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OSCILLATION OF FORCED IMPULSIVE DIFFERENTIAL EQUATIONS WITH γ-LAPLACIAN AND NONLINEARITIES GIVEN BY RIEMANN-STIELTJES INTEGRALS

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ABSTRACT. In this article, we study the oscillation of second order forced impulsive differential equation with γ-Laplacian and nonlinearities given by Riemann-Stieltjes integrals of the form

$$(p(t)\phi_\gamma(x'(t)))' + q_0(t)\,\phi_\gamma(x(t)) + \int_0^b q(t,s)\,\phi_{\alpha(s)}(x(t))\,d\zeta(s) = e(t), \quad t \neq \tau_k,$$

with impulsive conditions

$$x\left(\tau_k^+\right) = \lambda_k\; x(\tau_k), \quad x'\left(\tau_k^+\right) = \eta_k\; x'(\tau_k),$$

where $\phi_\gamma(u) := |u|^\gamma \operatorname{sgn} u$, $\gamma,\ b \in (0,\infty)$, $\alpha \in C[0,b)$ is strictly increasing such that $0 \leq \alpha(0) < \gamma < \alpha(b-)$, and $\{\tau_k\}_{k\in\mathbb{N}}$ is the the impulsive moments sequence. Using the Riccati transformation technique, we obtain sufficient conditions for this equation to be oscillatory.

1. Introduction

We are concerned with the oscillatory behavior of forced second order impulsive differential equations with γ-Laplacian and nonlinearities given by a Riemann-Stieltjes integrals in the form of

$$(p(t)\phi_\gamma(x'(t)))' + q_0(t)\,\phi_\gamma(x(t)) + \int_0^b q(t,s)\,\phi_{\alpha(s)}(x(t))\,d\zeta(s) = e(t),\ t \geq t_0,\ t \neq \tau_k, \tag{1.1}$$

with the impulsive conditions

$$x\left(\tau_k^+\right) = \lambda_k\; x(\tau_k), \quad x'\left(\tau_k^+\right) = \eta_k\; x'(\tau_k),$$

where

(a) $\phi_\gamma(u) := |u|^\gamma \operatorname{sgn} u$, $\gamma,\ b \in (0,\infty)$, and $t_0 \in \mathbb{R}$;
(b) $\zeta : [0,b) \to \mathbb{R}$ is nondecreasing and $\int_0^b f(s)d\zeta$ denotes the Riemann-Stieltjes integral of the function f on $[0,b)$ with respect to ζ;
(c) $\alpha \in C[0,b)$ is strictly increasing such that $0 \leq \alpha(0) < \gamma < \alpha(b-)$ and $\{\tau_k\}_{k\in\mathbb{N}}$ is the the impulsive moments sequence with

$$t_0 < \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \cdots, \quad \lim_{k\to\infty} \tau_k = \infty.$$

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(d) we denote

$$x\left(\tau_k^{\pm}\right)=\lim_{t\to\tau_k^{\pm}}x\left(t\right)\quad\text{and}\quad x'\left(\tau_k^{\pm}\right)=\lim_{h\to 0^{\pm}}\frac{x\left(\tau_k+h\right)-x\left(\tau_k\right)}{h}.$$

Throughout this paper and without further mention, we assume that the following conditions hold:

(A_1) $p,\ q_0,\ e\in C\left([t_0,\infty),\mathbb{R}\right)$ with $p\left(t\right)>0$ on $[t_0,\infty)$, $q\in C\left([0,\infty)\times[0,b)\right)$;

(A_2) $\lambda_k,\eta_k\in\mathbb{R}$ and $\lambda_k\neq 0$, for $k\in\mathbb{N}$ with $\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right)\geq 1$.

By an extendible solution of Eq. (1.1), we mean a function $x\in$PLC$[t_0,\infty):=\{y:[t_0,\infty)\to\mathbb{R}$ is continuous on each interval $\left(\tau_k,\tau_{k+1}\right),\ y\left(\tau_k^{\pm}\right)$ exist, $y\left(\tau_k\right)=y\left(\tau_k^{-}\right)$ for $k\in\mathbb{N}\}$ such that $x'\in$PLC$[t_0,\infty)$ and x satisfies Eq. (1.1) for $t\geq t_0$. An extendible solution of Eq. (1.1) is said to be oscillatory if it is not eventually positive or negative. Eq. (1.1) is said to be oscillatory if every extendible solution of Eq. (1.1) is oscillatory.

We note that as special cases, when $\zeta(s)$ is a step function, the integral term in the equation reduces to a finite sum and hence Eq. (1.1) becomes the equation

$$\left(p(t)\phi_\gamma\left(x'(t)\right)\right)'+q_0(t)\phi_\gamma(x(t))+\sum_{j=1}^{n}q_j(t)\phi_{\alpha j}\left(x(t)\right)=e(t);\tag{1.2}$$

and when $\zeta(s)=s$, the integral term in the equation reduces to a Riemann integral and hence Eq. (1.1) becomes the equation

$$\left(p(t)\phi_\gamma\left(x'(t)\right)\right)'+q_0(t)\phi_\gamma(x(t))+\int_0^b q\left(t,s\right)\phi_{\alpha(s)}\left(x(t)\right)ds=e(t).$$

The oscillation of Eq. (1.1) with the impulsive conditions removed has been studied widely. For instance, Sun and Wong [25] investigated Eq. (1.2) with $\gamma=1$, Hassan, Erbe, and Peterson [9] and Hassan and Kong [10] discussed Eq. (1.2) with a general $\gamma>0$, Sun and Kong [26] further investigated Eq. (1.1) with $\gamma=1$, and Hassan and Kong [11] studied Eq. (1.1) with a general $\gamma>0$. Oscillation criteria, especially criteria of El-Sayed-type and Kong-type, were established for the above equations. However, no impulses were involved in any of the above papers.

Differential equations with impulses are used to characterize motions subject to perturbations at a sequence of impulsive moments. Such motions are often encountered in various fields of science and technology such as physics, engineering, population dynamics, ecology, biological systems, and optimal controls, see [2, 3, 7, 16, 27–29, 31] and the references therein.

Recently, progress has been made for oscillation of impulsive differential equations. In particular, Özbekler and Zafer [21] studied

$$\begin{cases}\left(p(t)\phi_\gamma\left(x'(t)\right)\right)'+q_0\left(t\right)\phi_\gamma\left(x(t)\right)+\sum\limits_{k=1}^{n}q_k\left(t\right)\phi_{\alpha_k}\left(x\left(t\right)\right)=e(t), & t\neq\tau_k,\\ x\left(\tau_k^{+}\right)=\lambda_k\ x\left(t_k\right), \quad x'\left(\tau_k^{+}\right)=\eta_k\ x'\left(\tau_k\right), & \end{cases}\tag{1.3}$$

where $\alpha_1 > \alpha_2 > ..... > \alpha_m > \gamma > \alpha_{m+1} > ... > \alpha_n > 0$, and in [22, 23], they discussed

$$\begin{cases} (p(t)\phi_\gamma(x'(t)))' + q_0(t)\,\phi_\gamma(x'(t)) + q(t)\,\phi_\alpha(x(t)) = e(t), & t \neq \tau_k, \\ \Delta(p(t)\phi_\gamma(x'(t))) + q_i\phi_\alpha(x(t)) = e_i, & t = \tau_k, \end{cases} \tag{1.4}$$

where $\alpha > \gamma$. Oscillation criteria of the El-Sayed-type were derived for both Eqs. (1.3) and (1.4). Muthulakshmi and Thandapani [20] obtained oscillation criteria of the Philos and Kong type for the equation

$$\begin{cases} (p(t)x'(t))' + r(t)(x'(t)) + q_0(t)\,x(t) + \sum\limits_{k=1}^{n} q_k(t)\,\phi_{\alpha_k}(x(t)) = e(t), & t \neq \tau_k, \\ x(\tau_k^+) = \lambda_k\ x(t_k), \quad x'(\tau_k^+) = \eta_k\ x'(\tau_k), \end{cases} \tag{1.5}$$

Motivated by above, in this paper, we will establish interval oscillation criteria of both the El-Sayed-type and the Kong-type for the more general equation (1.1). Our work is of significance because Eq. (1.1) not only contains a γ-Laplacian term but also allows an infinite number and even a continuum of nonlinearities determined by the function ζ together with a sequence of impulses.

This paper is organized as follows: after this introduction, we state our main results for Eq. (1.1) in section 2, followed by a demonstrating example. All proofs are given in section 3.

2. Main Results

We denote by $L_\zeta(0,b)$ the set of Riemann-Stieltjes integrable functions on $[0,b)$ with respect to ζ. Let $a \in (0,b)$ such that $\alpha(a) = \gamma$, and let α^{-1} be the reciprocal of α. We further assume that

$$\alpha^{-1} \in L_\zeta(0,b) \quad \text{and} \quad \int_0^a d\zeta(s) > 0 \text{ and } \int_a^b d\zeta(s) > 0.$$

We see that the condition $\alpha^{-1} \in L_\zeta(0,b)$ is satisfied if either $\alpha(0) > 0$ or $\alpha(s) \to 0$ "slowly" as $s \to 0^+$ (for example, $\alpha(s) \geq s^l$ for some $l \in (0,1)$ when $\zeta(s) = s$), or $\zeta(s)$ is constant in a right neighborhood of 0.

We denote

$$m := \gamma \left(\int_a^b d\zeta(s) \right)^{-1} \int_a^b \alpha^{-1}(s)\, d\zeta(s)$$

and

$$n := \gamma \left(\int_0^a d\zeta(s) \right)^{-1} \int_0^a \alpha^{-1}(s)\, d\zeta(s).$$

Then it is easy to see that $m < 1 < n$. In fact, since $\alpha(s)$ is strictly increasing, $\alpha^{-1}(s)$ is strictly decreasing. Hence

$$m < \gamma \left(\int_a^b d\zeta(s) \right)^{-1} \alpha^{-1}(a) \int_a^b d\zeta(s) = 1$$

and

$$n > \gamma \left(\int_0^a d\zeta(s) \right)^{-1} \alpha^{-1}(a) \int_0^a d\zeta(s) = 1.$$

The following lemma, which is a generalization of [26, Lemma 1] and is established in [11, Lemma 1], will be used in the statement of our main results.

Lemma 2.1. *For any $\delta \in (m,n)$, there exists $\eta \in L_\zeta(0,b)$ such that $\eta(s) > 0$ on $[0,b)$, and*

$$\int_0^b \alpha(s)\eta(s)\,d\zeta(s) = \gamma \quad \text{and} \quad \int_0^b \eta(s)\,d\zeta(s) = \delta. \tag{2.1}$$

To present our main results, we denote

$$\widehat{s} := \max\{k : \ t_0 < \tau_k < s\}.$$

Let $\rho(t)$ be a positive continuous function on $[t_0,\infty)$ given later. Then for any $a,b \in \mathbb{T}$ with $a < b$ and $k = \widehat{a}+1,\ldots,\widehat{b}$ we denote

$$(\rho p)_k := \max\{\rho(t) : t \in [\tau_{k-1},\tau_k] \cap [a,b]\} \max\{p(t) : t \in [\tau_{k-1},\tau_k] \cap [a,b]\};$$

and define an operator $\Phi : C([a,b],\mathbb{R}) \to \mathbb{R}$ as

$$\Phi[u;a,b] := \begin{cases} 0, & \widehat{a} = \widehat{b}, \\ u(\tau_{\widehat{a}+1})(\rho p)_{\widehat{a}+1} \ (\tau_{\widehat{a}+1} - a)^{-\gamma}\left[\phi_\gamma\left(\frac{\eta_{\widehat{a}+1}}{\lambda_{\widehat{a}+1}}\right) - 1\right] & \\ +\sum_{k=\widehat{a}+2}^{\widehat{b}} u(\tau_k)(\rho p)_k \ (\tau_k - \tau_{k-1})^{-\gamma}\left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1\right], & \widehat{a} < \widehat{b}; \end{cases}$$

with the convention that $\sum_{k=m}^n c_k = 0$ when $m > n$.

Our first result provides an oscillation criterion of the El-Sayed-type.

Theorem 2.1. *Suppose that for any $T \geq 0$ and for $i = 1,2$, there exist constants a_i and b_i with $T \leq a_i < b_i$ and $(a_1,b_1) \cap (a_2,b_2) = \emptyset$ such that for $i = 1,2$*

$$q_0(t) \geq 0 \text{ and } q(t,s) \geq 0 \quad \text{and} \quad (-1)^i e(t) \geq 0 \tag{2.2}$$

for $t \in [a_i,b_i] \setminus \{\tau_k\}$ and $s \in [0,b)$. Assume further that for $i = 1,2$, there exists $u_i \in C^1[a_i,b_i]$ satisfying $u_i(a_i) = u_i(b_i) = 0$ and $u_i(t) \not\equiv 0$ on $[a_i,b_i]$ and a positive continuous function ρ such that

$$\sup_{\delta \in (m,1]} \int_{a_i}^{b_i} \left[Q(t)|u_i(t)|^{\gamma+1} - p(t)\widetilde{u}_i(t)\right] dt > \Phi\left[|u_i|^{\gamma+1}; a_i, b_i\right], \tag{2.3}$$

where

$$Q(t) := (\rho q_0)(t) + \left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta} \rho(t) \exp\left(\int_0^b \eta(s) \ln \frac{q(t,s)}{\eta(s)} d\zeta(s)\right) \tag{2.4}$$

and

$$\widetilde{u}_i(t) := \frac{\rho(t)}{(\gamma+1)^{\gamma+1}} \left|(\gamma+1)u_i'(t) + \frac{\rho'(t)}{\rho(t)} u_i(t)\right|^{\gamma+1} \tag{2.5}$$

with $\eta(s)$ defined as in Lemma 2.1 based on δ. Here we use the convention that $\ln 0 = -\infty$, $e^{-\infty} = 0$, and $0^{1-\delta} = 1$ and $(1-\delta)^{1-\delta} = 1$ for $\delta = 1$. Then Eq. (1.1) is oscillatory.

Following Philos [17], Kong [12], and Kong [13], we say that for any $a, b \in \mathbb{R}$ such that $a < b$, a function $H(t,s)$ belongs to a function class $\mathcal{H}(a,b)$, denoted by $H \in \mathcal{H}(a,b)$, if $H \in C(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} := \{(t,s) : b \geq t \geq s \geq a\}$, which satisfies

$$H(t,t) = 0, \quad H(b,s) > 0 \quad \text{and} \quad H(s,a) > 0 \quad \text{for } b > s > a, \tag{2.6}$$

and $H(t,s)$ has continuous partial derivatives $\partial H(t,s)/\partial t$ and $\partial H(t,s)/\partial s$ on $[a,b] \times [a,b]$ such that

$$\frac{\partial H(t,s)}{\partial t} + \frac{\rho'(t)}{\rho(t)} H(t,s) = (\gamma+1) h_1(t,s) H^{\frac{\gamma}{\gamma+1}}(t,s) \tag{2.7}$$

and

$$\frac{\partial H(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)} H(t,s) = (\gamma+1) h_2(t,s) H^{\frac{\gamma}{\gamma+1}}(t,s), \tag{2.8}$$

where $h_1, h_2 \in L_{loc}(\mathbb{D}, \mathbb{R})$. Next, we use the function class $\mathcal{H}(a,b)$ to establish an oscillation criterion for Eq. (1.1) of the Kong-type.

Theorem 2.2. *Suppose that for any $T \geq 0$ and for $i = 1, 2$, there exist constants a_i and b_i with $T \leq a_i < b_i$ such that (2.2) holds. Assume further that for $i = 1, 2$, there exists $c_i \in (a_i, b_i)$ and $H_i \in \mathcal{H}(a_i, b_i)$ and a positive continuous function ρ such that*

$$\begin{aligned} \sup_{\delta \in (m,1]} \Bigg\{ & \frac{1}{H_i(c_i, a_i)} \int_{a_i}^{c_i} \left[Q(s) H_i(s, a_i) - (\rho p)(s) \left| h_{i1}(s, a_i) \right|^{\gamma+1} \right] ds \\ & + \frac{1}{H_i(b_i, c_i)} \int_{c_i}^{b_i} \left[Q(s) H_i(b_i, s) - (\rho p)(s) \left| h_{i2}(b_i, s) \right|^{\gamma+1} \right] ds \Bigg\} \\ & > \frac{1}{H_1(c_i, a_i)} \Phi\left[H_1(\cdot, a_i); a_i, c_i \right] + \frac{1}{H_1(b_i, c_i)} \Phi\left[H_1(b_i, \cdot); c_i, b_i \right], \end{aligned} \tag{2.9}$$

where $Q(t)$ is defined by (2.4). Then Eq. (1.1) is oscillatory.

The example below is to demonstrate the application of Theorem 2.1, especially, how the non-constant function ρ plays a role in the determination of oscillation. Similar example can be constructed to demonstrate the application of Theorem 2.2, but we leave it to interested readers.

Example 2.1. Consider Eq.

$$\left((2 + \cos 4t) \; x'(t) \right)' + \sin t \; x(t) + \int_0^1 \sin t \; \phi_{3s}(x(t)) \, ds = -re^t \cos 2t, \;\; t \geq 0, t \neq \tau_k,$$

where $\tau_k = (2k-1)\pi/16$ with the impulsive condition

$$x\left(\tau_k^+\right) = (-1)^k \sigma_1 x(t_k), \quad x'\left(\tau_k^+\right) = (-1)^k \sigma_2 x'(\tau_k)$$

for some $\sigma_1, \sigma_2 > 0$ such that $\sigma_2/\sigma_1 \geq 1$. Here we have

(i) $\alpha(s) = 3s, \; \xi(s) = s, \; \gamma = 1$, and $b = 1$;
(ii) $p(t) = 2 + \cos 4t, \; q_0(t) = q(t,s) = \sin t$, and $e(t) = -re^t \cos 2t$ for $r > 0$;
(iii) $\lambda_k = (-1)^k \sigma_1$ and $\eta_k = (-1)^k \sigma_2$.

Note that

$$m=\left(\int_{\frac{1}{3}}^{1} ds\right)^{-1}\left(\int_{\frac{1}{3}}^{1}\frac{1}{3s}\,ds\right)=\ln\sqrt{3}.$$

For any $\delta\in\left(\ln\sqrt{3},1\right]$, we set

$$\eta\left(s\right):=\frac{\delta}{3\delta-1}s^{\frac{2-3\delta}{3\delta-1}},$$

then (2.1) is satisfied. For any $T\in\mathbb{R}$, we choose $n\in\mathbb{N}$ so large that $2n\pi\geq T$ and let

$$a_1=2n\pi,\ a_2=b_1=2n\pi+\frac{\pi}{4},\ b_2=2n\pi+\frac{\pi}{2}.$$

Let $\rho\left(t\right)=2-\cos 4t$, and for $i=1,2$ let $u_i\left(t\right)=\sin 4t$ and $\tilde{u}_i$ be defined by (2.5). Then

$$\begin{aligned}\int_{a_i}^{b_i}p(t)\widetilde{u}_i\left(t\right)dt &= \frac{1}{4}\int_0^{\frac{\pi}{4}}\left(4-\cos^2 4t\right)\left\{8\cos 4t+\frac{4\sin^2 4t}{2-\cos 4t}\right\}^2 dt\\ &= \frac{1}{8}\left(96\sqrt{3}-107\right)\pi.\end{aligned}$$

With $t_0=0$ we have

$$\widehat{a_1}=16n,\ \widehat{a_1}+1=16n+1,\ \widehat{a_1}+2=16n+2=\widehat{b_1},$$

and

$$\tau_{\widehat{a_1}}=2n\pi-\frac{\pi}{16},\ \tau_{\widehat{a_1}+1}=2n\pi+\frac{\pi}{16},\ \tau_{\widehat{a_1}+2}=2n\pi+\frac{3\pi}{16}=\tau_{\widehat{b_1}},\ \tau_{\widehat{a_1}+3}=2n\pi+\frac{5\pi}{16}.$$

It is easy to see that

$$\max\{\rho(t):\ t\in[\tau_{\widehat{a_1}},\tau_{\widehat{a_1}+1}]\}=2-\frac{1}{\sqrt{2}},\ \max\{p(t):\ t\in[\tau_{\widehat{a_1}},\tau_{\widehat{a_1}+1}]\}=3,$$

$$u_i(\tau_{\widehat{a_1}+1})=\frac{1}{\sqrt{2}},\ (\rho p)_{\widehat{a_1}+1}=3\left(2-\frac{1}{\sqrt{2}}\right),\ (\tau_{\widehat{a_1}+1}-a_1)^{-\gamma}=\frac{16}{\pi};$$

and

$$\max\{\rho(t):\ t\in[\tau_{\widehat{a_1}+1},\tau_{\widehat{a_1}+2}]\}=2+\frac{1}{\sqrt{2}},\ \max\{p(t):\ t\in[\tau_{\widehat{a_1}+1},\tau_{\widehat{a_1}+2}]\}=2+\frac{1}{\sqrt{2}},$$

$$u_i(\tau_{\widehat{a_1}+2})=\frac{1}{\sqrt{2}},\ (\rho p)_{\widehat{a_1}+2}=\left(2+\frac{1}{\sqrt{2}}\right)^2,\ (\tau_{\widehat{a_1}+2}-\tau_{\widehat{a_1}+1})^{-\gamma}=\frac{8}{\pi}.$$

Note that

$$\phi_\gamma\left(\frac{\eta_{\widehat{a_1}+1}}{\lambda_{\widehat{a_1}+1}}\right)-1=\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right)-1=\frac{\sigma_2}{\sigma_1}-1.$$

Then we have

$$\begin{aligned}\Phi\left[(u_1)^2;a_1,b_1\right] &= \left[\frac{\sigma_2}{\sigma_1}-1\right]\left\{\frac{3}{2}\left(2-\frac{1}{\sqrt{2}}\right)\frac{16}{\pi}+\frac{8}{\pi}\sum_{k=16n+2}^{16n+2}\left(u_i\left(\tau_k\right)\right)^2(\rho p)_k\right\}\\ &= \left[\frac{\sigma_2}{\sigma_1}-1\right]\left\{\frac{3}{2}\left(2-\frac{1}{\sqrt{2}}\right)\frac{16}{\pi}+\frac{8}{\pi}\frac{1}{2}\left(2+\frac{1}{\sqrt{2}}\right)^2\right\}\\ &= \frac{1}{\pi}\left[\frac{\sigma_2}{\sigma_1}-1\right]\left[66-4\sqrt{2}\right].\end{aligned}$$

Similarly,

$$\Phi\left[(u_2)^2; a_2, b_2\right] = \frac{1}{\pi}\left[\frac{\sigma_2}{\sigma_1} - 1\right]\left[66 - 4\sqrt{2}\right].$$

Therefore, there exists an $r_0 > 0$ such that for $r \geq r_0$

$$\sup_{\delta\in\left(\ln\sqrt{3},1\right]} \int_0^{\frac{\pi}{4}} (2-\cos 4t)\left[\sin t + g(t)\right] \geq \frac{1}{8}\left(96\sqrt{3} - 107\right)\pi + \frac{1}{\pi}\left[\frac{\sigma_2}{\sigma_1} - 1\right]\left[66 - 4\sqrt{2}\right],$$

where

$$g(t) := \left[\frac{re^t \cos 2t}{1-\delta}\right]^{1-\delta} \exp\left(\int_0^1 \eta(s)\ln\frac{\sin t}{\eta(s)}ds\right).$$

Hence (2.3) holds and Eq. (1.1) is oscillatory for $r \geq r_0$. We comment that the value of r_0 can be easily obtained by a numerical calculation.

3. Proofs

The Lemma below, established in [26], provides a generalized Arithmetic-Geometric mean inequality.

Lemma 3.1. *Let $u \in C[0,b)$ and $\eta \in L_\zeta(0,b)$ satisfying $u \geq 0$, $\eta > 0$ on $[0,b)$ and $\int_0^b \eta(s)\,d\zeta(s) = 1$. Then*

$$\int_0^b \eta(s)\,u(s)\,d\zeta(s) \geq \exp\left(\int_0^b \eta(s)\ln\left[u(s)\right]d\zeta(s)\right), \tag{3.1}$$

where we use the convention that $\ln 0 = -\infty$ and $e^{-\infty} = 0$.

Indeed, Lemma 3.1 is a generalization of the original Arithmetic-Geometric mean inequality. This can be seen as follows: For $n \in \mathbb{N}$ let $b = 1 + 1/n$, $\eta(s) = 1$ for $s \in [0,1)$, and $\zeta(s) = i/n$ for $s \in [i-1, i)$ and $i = 1, \ldots, n+1$. For $u \in C[0, 1+1/n)$ denote $u_i = u(i/n)$ for $i = 1, \ldots, n$. It is easy to see that $\int_0^b \eta(s)\,d\zeta(s) = 1$. Also,

$$\int_0^b \eta(s)\,u(s)\,d\zeta(s) = \sum_{i=1}^n u_i/n$$

and

$$\exp\left(\int_0^b \eta(s)\ln\left[u(s)\right]d\zeta(s)\right) = \exp\left(\sum_{i=1}^n \ln u_i/n\right) = \prod_{i=1}^n u_i^{1/n}.$$

Hence (3.1) becomes

$$\sum_{i=1}^n u_i/n \geq \prod_{i=1}^n u_i^{1/n}$$

which is the original Arithmetic-Geometric mean inequality. Some other Arithmetic-Geometric mean inequalities can also be deduced from (3.1) by using different $\eta(s)$ and $\zeta(s)$.

Proof of Theorem 2.1. Assume Eq. (1.1) has an extendible solution $x(t)$ which is eventually positive or negative. Then, without loss of generality, assume $x(t) > 0$ for all $t \geq T \geq 0$, where T depends on the solution $x(t)$. When $x(t)$ is an eventually negative, the proof follows the same way except that the interval $[a_2, b_2]$, instead of $[a_1, b_1]$, is used. Define

$$z(t) := \rho(t) \frac{p(t)\phi_\gamma(x'(t))}{\phi_\gamma(x(t))} \quad \text{for } t \geq T, t \neq t_k,\ k \in \mathbb{N}. \tag{3.2}$$

It follows from (1.1) that for $t \geq T$ and $t \neq \tau_k$, $z(t)$ satisfies the first order nonlinear Riccati equation

$$\begin{aligned} z'(t) &= -\rho(t) q_0(t) - \rho(t) \int_0^b q(t,s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) \\ &\quad + \rho(t) e(t) x^{-\gamma}(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t) p(t))^{\frac{1}{\gamma}}}. \end{aligned} \tag{3.3}$$

(I) We first consider the case where the supremum in (2.3) is assumed at $\delta = 1$. From (2.2) and (3.3), we have that for $t \in (a_1, b_1]$ and $t \neq \tau_k$

$$z'(t) \leq -\rho(t) q_0(t) - \rho(t) \int_0^b q(t,s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t) p(t))^{\frac{1}{\gamma}}}. \tag{3.4}$$

Let $\eta \in L_\zeta(0,b)$ be defined as in Lemma 2.1 with $\delta = 1$. Then η satisfies (2.1) with $\delta = 1$. This follows that

$$\int_0^b \eta(s) [\alpha(s) - \gamma] d\zeta = 0.$$

Then, from Lemma 3.1, we get, for $t \in (a_1, b_1]$ and $t \neq \tau_k$

$$\begin{aligned} &\int_0^b q(t,s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) \\ = &\int_0^b \eta(s) \frac{q(t,s)}{\eta(s)} [x(t)]^{\alpha(s)-\gamma} d\zeta(s) \\ \geq &\exp\left(\int_0^b \eta(s) \ln\left(\frac{q(t,s)}{\eta(s)} [x(t)]^{\alpha(s)-\gamma} \right) d\zeta(s) \right) \\ = &\exp\left(\int_0^b \eta(s) \ln\left[\frac{q(t,s)}{\eta(s)} \right] d\zeta(s) + \ln(x(t)) \int_0^b \eta(s) [\alpha(s) - \gamma] d\zeta(s) \right) \\ = &\exp\left(\int_0^b \eta(s) \ln\left[\frac{q(t,s)}{\eta(s)} \right] d\zeta(s) \right). \end{aligned}$$

This together with (3.4) shows that

$$z'(t) \leq -Q(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t) p(t))^{\frac{1}{\gamma}}}, \quad \text{for } t \in (a_1, b_1] \text{ and } t \neq \tau_k, \tag{3.5}$$

where $Q(t)$ is defined by (2.4) with $\delta = 1$.

(II) Now, we consider the case where the supremum in (2.3) is assumed at $\delta \in (m, 1)$. Then from (2.2) we see that for $t \in (a_1, b_1]$ and $t \neq \tau_k$,

$$\begin{aligned} z'(t) \quad \leq \quad & -\rho(t) q_0(t) - \rho(t) \int_0^b q_1(t,s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) \\ & -\rho(t) |e(t)| x^{-\gamma}(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t) p(t))^{\frac{1}{\gamma}}}. \end{aligned} \tag{3.6}$$

Let $\widetilde{\eta}(s) = \delta^{-1}\eta(s)$. Then from (2.1) we have

$$\int_0^b \widetilde{\eta}(s) d\zeta(s) = 1 \quad \text{and} \quad \int_0^b \widetilde{\eta}(s) [\delta\alpha(s) - \gamma] d\zeta = 0. \tag{3.7}$$

Hence for $t \in (a_1, b_1]$ and $t \neq \tau_k$

$$\begin{aligned} & \int_0^b q_1(t,s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) + |e(t)| x^{-\gamma}(t) \\ = & \int_0^b \widetilde{\eta}(s) \left(\delta\eta^{-1}(s) q_1(t,s) [x(t)]^{\alpha(s)-\gamma} + |e(t)| x^{-\gamma}(t) \right) d\zeta(s). \end{aligned} \tag{3.8}$$

Using the Arithmetic-geometric mean inequality, see [4, Page 17],

$$ch + dk \geq c^h d^k, \quad \text{where } c, d \geq 0,\ h, k > 0 \text{ and } h + k = 1,$$

with

$$c = \eta^{-1}(s) q_1(t,s) [x(t)]^{\alpha(s)-\gamma},\ d = \frac{1}{1-\delta} |e(t)| x^{-\gamma}(t),\ h = \delta \text{ and } k = 1 - \delta,$$

we have that for for $t \in (a_1, b_1]$, and $t \neq \tau_k$ and $s \in [0, b)$

$$\begin{aligned} & \delta\eta^{-1}(s) q_1(t,s) [x(t)]^{\alpha(s)-\gamma} + (1-\delta) \frac{|e(t)|}{1-\delta} x^{-\gamma}(t) \\ \geq & \left[\frac{q_1(t,s)}{\eta(s)} \right]^{\delta} \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} [x(t)]^{\delta\alpha(s)-\gamma}. \end{aligned}$$

Substituting this into (3.8) and using Lemma 3.1 and (3.7), we see that for $t \in (a_1, b_1]$, $t \neq \tau_k$ and $e(t) \neq 0$,

$$\begin{aligned} & \int_0^b q_1(t,s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) + |e(t)| x^{-\gamma}(t) \\ \geq \ & \exp\left(\int_0^b \widetilde{\eta}(s) \ln\left(\left[\frac{q_1(t,s)}{\eta(s)} \right]^{\delta} \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} [x(t)]^{\delta\alpha(s)-\gamma} \right) d\zeta(s) \right) \\ = \ & \exp\left(\int_0^b \widetilde{\eta}(s) \left(\ln \left[\frac{q_1(t,s)}{\eta(s)} \right]^{\delta} + \ln \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} + [\delta\alpha(s) - \gamma] \ln x(t) \right) d\zeta(s) \right) \\ = \ & \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} \exp\left(\int_0^b \eta(s) \ln \frac{q_1(t,s)}{\eta(s)} d\zeta(s) \right). \end{aligned} \tag{3.9}$$

Note that (3.9) also holds when $e(t) = 0$ since $\left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta} = 0$ for $\delta \in (0,1)$. It follows from (3.6) and (3.9) that for $t \in (a_1, b_1]$ and $t \neq \tau_k$

$$\begin{aligned} z'(t) &\leq -\rho(t) q_0(t) - \rho(t)\left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta} \exp\left(\int_0^b \eta(s) \ln \frac{q_1(t,s)}{\eta(s)} d\zeta(s)\right) \\ &\quad + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho p)^{\frac{1}{\gamma}}(t)} \\ &= -Q(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho p)^{\frac{1}{\gamma}}(t)}, \end{aligned} \tag{3.10}$$

where $Q(t)$ is defined by (2.4) with $\delta \in (m, 1)$.

For both cases (I) and (II), from (3.5) and (3.10), we have

$$z'(t) \leq -Q(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho p)^{\frac{1}{\gamma}}(t)}, \quad \text{for } t \in (a_1, b_1] \text{ and } t \neq \tau_k. \tag{3.11}$$

Multiplying both sides of (3.5) by $|u_1(t)|^{\gamma+1}$, integrating from a_1 to b_1, and using integration by parts, we find that

$$\begin{aligned} &\int_{a_1}^{b_1} Q(t) |u_1(t)|^{\gamma+1} dt \\ \leq\ & \sum_{k=\widehat{a_1}+1}^{\widehat{b_1}} |u_1(\tau_k)|^{\gamma+1} z(\tau_k) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1\right] + \\ & \int_{a_1}^{b_1} \left[\phi_\gamma(u_1(t)) \left[(\gamma+1) u_1'(t) + \frac{\rho'(t)}{\rho(t)} u_1(t)\right] z(t) - \frac{\gamma |u_1(t)|^{\gamma+1}}{(\rho(t) p(t))^{\frac{1}{\gamma}}} |z(t)|^{\frac{\gamma+1}{\gamma}}\right] dt \\ \leq\ & \sum_{k=\widehat{a_1}+1}^{\widehat{b_1}} |u_1(\tau_k)|^{\gamma+1} z(\tau_k) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1\right] + \\ & \int_{a_1}^{b_1} \left[|u_1(t)|^{\gamma} \left|(\gamma+1) u_1'(t) + \frac{\rho'(t)}{\rho(t)} u_1(t)\right| |z(t)| - \frac{\gamma |u_1(t)|^{\gamma+1}}{(\rho(t) p(t))^{\frac{1}{\gamma}}} |z(t)|^{\frac{\gamma+1}{\gamma}}\right] dt. \end{aligned} \tag{3.12}$$

Let $\lambda := \frac{\gamma+1}{\gamma}$. Define A and B by

$$A^{\lambda} := \frac{\gamma |u_1(t)|^{\gamma+1}}{(\rho(t) p(t))^{\frac{1}{\gamma}}} |z(t)|^{\lambda},$$

and

$$B^{\lambda-1} := \frac{(\gamma \rho(t) p(t))^{\frac{1}{\gamma+1}}}{\gamma+1} \left|(\gamma+1) u_1'(t) + \frac{\rho'(t)}{\rho(t)} u_1(t)\right|.$$

Using the inequality in [8] we have

$$\lambda A B^{\lambda-1} - A^{\lambda} \leq (\lambda-1) B^{\lambda}, \tag{3.13}$$

i.e.,

$$\begin{aligned}
&|u_1(t)|^\gamma \left|(\gamma+1)\,u_1'(t)+\frac{\rho'(t)}{\rho(t)}u_1(t)\right|\;|z(t)| - \frac{\gamma\,|u_1(t)|^{\gamma+1}}{(\rho(t)\,p(t))^{\frac{1}{\gamma}}}\,|z(t)|^{\frac{\gamma+1}{\gamma}} \\
\le\;& \frac{\rho(t)\,p(t)}{(\gamma+1)^{\gamma+1}}\left|(\gamma+1)\,u_1'(t)+\frac{\rho'(t)}{\rho(t)}u_1(t)\right|^{\gamma+1},
\end{aligned}$$

which together with (3.12) and (2.5) implies that

$$\begin{aligned}
&\int_{a_1}^{b_1}\left[Q(t)\,|u_1(t)|^{\gamma+1}-p(t)\widetilde{u}_1(t)\right]dt \\
\le\;& \sum\nolimits_{k=\widehat{a_1}+1}^{\widehat{b_1}} |u_1(\tau_k)|^{\gamma+1}\,z(\tau_k)\left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right)-1\right]. \qquad (3.14)
\end{aligned}$$

There are two cases to consider: either $\widehat{a_1}=\widehat{b_1}$ or $\widehat{a_1}<\widehat{b_1}$.

Case (i) $\widehat{a_1}=\widehat{b_1}$. There is no impulsive moments in $[a_1,b_1]$ and (3.14) yields

$$\begin{aligned}
&\int_{a_1}^{b_1}\left[Q(t)\,|u_1(t)|^{\gamma+1}-p(t)\widetilde{u}_1(t)\right]dt \\
\le\;& \sum\nolimits_{k=\widehat{a_1}+1}^{\widehat{b_1}} |u_1(\tau_k)|^{\gamma+1}\,z(\tau_k)\left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right)-1\right] \\
=\;& 0=\Phi\left[|u_1|^{\gamma+1};a_1,b_1\right]. \qquad (3.15)
\end{aligned}$$

Case (ii) $\widehat{a_1}<\widehat{b_1}$. There are impulsive moments $\tau_{\widehat{a_1}+1},\ \tau_{\widehat{a_1}+2},\ \ldots,\ \tau_{\widehat{b_1}}$ in $[a_1,b_1]$. For $t\in(a_1,\tau_{\widehat{a_1}+1}]$,

$$(p(t)\phi_\gamma(x'(t)))'=e(t)-q_0(t)\,\phi_\gamma(x(t))-\int_0^b q(t,s)\,\phi_{\alpha(s)}(x(t))\,d\zeta(s)\le 0.$$

Hence $p(t)\phi_\gamma(x'(t))$ is nonincreasing on $(a_1,\tau_{\widehat{a_1}+1}]$. Note that for any $t\in(a_1,\tau_{\widehat{a_1}+1}]$

$$x(t)-x(a_1)=x'(\zeta)(t-a_1)\qquad \text{for some } \zeta\in(a_1,t).$$

Since $x(a_1)>0$ and ϕ_γ is an increasing function, then

$$\begin{aligned}
\phi_\gamma(x(t)) \;&>\; \phi_\gamma(x(t)-x(a_1))=\phi_\gamma(x'(\zeta))(t-a_1)^\gamma \ge \frac{p(t)\;\phi_\gamma(x'(t))}{p(\zeta)}(t-a_1)^\gamma \\
&=\; \frac{(\rho p)(t)\;\phi_\gamma(x'(t))}{\rho(t)\,p(\zeta)}(t-a_1)^\gamma \ge \frac{(\rho p)(t)\;\phi_\gamma(x'(t))}{(\rho p)_{\widehat{a_1}+1}}(t-a_1)^\gamma,
\end{aligned}$$

which yields that for $t\in(a_1,\tau_{\widehat{a_1}+1}]$

$$z(t)=\frac{(\rho p)(t)\,\phi_\gamma(x'(t))}{\phi_\gamma(x(t))}<(\rho p)_{\widehat{a_1}+1}\;(t-a_1)^{-\gamma}.$$

In particular, when $t=\tau_{\widehat{a_1}+1}^-$, we get

$$z(\tau_{\widehat{a_1}+1})<(\rho p)_{\widehat{a_1}+1}\;(\tau_{\widehat{a_1}+1}-a_1)^{-\gamma}.$$

In the same way, we have, for $t\in(\tau_{k-1},\tau_k]$ with $k=\widehat{a_1}+2,\ldots,\widehat{b_1}$

$$z(\tau_k)<(\rho p)_k\;(\tau_k-\tau_{k-1})^{-\gamma}.$$

Hence

$$\begin{aligned}
& \sum\nolimits_{k=\widehat{a_1}+1}^{\widehat{b_1}} \left|u_1\left(\tau_k\right)\right|^{\gamma+1} z\left(\tau_k\right)\left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right)-1\right] \\
< \; & \left|u_1\left(\tau_{\widehat{a_1}+1}\right)\right|^{\gamma+1}(\rho p)_{\widehat{a_1}+1}\ \left(\tau_{\widehat{a_1}+1}-a_1\right)^{-\gamma}\left[\phi_\gamma\left(\frac{\eta_{\widehat{a}_1+1}}{\lambda_{\widehat{a_1}+1}}\right)-1\right] \\
& +\sum\nolimits_{k=\widehat{a_1}+2}^{\widehat{b_1}}\left|u_1\left(\tau_k\right)\right|^{\gamma+1}(\rho p)_k\ \left(\tau_k-\tau_{k-1}\right)^{-\gamma}\left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right)-1\right].
\end{aligned}$$

Then from (3.14), we get

$$\int_{a_1}^{b_1}\left[Q(t)\left|u_1(t)\right|^{\gamma+1}-p(t)\widetilde{u}_1(t)\right] dt<\Phi\left[\left|u_1\right|^{\gamma+1} ; a_1, b_1\right].$$

This leads to a contradiction to (2.3). This completes the proof. □

Proof of Theorem 2.2. Assume Eq. (1.1) has an extendible solution $x(t)$ which is eventually positive or negative. Then, without loss of generality, assume $x(t)>0$ for all $t \geq T \geq 0$, where T depends on the solution $x(t)$. Define $z(t)$ by (3.2). From (3.11), we get that

$$z'(t) \leq -Q(t)+\frac{\rho'(t)}{\rho(t)} z(t)-\frac{\gamma|z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho p)^{\frac{1}{\gamma}}(t)} \quad \text{for } t \in\left(a_1, b_1\right] \text{ and } t \neq \tau_k. \tag{3.16}$$

Multiplying both sides of (3.16), with t replaced by s, by $H_1\left(b_1, s\right)$ and integrating with respect to s from c_1 to b_1, we find that

$$\begin{aligned}
\int_{c_1}^{b_1} Q(s) H_1\left(b_1, s\right) ds & \leq-\int_{c_1}^{b_1} z'(s) H_1\left(b_1, s\right) ds \\
+\int_{c_1}^{b_1} \frac{\rho'(s)}{\rho(s)} z(s) H_1\left(b_1, s\right) ds & -\int_{c_1}^{b_1} \frac{\gamma|z(s)|^{\frac{\gamma+1}{\gamma}}}{(\rho p)^{\frac{1}{\gamma}}(s)} H_1\left(b_1, s\right) ds.
\end{aligned}$$

Using integration by parts and from (2.6) and (2.8), we obtain that

$$\begin{aligned}
& \int_{c_1}^{b_1} Q(s) H_1\left(b_1, s\right) ds \\
\leq \; & -\sum\nolimits_{k=\widehat{c_1}+1}^{\widehat{b_1}} H_1\left(b_1, \tau_k\right) z\left(\tau_k\right)\left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right)-1\right] \\
& +\int_{c_1}^{b_1}\left[(\gamma+1) h_{12}\left(b_1, s\right) H_1^{\frac{\gamma}{\gamma+1}}\left(b_1, s\right) z(s)-\frac{\gamma|z(s)|^{\frac{\gamma+1}{\gamma}} H_1\left(b_1, s\right)}{(\rho p)^{\frac{1}{\gamma}}(s)}\right] ds \\
\leq \; & -\sum\nolimits_{k=\widehat{c_1}+1}^{\widehat{b_1}} H_1\left(b_1, \tau_k\right) z\left(\tau_k\right)\left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right)-1\right] \\
& +\int_{c_1}^{b_1}\left[(\gamma+1)\left|h_{12}\left(b_1, s\right)\right| H_1^{\frac{\gamma}{\gamma+1}}\left(b_1, s\right)|z(s)|-\frac{\gamma|z(s)|^{\frac{\gamma+1}{\gamma}} H_1\left(b_1, s\right)}{(\rho p)^{\frac{1}{\gamma}}(s)}\right] ds.
\end{aligned} \tag{3.17}$$

Let $\lambda = \frac{\gamma+1}{\gamma}$. Define A and B by

$$A^{\lambda} := \frac{\gamma \left| z\left(s\right) \right|^{\lambda} H_1\left(b_1, s\right)}{p^{\frac{1}{\gamma}}\left(t\right)} \quad \text{and} \quad B^{\lambda-1} := \left(\gamma \rho\left(s\right) p(s)\right)^{\frac{1}{\gamma+1}} \left| h_{12}\left(b_1, s\right) \right|.$$

Then by the inequality (3.13), we get that

$$\begin{aligned} & (\gamma+1) \left| h_{12}\left(b_1, s\right) \right| H_1^{\frac{\gamma}{\gamma+1}}\left(b_1, s\right) \left| z\left(s\right) \right| - \frac{\gamma \left| z\left(s\right) \right|^{\frac{\gamma+1}{\gamma}} H_1\left(b_1, s\right)}{p^{\frac{1}{\gamma}}\left(s\right)} \\ \leq \ & (\rho p)^{\frac{1}{\gamma}}\left(s\right) \left| h_{12}\left(b_1, s\right) \right|^{\gamma+1}. \end{aligned}$$

This together with (3.17) shows that

$$\begin{aligned} & \int_{c_1}^{b_1} \left[Q\left(s\right) H_1\left(b_1, s\right) - (\rho p)^{\frac{1}{\gamma}}\left(s\right) \left| h_{12}\left(b_1, s\right) \right|^{\gamma+1} \right] ds \\ \leq \ & -\sum_{k=\widehat{c_1}+1}^{\widehat{b_1}} H_1\left(b_1, \tau_k\right) z\left(\tau_k\right) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1 \right]. \end{aligned} \tag{3.18}$$

Similarly, multiplying both sides of (3.16), with t replaced by s, by $H_1\left(s, a_1\right)$ and integrating by parts from a_1 to c_1, we see that

$$\begin{aligned} & \int_{a_1}^{c_1} \left[Q\left(s\right) H_1\left(s, a_1\right) - (\rho p)^{\frac{1}{\gamma}}\left(s\right) \left| h_{11}\left(s, a_1\right) \right|^{\gamma+1} \right] ds \\ \leq \ & -\sum_{k=\widehat{a_1}+1}^{\widehat{c_1}} H_1\left(\tau_k, a_1\right) z\left(\tau_k\right) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1 \right]. \end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) we obtain that

$$\begin{aligned} & \frac{1}{H_1\left(c_1, a_1\right)} \int_{a_1}^{c_1} \left[Q\left(s\right) H_1\left(s, a_1\right) - (\rho p)^{\frac{1}{\gamma}}(s) h_{11}^{\gamma+1}\left(s, a_1\right) \right] ds \\ & + \frac{1}{H_1\left(b_1, c_1\right)} \int_{c_1}^{b_1} \left[Q\left(s\right) H_1\left(b_1, s\right) - (\rho p)^{\frac{1}{\gamma}}(s) h_{12}^{\gamma+1}\left(b_1, s\right) \right] ds \\ \leq \ & -\frac{1}{H_1\left(c_1, a_1\right)} \sum_{k=\widehat{a_1}+1}^{\widehat{c_1}} H_1\left(\tau_k, a_1\right) z\left(\tau_k\right) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1 \right] \\ & -\frac{1}{H_1\left(b_1, c_1\right)} \sum_{k=\widehat{c_1}+1}^{\widehat{b_1}} H_1\left(b_1, \tau_k\right) z\left(\tau_k\right) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1 \right]. \end{aligned} \tag{3.20}$$

There are several cases to consider: (i) $\widehat{a_1} = \widehat{c_1} = \widehat{b_1}$, (ii) $\widehat{a_1} < \widehat{c_1} = \widehat{b_1}$, (iii) $\widehat{a_1} = \widehat{c_1} < \widehat{b_1}$, and (iv) $\widehat{a_1} < \widehat{c_1} < \widehat{b_1}$.

Case (i) $\widehat{a_1} = \widehat{c_1} = \widehat{b_1}$. There are no impulsive moments in $[a_1, b_1]$. Thus

$$\sum_{k=\widehat{a_1}+1}^{\widehat{c_1}} H_1\left(\tau_k, a_1\right) z\left(\tau_k\right) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1 \right] = 0 = \Phi\left[H_1\left(\cdot, a_1\right); a_1, c_1\right] \tag{3.21}$$

and

$$\sum_{k=\widehat{c_1}+1}^{\widehat{b_1}} H_1\left(b_1, \tau_k\right) z\left(\tau_k\right) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1 \right] = 0 = \Phi\left[H_1\left(b_1, \cdot\right); c_1, b_1\right]. \tag{3.22}$$

Case (ii) $\widehat{a_1} < \widehat{c_1} = \widehat{b_1}$. There are impulsive moments $\tau_{\widehat{a_1}+1},\ \tau_{\widehat{a_1}+2},\ ...,\ \tau_{\widehat{c_1}}$ in $[a_1, c_1]$ and no impulsive moments in $[c_1, b_1]$. Thus, (3.22) holds. As in Theorem 2.1, we have

$$z\left(\tau_{\widehat{a_1}+1}\right) < (\rho p)_{\widehat{a_1}+1}\ \left(\tau_{\widehat{a_1}+1} - a_1\right)^{-\gamma}$$

and

$$z\left(\tau_{k+1}\right) < (\rho p)_k\ \left(\tau_k - \tau_{k-1}\right)^{-\gamma}, \quad k = \widehat{a_1}+1, \ldots, \widehat{c_1}-1.$$

Thus

$$\sum\nolimits_{k=\widehat{a_1}+1}^{\widehat{c_1}} H_1\left(\tau_k, a_1\right) z\left(\tau_k\right) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1\right] < \Phi\left[H_1\left(\cdot, a_1\right); a_1, c_1\right].$$

Case (iii) $\widehat{a_1} = \widehat{c_1} < \widehat{b_1}$. there are no impulsive moments in $[a_1, c_1]$ and there are impulsive moments $\tau_{\widehat{c_1}+1},\ \tau_{\widehat{c_1}+2},\ ...,\ \tau_{\widehat{b_1}}$ in $[c_1, b_1]$. Similar to Case (ii), we have that (3.21) holds, and

$$\sum\nolimits_{k=\widehat{c_1}+1}^{\widehat{b_1}} H_1\left(b_1, \tau_k\right) z\left(\tau_k\right) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1\right] < \Phi\left[H_1\left(b_1, \cdot\right); c_1, b_1\right].$$

Case (iv) $\widehat{a_1} < \widehat{c_1} < \widehat{b_1}$. There are impulsive moments $\tau_{\widehat{a_1}+1},\ \tau_{\widehat{a_1}+2},\ ...,\ \tau_{\widehat{c_1}}$ in $[a_1, c_1]$ and impulsive moments $\tau_{\widehat{c_1}+1},\ \tau_{\widehat{c_1}+2},\ ...,\ \tau_{\widehat{b_1}}$ in $[c_1, b_1]$. Similar to Cases (ii) and (iii), we have that

$$\sum\nolimits_{k=\widehat{a_1}+1}^{\widehat{c_1}} H_1\left(\tau_k, a_1\right) z\left(\tau_k\right) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1\right] < \Phi\left[H_1\left(\cdot, a_1\right); a_1, c_1\right]$$

and

$$\sum\nolimits_{k=\widehat{c_1}+1}^{\widehat{b_1}} H_1\left(b_1, \tau_k\right) z\left(\tau_k\right) \left[\phi_\gamma\left(\frac{\eta_k}{\lambda_k}\right) - 1\right] < \Phi\left[H_1\left(b_1, \cdot\right); c_1, b_1\right].$$

For all the cases, from (3.20) we have

$$\begin{aligned} & \frac{1}{H_1\left(c_1, a_1\right)} \int_{a_1}^{c_1} \left[Q\left(s\right) H_1\left(s, a_1\right) - (\rho p)\left(s\right) h_{11}^{\gamma+1}\left(s, a_1\right)\right] ds \\ & + \frac{1}{H_1\left(b_1, c_1\right)} \int_{c_1}^{b_1} \left[Q\left(s\right) H_1\left(b_1, s\right) - (\rho p)\left(s\right) h_{12}^{\gamma+1}\left(b_1, s\right)\right] ds \\ \leq\ & \frac{1}{H_1\left(c_1, a_1\right)} \Phi\left[H_1\left(\cdot, a_1\right); a_1, c_1\right] + \frac{1}{H_1\left(b_1, c_1\right)} \Phi\left[H_1\left(b_1, \cdot\right); c_1, b_1\right]. \end{aligned}$$

This contradicts (2.9) with $i = 1$ and hence completes the proof. □

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