t) - (Ay)(t)| \quad &\leq \quad \beta_n |x(\nu(t)) - y(\nu(t))| + \\ &+ \frac{k}{t^{\alpha_n}} \int_0^t |x(\sigma(s)) - y(\sigma(s))|\, ds, \end{aligned} \tag{3.1}$$

*where $\nu, \sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions such that $\nu(t) \leq t$, $\sigma(t) \leq t$, $\forall t \in \mathbb{R}_+$, then $A$ has a unique fixed point in $M$.*

**Proof.** As we have seen within the proof of Theorem 2.1, by choosing conveniently $\gamma_n \in (0, n)$ and $\lambda_n > 0$, there exists $\delta_n \in [0,1)$ such that for any $x, y \in M$,

$$\|Ax - Ay\|_n \leq \delta_n \|x - y\|_n,\ \forall n \in \mathbb{N}^*. \tag{3.2}$$

The proof of Theorem 3.1 is similar to the proof of the Banach Contraction Principle. We build the iterative sequence $x_{m+1} = Ax_m$, $\forall m \in \mathbb{N}$, where $x_0 \in M$ is arbitrary.

Let $n \in \mathbb{N}^*$ be arbitrary. One has

$$\|x_{m+1} - x_m\|_n = \|Ax_m - Ax_{m-1}\|_n \leq \delta_n \|x_m - x_{m-1}\|_n,\ \forall m \in \mathbb{N}^*$$

and therefore

$$\|x_{m+1} - x_m\|_n \leq \delta_n^m \|x_1 - x_0\|_n,\ \forall m \in \mathbb{N}.$$

Similarly,

$$\begin{aligned} \|x_{m+p} - x_m\|_n \quad &\leq \quad \left(\delta_n^{m+p} + \ldots + \delta_n^m\right) \|x_1 - x_0\|_n < \\ &< \quad \frac{\delta_n^m}{1 - \delta_n} \|x_1 - x_0\|_n,\ \forall m \in \mathbb{N},\ p \in \mathbb{N}^*. \end{aligned}$$

So, $\{x_m\}_{m\in\mathbb{N}}$ is fundamental and hence it will be convergent. Let $x_* := \lim_{m\to\infty} x_m \in M$. By (3.2) it follows that $Ax_m \to Ax_*$ or, equivalently, $x_m \to Ax_*$. Therefore, $x_* = Ax_*$.

If $A$ would have another fixed point in $M$, say $x_{**}$, it would follow that

$$\|x_* - x_{**}\|_n = \|Ax_* - Ax_{**}\|_n \leq \delta_n \|x_* - x_{**}\|_n$$

and so $\|x_* - x_{**}\|_n (1 - \delta_n) \leq 0$, $\forall n \in \mathbb{N}^*$. But $\delta_n \in [0,1)$. It follows that $x_* = x_{**}$.

The proof of Theorem 3.1 is now complete. ■

**Remark 3.1** *If the relation* (1.1) *holds for all* $t \in \mathbb{R}_+$, *then the relation* (3.1) *holds.*

*In particular, the condition* (3.1) *is fulfilled if for every* $x, y \in M$ *and* $t \in [0, n]$,

$$\begin{aligned} |(Ax)(t) - (Ay)(t)| &\leq \beta(t) |x(\nu(t)) - y(\nu(t))| + \\ &+ \frac{k(t)}{t^{\alpha(t)}} \int_0^t |x(\sigma(s)) - y(\sigma(s))| \, ds, \end{aligned}$$

*where* $\alpha : \mathbb{R}_+ \to [0,1)$, $\beta : \mathbb{R}_+ \to [0,1)$, *and* $k : \mathbb{R}_+ \to \mathbb{R}_+$, *are continuous functions.*

Indeed, in this case we can set

$$\beta_n := \sup_{t \in [0,n]} \{\beta(t)\}, \; k_n := \sup_{t \in [0,n]} \{k(t)\}, \; \alpha_n := \inf_{t \in [0,n]} \{\alpha(t)\}, \forall n \in \mathbb{N}^*.$$

**Remark 3.2** *Within the proof of Theorem 3.1 we have get the fixed point of* $A$ *as limit of the iterative sequence. It is interesting to remark that the fixed point of* $A$ *can be obtained as limit of other sequences.*

We present in the sequel an example.

Consider the space $C([0, n], X)$ and let

$$M_n := \left\{ x \mid_{[0,n]}, \; x \in M \right\}$$

i.e. $M_n$ is the set of the restrictions of $x \in M$ to $[0, n]$, $\forall n \in \mathbb{N}^*$.

Let $n \in \mathbb{N}^*$ be arbitrary. One has obviously $AM_n \subset M_n$. By applying Theorem 2.1, $A$ has a unique fixed point $x_n \in M_n$. We extend $x_n$ to $\mathbb{R}_+$ by continuity: for example, one could set

$$\widetilde{x}_n(t) := \begin{cases} x_n(t), & \text{if } t \in [0, n] \\ x_n(n), & \text{if } t \geq n \end{cases}$$

and hence $\widetilde{x}_n \in C(\mathbb{R}_+, X)$.

By the uniqueness property of the fixed point we have

$$\widetilde{x}_n(t) = \widetilde{x}_m(t), \; \forall m \leq n, \; \forall t \in [0, m], \tag{3.3}$$

which allows us to conclude that $\{\widetilde{x}_n\}_{n \in \mathbb{N}^*}$ converges in $C(\mathbb{R}_+, X)$ to the function $x^* : \mathbb{R}_+ \to X$ defined by

$$x^*(t) = \widetilde{x}_n(t), \; \forall t \in [0, n]. \tag{3.4}$$

Notice that $x^*$ is well defined due to $(3.3)$.

Let $t \in \mathbb{R}_+$ be arbitrary. Then there exists $n_0 \in \mathbb{N}^*$ such that $t \in [0, n_0]$. But

$$x^*(t) = \widetilde{x}_{n_0}(t) = (A\widetilde{x}_{n_0})(t) = (Ax^*)(t),$$

and so $x^*(t) = (Ax^*)(t)$. Since $t$ was arbitrary in $\mathbb{R}_+$, it follows $x^* = Ax^*$.

# 4. Applications

A particular case when the previous existence results can be applied is the following.

Consider an integral equation of the type

$$x(t) = F(t, x(\nu(t))) + \frac{1}{t^{\alpha(t)}} \int_0^t \mathcal{K}(t, s, x(\sigma(s)))\, ds, \tag{4.1}$$

where $\alpha \in [0,1)$ and $F : J \times \mathbb{R}^N \to \mathbb{R}^N$, $K : \Delta \to \mathbb{R}^N$, $\alpha : J \to [0,1)$ are continuous functions. Here,

$$J = [0,T] \text{ or } J = \mathbb{R}_+,\ \Delta = \left\{(t,s,x) \mid t, s \in J,\ 0 \le s \le t,\ x \in \mathbb{R}^N\right\}$$

and $\nu$, $\sigma : J \to J$ are continuous functions such that $\nu(t) \le t$, $\sigma(t) \le t$, $\forall t \in J$.

Consider the continuous functions $\beta : J \to [0,1)$, $\gamma : J \to \mathbb{R}_+$. If

$$\begin{aligned} |F(t,x) - F(t,y)| &\le \beta(t)\,|x-y|\,,\ \forall x, y \in \mathbb{R}^N,\ t \in J, \\ |\mathcal{K}(t,s,x) - \mathcal{K}(t,s,y)| &\le k(t)\,|x-y|\,,\ \forall (t,s,x), (t,s,y) \in \Delta, \end{aligned}$$

then the equation (4.1) has exactly one solution.

Indeed, it is easily checked the hypotheses of Theorem 2.1 and Theorem 3.1.

# References


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