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Qualitative aspects of a Volterra integro-dynamic system on time scales

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Abstract. This paper deals with the resolvent, asymptotic stability and boundedness of the solution of time-varying Volterra integro-dynamic system on time scales in which the coefficient matrix is not necessarily stable. We generalize to a time scale some known properties about asymptotic behavior and boundedness from the continuous case. Some new results for the discrete case are obtained.
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1 Introduction and preliminaries

Basic qualitative results about Volterra integro-differential equations have been studied by many authors. Notable exceptions that have dispensed with the stability condition on the coefficient matrix have been the works of Burton [4, 5], Corduneanu [7], Choi and Koo [8], Mahfoud [21], Medina [22], Rao and Srinivas [24], among others. In [4], the author investigates the stability

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and boundedness of the solution involving the anti-derivatives of the kernel. Sufficient conditions for uniformly bounded solution are developed in [21]. In [24], the asymptotic behavior of the solution of a Volterra integro-differential equation is discussed in which the coefficient matrix is not necessarily stable. The resolvent of a Volterra integro-differential equation was first investigated by Grossman and Miller in [14]. In the discrete case resolvent equation was obtained by Elaydi in [10].

The area of dynamic equations on time scales is a new, modern and progressive component of applied analysis that acts as the framework to effectively describe processes that feature both continuous and discrete elements (see e.g. [2, 18, 19, 20, 25]). Created by Hilger in 1988 [15] and developed by Bohner and Peterson [6]. Volterra type equations (both integral and intgro-dynamic) on time scales become a new field of interest. In [16], Kulik and Tisdell obtained basic qualitative and quantitative results for Volterra integral equations. Furthermore, in [17] Karpuz studied the existence and the uniqueness of solutions to generalized Volterra integral equations.

In a very recent paper [1], Adivar introduces the principal matrix solutions and variation of parameters for Volterra integro-dynamic equations. Motivated by the interesting nature of this problem, an attempt has been made to study some stability and boundedness properties of the following system

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s + F(t), \ \ t \in \mathbb{T}_0 = [t_0, \infty) \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

where $0 \leq t_0 \in \mathbb{T}^k$ is fixed, A (not necessarily stable) is an $n \times n$ matrix function and F is an $n-$vector function, which are both continuous on $\mathbb{T}_0$, K is an $n \times n$ matrix function, which is continuous on $\Omega := \{(t,s) \in \mathbb{T}_0 \times \mathbb{T}_0 : t_0 \leq s \leq t < \infty\}$.

We will briefly recall some basic definitions and facts from the times scales calculus that we will use in the sequel, for readers convenience.

A time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) =$

$t, \rho(t) < t, \sigma(t) = t, \sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. $\mu(t) = \sigma(t) - t$ is called the *graininess function*. The notations $[a, b]$, $[a, b)$, and so on, will denote time scales intervals such as $[a, b] := \{t \in \mathbb{T}; a \leq t \leq b\}$, where $a, b \in \mathbb{T}$. Throughout this article we assume that $\sup\mathbb{T} = \infty$ and the graininess function $\mu(t)$ is bounded.

Definition 1.1 *Let X be a Banach space. The function $f : \mathbb{T} \to X$ is called* rd-continuous *provided it is continuous at each right-dense point and has a left-sided limit at each point.*

The set of all rd-continuous functions $f : \mathbb{T} \to X$ is denoted by $C_{rd}(\mathbb{T}, X)$.

Definition 1.2 *For $t \in \mathbb{T}^k$, let the Δ-derivative of f at t, denoted $f^\Delta(t)$, be the number (provided it exists), such that for all $\varepsilon > 0$ there exists a neighborhood U of t such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

The set of all functions $f : \mathbb{T} \to X$ that are differentiable on $\mathbb{T}$ and its Δ-derivative $f^\Delta(t) \in C_{rd}(\mathbb{T}, X)$ is denoted by $C^1_{rd}(\mathbb{T}, X)$.

Definition 1.3 *A function F is called an antiderivative of $f : \mathbb{T} \to X$ provided*

$$F^\Delta(t) = f(t) \quad \textit{for each } t \in \mathbb{T}^k.$$

Remark 1.4 (i) If f is continuous, then f is $rd-$continuous.

(ii) If f is delta differentiable at t, then f is continuous at t.

(iii) From the definition of the operator σ it follows that

$$\sigma(t) = \begin{cases} t, & \mathbb{T} = \mathbb{R} \\ t + 1, & \mathbb{T} = \mathbb{Z} \\ qt, & \mathbb{T} = \overline{q^{\mathbb{Z}}} \end{cases}$$

where $\overline{q^{\mathbb{Z}}} = \{q^k : k \in Z\} \cup \{0\}$ and $q > 1$. Hence the Δ-derivative $f^\Delta(t)$ turns into ordinary derivative $f'(t)$ if $\mathbb{T} = \mathbb{R}$ and it becomes the forward difference operator $\Delta f(t) = f(t+1) - f(t)$ whenever $\mathbb{T} = \mathbb{Z}$. For the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ we have $f^{\Delta}(t) = D^q f(t)$, where $D^q f(t) = \dfrac{f(qt) - f(t)}{(q-1)t}$. Thus, one can consider the differential, difference and q-difference equations as special cases of the dynamic equations on time scale.

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* (respectively *positively regressive*) if $1 + \mu(t)p(t) \neq 0$ (respectively $1 + \mu(t)p(t) > 0$) for all $t \in \mathbb{T}^k$. The space of all regressive functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$ (respectively $\mathcal{R}^+(\mathbb{T}, \mathbb{R})$). The space of all rd-continuous and regressive functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$. The generalized exponential function e_p is defined as the unique solution $y(t) = e_p(t, a)$ of the initial value problem $y^{\Delta} = p(t)y,\ y(a) = 1$, where p is regressive function. An explicit formula for $e_p(t, a)$, is given by

$$e_p(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \text{ with } \xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1+hz)}{h} & h \neq 0 \\ z & h = 0. \end{cases}$$

For more details, see [6]. Clearly, $e_p(t, s)$ never vanishes. The following results will be used throughout this work.

Lemma 1.5 *([6, Theorem 2.38]) If $p,\ q \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$. Then $e^{\Delta}_{p\ominus q}(\cdot, t_0) = (p - q)\dfrac{e_p(\cdot, t_0)}{e^{\sigma}_q(\cdot, t_0)}$.*

Lemma 1.6 *([6, Theorem 6.2]) Let $\alpha \in \mathbb{R}$ with $\alpha \in C^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$. Then*

$$e_\alpha(t, s) \geq 1 + \alpha(t - s) \text{ for all } t \geq s.$$

Lemma 1.7 *([6, Corollary 6.7]) Let $y \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $p \in C^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $p \geq 0$ and $\alpha \in \mathbb{R}$. Then*

$$y(t) \leq \alpha + \int_\tau^t y(s)p(s)\Delta s \text{ for all } t \in \mathbb{T},$$

implies

$$y(t) \leq \alpha e_p(t, \tau) \text{ for all } \ t \in \mathbb{T}.$$

Theorem 1.8 *([17, Theorem 7]) Let $a, b \in \mathbb{T}$ with $b > a$ and assume that $f : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is integrable on $\{(t, s) \in \mathbb{T} \times \mathbb{T} : b > t > s \geq a\}$. Then*

$$\int_a^b \int_a^\eta f(\eta, \xi)\Delta\xi\Delta\eta = \int_a^b \int_{\sigma(\xi)}^b f(\eta, \xi)\Delta\eta\Delta\xi.$$

It is easy to verify that the above result holds for $f \in C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{R}^n)$.

Also we consider the discrete time scale (see [9, 23])

$$\mathbb{T}^r_{(q,h)} = \{rq^k + [k]_q h : k \in \mathbb{Z}\} \cup \left\{\frac{h}{1-q}\right\},$$

where $r \in \mathbb{R}$, $q \geq 1$, $h \geq 0$, $q + h > 1$, $[k]_q = \frac{q^k-1}{q-1}$, $k \in \mathbb{R}$, $q \neq 1$ *and* $[k]_1 = k$ $\left(= \lim\limits_{q\to 1} \frac{q^k-1}{q-1}\right)$. It is easy to see that $\mathbb{T}^r_{(q,h)} = \mathbb{T}^r_q = \{rq^k : k \in \mathbb{Z}\} \cup \{0\}$ provided $h = 0$ and $\mathbb{T}^r_{(q,h)} = \mathbb{T}^r_h = \{r + kh : k \in \mathbb{Z}\}$ provided $q = 1$ (in this case we put $h/(1 - q) = -\infty$). It is clear that, for $t \in \mathbb{T}^r_{(q,h)}$, we have

$$\sigma(t) = qt + h \text{ and } \mu(t) = (q - 1)t + h.$$

Let $t \in \mathbb{T}^r_{(q,h)}$ and $f : \mathbb{T}^r_{(q,h)} \to \mathbb{R}$. Then the delta (q, h)-derivative of f at t is

$$\Delta_{(q,h)} f(t) := \frac{f(qt + h) - f(t)}{(q - 1)t + h}$$

and the (q, h)-integral is

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} f(t)\mu(t).$$

For $z \neq -1/(q't + h)$, where $q' = q - 1$, the exponential function has form

$$e_z(t, s) = \prod_{r\in[s,t)} (1 + \mu(r)z), \text{ for all } t, s \in \mathbb{T}^r_{(q,h)}$$

and

$$e_{\ominus z}(t, s) = \prod_{r\in[s,t)} \frac{1}{(1 + \mu(r)z)}, \text{ for all } t, s \in \mathbb{T}^r_{(q,h)}.$$

Then, from (1.1), we obtain the following discrete variant

$$\begin{cases} \Delta_{(q,h)} x(t) = A(t)x(t) + \sum_{s\in[t_0,t)} K(t,s)x(s)\mu(s) + F(t), \\ x(t_0) = x_0, \end{cases} \tag{1.2}$$

where A is an $n\times n$ matrix function, F is an $n-$vector function on $\mathbb{T}^r_{(q,h)}$ and K is an $n \times n$ matrix function on $\Omega_{(q,h)} := \{(t,s) \in \mathbb{T}^r_{(q,h)} \times \mathbb{T}^r_{(q,h)} : t_0 \le s \le t < \infty\}$.

The rest of the paper is organized as follows: Section 2 is devoted to the study of the relation between principal matrix and resolvent of (1.1). In section 3 we investigate the asymptotic behavior of the solutions of the system (1.1). The main aim in this section is to develop an equivalence system of (1.1) having a potential to give the sufficient conditions for asymptotic stability. In Section 4 we first discuss the uniform boundedness of the solutions of (1.1) by constructing a Lyapunov functional. Further results for boundedness, the uniform boundedness and stability of the solution are developed by an equivalence system of (1.1), which is constructed by using the antiderivative of the kernel. For the discrete time scale $\mathbb{T}^r_{(q,h)}$ we give the related results as corollaries.

2 Resolvent

Lemma 2.1 *If $A(t)$ and $K(t,s)$ are the continuous functions given in equation (1.1), then*

$$\Delta_s R(t,s) = -R(t,\sigma(s))A(s) - \int_{\sigma(s)}^{t} R(t,\sigma(u))K(u,s)\Delta u \tag{2.1}$$

is equivalent to

$$R(t,s) = I + \int_s^t R(t,\sigma(u))W(u,s)\Delta u, \tag{2.2}$$

where

$$W(t,s) = A(t) + \int_s^t K(t,u)\Delta u. \tag{2.3}$$

Proof. Substituting (2.3) in (2.2) and using Theorem 1.8, we obtain

$$\begin{aligned} R(t,s) &= I + \int_s^t R(t,\sigma(u)) \left[A(u) + \int_s^u K(u,v)\Delta v \right] \Delta u \\ &= I + \int_s^t R(t,\sigma(u))A(u)\Delta u + \int_s^t R(t,\sigma(u)) \int_s^u K(u,v)\Delta v \Delta u \\ &= I + \int_s^t R(t,\sigma(u))A(u)\Delta u + \int_s^t \int_{\sigma(v)}^t R(t,\sigma(u))K(u,v)\Delta u \Delta v. \end{aligned}$$

Differentiating with respect to s, we obtain (2.1).

Conversely, we have to show that (2.1) implies (2.2). So, integrating (2.1) from s to t, we obtain

$$R(t,t) - R(t,s) = -\int_s^t R(t,\sigma(u))A(u)\Delta u - \int_s^t \int_{\sigma(v)}^t R(t,\sigma(u))K(u,v)\Delta u \Delta v,$$

which implies that

$$R(t,s) = I + \int_s^t R(t,\sigma(u))A(u)\Delta u + \int_s^t \int_{\sigma(v)}^t R(t,\sigma(u))K(u,v)\Delta u \Delta v.$$

Furthermore, using Theorem 1.8, we have

$$\begin{aligned} R(t,s) &= I + \int_s^t R(t,\sigma(u))A(u)\Delta u + \int_s^t R(t,\sigma(u)) \int_s^u K(u,v)\Delta v \Delta u \\ &= I + \int_s^t R(t,\sigma(u)) \left[A(u) + \int_s^u K(u,v)\Delta v \right] \Delta u \\ &= I + \int_s^t R(t,\sigma(u))W(u,s)\Delta u. \end{aligned}$$

Hence, (2.1) and (2.2) are equivalent systems, and the proof is completed. ■

Theorem 2.2 *Assume A and K are continuous functions given in (1.1). Then the function $R(t,s)$, as defined in (2.2), exists on $t_0 \le s \le t$ and is continuous in (t,s). $\Delta_s R(t,s)$ exists, is continuous and satisfies the equation (2.1) on $t_0 \le s \le t$, for each $t > t_0$. Moreover, given any vector x_0 and any continuous function $F(t)$, equation (1.1) is equivalent to the system*

$$x(t) = R(t,t_0)x_0 + \int_{t_0}^t R(t,\sigma(s))F(s)\Delta s. \tag{2.4}$$

Proof. Since $W(t,s)$ is continuous in s for each fixed t, the existence of $R(t,s)$ on $t_0 \le s \le t$ is trivial (see [17, Theorem 1]). From the above calculations, it follows that for each fixed t, $\Delta_s R(t,s)$ exists and satisfies (2.1) by Lemma 2.1. Since K is continuous on $t_0 \le s \le t < \infty$, we have

$$|W(t,s)| \le |A(t)| + \int_{t_0}^{t} |K(t,u)|\,\Delta u = w(t)$$

and w is continuous. Application of the Gronwall inequality (see [6, Theorem 6.4]) in (2.2), yields the estimate

$$\begin{aligned} |R(t,\sigma(s))| &= \left| I + \int_{\sigma(s)}^{t} R(t,\sigma(u))W(u,\sigma(s))\Delta u \right| \\ &\le 1 + \int_{t_0}^{t} |R(t,\sigma(u))|\, w(u)\Delta u \\ &\le 1 + \int_{t_0}^{t} e_w(t,\sigma(u))w(u)\Delta u \\ &= w_0(t), \end{aligned} \tag{2.5}$$

which implies that $R(t,\sigma(s))$ is continuous. Using this fact in (2.1) it is apparent that $\Delta_s R(t,s)$ is continuous, and that

$$\begin{aligned} |\Delta_s R(t,s)| &= \left| -R(t,\sigma(s))A(s) - \int_{\sigma(s)}^{t} R(t,\sigma(u))K(u,s)\Delta u \right| \\ &\le w_0(t)\,|A(s)| + \int_{\sigma(s)}^{t} w_0(t)\,|K(u,s)|\,\Delta u \\ &\le w_0(t)\left(A(s) + \int_{\sigma(s)}^{T} K(u,s)\Delta u \right) \end{aligned} \tag{2.6}$$

if $t_0 \le s \le t \le T$. Using (2.5) and dominated convergence, it follows the continuity of $R(t,s)$ in t for a fixed s. From (2.6), it is clear that $R(t,s)$ is uniformly continuous for $t_0 \le s \le t \le T$. Hence, by [13, Theorem 5, p. 102], $R(t,s)$ is continuous in the pair (t,s).

Now let $x(t)$ be a solution of (1.1) on an interval $t_0 \le t \le T$. If we take $p(s) = R(t,s)x(s)$, then we have

$$p^{\Delta}(s) = \Delta_s R(t,s)x(s) + R(t,\sigma(s))x^{\Delta}(s)$$

and by (1.1), it follows

$$p^{\Delta}(s) \;=\; \Delta_s R(t,s)x(s) + R(t,\sigma(s))A(s)x(s)$$

$$+R(t,\sigma(s))\int_{t_0}^{s} K(s,\tau)x(\tau)\Delta\tau + R(t,\sigma(s))F(s).$$

Integrating from t_0 to t we have

$$\begin{aligned} p(t)-p(t_0) &= \int_{t_0}^{t} \Delta_s R(t,s)x(s)\Delta s + \int_{t_0}^{t} R(t,\sigma(s))A(s)x(s)\Delta s \\ &\quad + \int_{t_0}^{t} R(t,\sigma(s)) \int_{t_0}^{s} K(s,\tau)x(\tau)\Delta\tau\Delta s \\ &\quad + \int_{t_0}^{t} R(t,\sigma(s))F(s)\Delta s. \end{aligned}$$

Using Theorem 1.8, we obtain

$$\begin{aligned} & x(t) - R(t,t_0)x_0 \\ = & \int_{t_0}^{t} \Delta_s R(t,s)x(s)\Delta s + \int_{t_0}^{t} R(t,\sigma(s))A(s)x(s)\Delta s \\ & + \int_{t_0}^{t} \left(\int_{\sigma(s)}^{t} R(t,\sigma(\tau))K(\tau,s)\Delta\tau \right) x(s)\Delta s + \int_{t_0}^{t} R(t,\sigma(s))F(s)\Delta s \\ = & \int_{t_0}^{t} \left[\Delta_s R(t,s) + R(t,\sigma(s))A(s) + \int_{\sigma(s)}^{t} R(t,\sigma(\tau))K(\tau,s)\Delta\tau \right] x(s)\Delta s \\ & + \int_{t_0}^{t} R(t,\sigma(s))F(s)\Delta s. \end{aligned}$$

Furthermore, by using (2.1) we obtain (2.4). Moreover, if $x(t)$ solves (2.4) on an interval $t_0 \le t \le \tau$, then it is easy to see that $x(t)$ solves (1.1), which completes the proof. ■

Consider the adjoint dynamical equation [6, Theorem 5.27],

$$y^{\Delta}(t) = -A^T(t)y^{\sigma}(t) - f(t) \tag{2.7}$$

where A^T is the transpose of A. Let us extend this definition to the integro-dynamic equation (1.1).

Definition 2.3 *For a fixed t the adjoint to (1.1) is*

$$\begin{cases} y^{\Delta}(s) = -A^T(s)y^{\sigma}(s) - \int_{\sigma(s)}^{t} K^T(u,s)y^{\sigma}(u)\Delta u - f(s) \\ y(t) = y_0, \end{cases} \tag{2.8}$$

where $s \in [t_0, t]$.

It is easy to see by Theorem 1.8 that (2.8) is equivalent to an integral equation

$$y(s) = y_0 + \int_s^t \left[A^T(u) + \int_s^u K(u,v)\Delta v\right] y^\sigma(u)\Delta u + \int_s^t f(u)\Delta u. \quad (2.9)$$

For the next result, we define, for a fixed t, the space of continuous function

$$C_{y_0}[t_0,t] := \{\varphi \in C[t_0,t] : \varphi(t) = y_0\}$$

and the metric

$$d^1_\beta(\varphi,\psi) := \sup\{|\varphi(s) - \psi(s)|\, e_\beta(s,t_0) : t_0 \le s \le t\}.$$

The metric space $(C_{y_0}[t_0,t], d^1_\beta)$ is complete by replacing β with $\ominus\beta$ in

$$d_\beta(\varphi,\psi) := \sup\left\{\frac{|\varphi(s) - \psi(s)|}{e_\beta(s,t_0)} : t_0 \le s \le t\right\},$$

where $\ominus\beta = -\beta/(1+\mu(t)\beta)$ (see [16, Lemma 3.1]).

Theorem 2.4 *For a fixed $t \in \mathbb{T}_0$ such that $t > t_0$ and a given $y_0 \in \mathbb{R}^n$, there is a unique solution $y(s)$ of (2.9) on the interval $[t_0,t]$ satisfying the condition $y(t) = y_0$.*

Proof. We define the mapping

$$(P\varphi)(s) := y_0 + \int_s^t \left[A^T(u) + \int_s^u K(u,v)\Delta v\right] \varphi^\sigma(u)\Delta u + \int_s^t f(u)\Delta u$$

for all $\varphi \in C_{y_0}[t_0,t]$. For a given $\varphi \in C_{y_0}[t_0,t]$, it follows that $P\varphi$ is continuous on $[t_0,t]$ and that $(P\varphi)(t) = y_0$. Thus, $P : C_{y_0}[t_0,t] \to C_{y_0}[t_0,t]$. For an arbitrary pair of functions $\varphi,\psi \in C_{y_0}[t_0,t]$,

$$\begin{aligned}
&|(P\phi)(s) - (P\psi)(s)| \\
= &\left|\int_s^t \left[A^T(u) + \int_s^u K^T(u,v)\Delta v\right](\varphi^\sigma(u) - \psi^\sigma(u))\Delta u\right| \\
\le &\int_s^t \left[\left|A^T(u)\right| + \int_s^u \left|K^T(u,v)\right|\Delta v\right] |(\varphi^\sigma(u) - \psi^\sigma(u))|\,\Delta u.
\end{aligned}$$

Since $A(u)$ and $K(u,v)$ are continuous for $t_0 \le s \le u \le t$, then there is $\beta > 1$ such that

$$\left|A^T(u)\right| + \int_s^u \left|K^T(u,v)\right| \Delta v \le \beta - 1.$$

Then we obtain the following estimation

$$|(P\phi)(s) - (P\psi)(s)| \le \int_s^t (\beta - 1) \left|(\varphi^\sigma(u) - \psi^\sigma(u))\right| \Delta u. \tag{2.10}$$

Now, we have to show that P is a contraction on $C_{y_0}[t_0, t]$. Multiplying (2.10) by $e_\beta(s, t_0)$ we obtain

$$\begin{aligned}
& |(P\phi)(s) - (P\psi)(s)|\, e_\beta(s, t_0) \\
\le\ & \int_s^t (\beta - 1) e_\beta(s, t_0) \left|(\varphi^\sigma(u) - \psi^\sigma(u))\right| \Delta u \\
\le\ & \int_s^t (\beta - 1) e_\beta(s, \sigma(u)) \left|(\varphi^\sigma(u) - \psi^\sigma(u))\right| e_\beta(\sigma(u), t_0) \Delta u \\
\le\ & d_\beta^1(\varphi, \psi) \int_s^t (\beta - 1) e_\beta(s, \sigma(u)) \Delta u \\
=\ & d_\beta^1(\varphi, \psi) \frac{(\beta - 1)}{-\beta} \int_s^t [e_\beta(s, u)]^\Delta \Delta u \\
=\ & d_\beta^1(\varphi, \psi) \frac{(\beta - 1)}{-\beta} [e_\beta(s, t) - 1)] \\
\le\ & d_\beta^1(\varphi, \psi) \frac{(\beta - 1)}{\beta}.
\end{aligned}$$

Taking supremum over s, we have

$$d_\beta^1(P\phi, P\psi) \le \frac{(\beta - 1)}{\beta} d_\beta^1(\varphi, \psi).$$

Therefore, by Banach fixed point theorem, P has a unique fixed point in $C_{y_0}[t_0, t]$. It follows that, (2.9) has a unique solution on the interval $[t_0, t]$. ■

Definition 2.5 *The principal matrix solution of*

$$y^\Delta(s) = -A^T(s) y^\sigma(s) - \int_{\sigma(s)}^t K^T(u, s) y^\sigma(u) \Delta u \tag{2.11}$$

is the $n \times n$ matrix function

$$Z_1(t,s) := [y^1(t,s) \quad y^2(t,s) \quad \dots \quad y^n(t,s)], \tag{2.12}$$

where $y^i(t,s)$ (t fixed) is the unique solution of (2.11) on $[t_0, t]$ that satisfies the condition $y^i(t,t) = e^i$.

By virtue of this definition, $Z_1(t,s)$ is the unique matrix solution of

$$\Delta_s Z_1(t,s) = -A^T(s) Z_1(t, \sigma(s)) - \int_{\sigma(s)}^{t} K^T(u,s) Z_1(t, \sigma(u)) \Delta u, \tag{2.13}$$

such that $Z_1(t,t) = I$, on the interval $[t_0, t]$. Reasoning as in the proof of [1, Theorem 12], we conclude that for a given $y_0 \in \mathbb{R}^n$, the unique solution of (2.11) satisfying the condition $y(t) = y_0$ is

$$y(s) = Z_1(t,s) y_0 \tag{2.14}$$

for $t_0 \leq s \leq t$.

Taking the transpose of (2.11) we obtain

$$(y^T)^{\Delta}(s) = -(y^T)^{\sigma}(s) A(s) - \int_{\sigma(s)}^{t} (y^T)^{\sigma}(u) K(u,s) \Delta u. \tag{2.15}$$

The solution satisfying the condition $y^T(t) = y_0^T$ is the transpose of (2.14), namely,

$$y^T(s) = y_0^T Z_1^T(t,s), \tag{2.16}$$

where

$$R(t,s) := Z_1^T(t,s).$$

Consequently, $R(t,s)$ is the principal matrix solution of the transposed equation. As a result, Lemma 18 from [1], has the following adjoint variant.

Theorem 2.6 *The solution of (2.15) on $[t_0, t]$ satisfying the condition $y^T(t) = y_0^T$ is*

$$y^T(s) = y_0^T R(t,s), \tag{2.17}$$

where $R(t,s)$ is the principal matrix solution of (2.15).

It follows from (2.13) that $R(t,s)$ is the unique matrix solution of (2.1).

The principal matrix $Z(t,s)$ ([1, Theorem 12]) and the solution of the adjoint equation (2.15) are related via the expression

$$\Delta_u[y^T(u)Z(u,s)] = (y^T)^\Delta(u)Z(u,s) + (y^T)^\sigma(u)\Delta_u Z(u,s) \tag{2.18}$$

for $t_0 \le s \le u \le t$.

Theorem 2.7 $R(t,s) \equiv Z(t,s)$.

Proof. Select any $t > t_0$. For a given row n-vector, let $y^T(s)$ be the unique solution of (2.15) on $[t_0,t]$ such that $y^T(t) = y_0^T$. Integrating (2.18) from s to t we have

$$y^T(t)Z(t,s) - y^T(s)Z(s,s) = \int_s^t \left[(y^T)^\Delta(u)Z(u,s) + (y^T)^\sigma(u)\Delta_u Z(u,s)\right]\Delta u.$$

By the use of (2.15), we obtain

$$\begin{aligned} y^T(t)Z(t,s) - y^T(s) &= \int_s^t \Big[(y^T)^\sigma(u)\Delta_u Z(u,s) - (y^T)^\sigma(u)A(u)Z(u,s) \\ &\quad - \left(\int_{\sigma(u)}^t (y^T)^\sigma(v)K(v,u)\Delta v\right) Z(u,s)\Big]\Delta u. \end{aligned}$$

Interchanging the order of integration by using Theorem 1.8, we obtain

$$\begin{aligned} y_0^T Z(t,s) - y^T(s) &= \int_s^t (y^T)^\sigma(u)\Big[\Delta_u Z(u,s) - A(u)Z(u,s) \\ &\quad - \int_s^u K(u,v)Z(v,s)\Delta v\Big]\Delta u. \end{aligned}$$

By [1, Theorem 19], the integrand is zero. Hence,

$$y^T(s) = y_0^T Z(t,s).$$

On the other hand,

$$y^T(s) = y_0^T R(t,s).$$

Therefore, by uniqueness of the solution $y^T(s)$,

$$y_0^T Z(t,s) = y_0^T R(t,s). \tag{2.19}$$

Now let y_0^T be the transpose of the i-th basis vector e^i. Then (2.19) implies that the i-th column of $R(t,s)$ and $Z(t,s)$ are equal for $t_0 \le s \le t$. The conclusion follows as t is arbitrary. ■

The continuous version ($\mathbb{T} = \mathbb{R}$) of the Theorem 2.6 can be found in [3, Theorem 2.7].

Now we are generalizing the idea of resolvent to discuss the asymptotic stability of (1.1) in the next section.

3 Asymptotic stability

Our first result in this section, is to present an equivalent system which involves an arbitrary function. A proper choice of this function has the potential to supply us a stable matrix B corresponding to A.

Theorem 3.1 *Let $L(t,s)$ be an $n\times n$ continuously differentiable matrix function on Ω. Then (1.1) is equivalent to the following system*

$$\begin{cases} y^{\Delta}(t) = B(t)y(t) + \int_{t_0}^{t} G(t,s)y(s)\Delta s + H(t), \quad t \in \mathbb{T}_0, \\ y(t_0) = y_0, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} B(t) &= A(t) - L(t,t), \\ H(t) &= F(t) + L(t,t_0)x_0 + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} G(t,s) = K(t,s) &+ \Delta_s L(t,s) + L(t,\sigma(s))A(s) \\ &+ \int_{\sigma(s)}^{t} L(t,\sigma(\tau))K(\tau,s)\Delta\tau. \end{aligned} \tag{3.3}$$

Proof. Let $x(t)$ be any solution of (1.1) on $\mathbb{T}_0$. If we take $p(s) = L(t,s)x(s)$, then we have

$$p^{\Delta}(s) = \Delta_s L(t,s)x(s) + L(t,\sigma(s))x^{\Delta}(s)$$

and by (1.1) it follows

$$p^{\Delta}(s) \quad = \quad \Delta_s L(t,s)x(s) + L(t,\sigma(s))A(s)x(s)$$

$$+L(t,\sigma(s))\int_{t_0}^{s} K(s,\tau)x(\tau)\Delta\tau + L(t,\sigma(s))F(s).$$

Integrating from t_0 to t we have

$$\begin{aligned} p(t)-p(t_0) &= \int_{t_0}^{t} \Delta_s L(t,s)x(s)\Delta s + \int_{t_0}^{t} L(t,\sigma(s))A(s)x(s)\Delta s \\ &\quad + \int_{t_0}^{t} L(t,\sigma(s))\int_{t_0}^{s} K(s,\tau)x(\tau)\Delta\tau\Delta s \\ &\quad + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s. \end{aligned}$$

Using Theorem 1.8, we obtain

$$\begin{aligned} p(t)-p(t_0) &= \int_{t_0}^{t} \Delta_s L(t,s)x(s)\Delta s + \int_{t_0}^{t} L(t,\sigma(s))A(s)x(s)\Delta s \\ &\quad + \int_{t_0}^{t} \left(\int_{\sigma(\tau)}^{t} L(t,\sigma(s))K(s,\tau)\Delta s\right) x(\tau)\Delta\tau \\ &\quad + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s. \end{aligned}$$

By change of variable, it follows

$$\begin{aligned} p(t)-p(t_0) &= \int_{t_0}^{t} \Bigg[\Delta_s L(t,s) + L(t,\sigma(s))A(s) \\ &\quad + \int_{\sigma(s)}^{t} L(t,\sigma(u))K(u,s)\Delta u\Bigg] x(s)\Delta s + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s. \end{aligned}$$

Further on, using (3.2) and (3.3), we obtain

$$(A(t)-B(t))x(t) = \int_{t_0}^{t} (G(t,s)-K(t,s))x(s)\Delta s + H(t) - F(t).$$

From (1.1) we have

$$x^{\Delta}(t) = B(t)x(t) + \int_{t_0}^{t} G(t,s)x(s)\Delta s + H(t),$$

for $t_0 \le s \le t < \infty$. Hence, $x(t)$ is a solution of (3.1).

Conversely, let $y(t)$ be any solution of (3.1) on $\mathbb{T}_0$. We shall show that it satisfies (1.1). Consider

$$Z(t) = y^\Delta(t) - F(t) - A(t)y(t) - \int_{t_0}^{t} K(t,s)y(s)\Delta s.$$

Then by (3.1) and (3.2) we have

$$\begin{aligned} Z(t) &= -L(t,t)y(t) + L(t,t_0)x_0 + \int_{t_0}^{t} G(t,s)y(s)\Delta s \\ &\quad + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s - \int_{t_0}^{t} K(t,s)y(s)\Delta s. \end{aligned}$$

Using (3.3), we obtain

$$\begin{aligned} Z(t) &= -L(t,t)y(t) + L(t,t_0)x_0 + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s \\ &\quad - \int_{t_0}^{t} K(t,s)y(s)\Delta s + \int_{t_0}^{t} \Bigg[K(t,s) + \Delta_s L(t,s) + L(t,\sigma(s))A(s) \\ &\quad + \int_{\sigma(s)}^{t} L(t,\sigma(\tau))K(\tau,s)\Delta\tau \Bigg] y(s)\Delta s. \end{aligned}$$

Again by Theorem 1.8 and change of variable, it follows

$$\begin{aligned} Z(t) &= -L(t,t)y(t) + \int_{t_0}^{t} \left[\Delta_s L(t,s) + L(t,\sigma(s))A(s)\right] y(s)\Delta s \\ &\quad + \int_{t_0}^{t} L(t,\sigma(s)) \left[\int_{t_0}^{s} K(s,\tau)y(\tau)\Delta\tau \right] \Delta s \\ &\quad + L(t,t_0)x_0 + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s. \end{aligned} \tag{3.4}$$

Now, by setting $q(s) = L(t,s)y(s)$, we get

$$q^\Delta(s) = \Delta_s L(t,s)y(s) + L(t,\sigma(s))y^\Delta(s). \tag{3.5}$$

Integrating (3.5) from t_0 to t, it becomes

$$q(t) - q(t_0) = \int_{t_0}^{t} \left[\Delta_s L(t,s)y(s) + L(t,\sigma(s))y^\Delta(s)\right] \Delta s$$

and therefore, we have

$$L(t,t)y(t) - L(t,t_0)x_0 = \int_{t_0}^{t} \left[\Delta_s L(t,s)y(s) + L(t,\sigma(s))y^{\Delta}(s)\right] \Delta s. \tag{3.6}$$

Substituting (3.6) in (3.4) we obtain

$$\begin{aligned} Z(t) &= -\int_{t_0}^{t} L(t,\sigma(s))y^{\Delta}(s)\Delta s + \int_{t_0}^{t} L(t,\sigma(s))A(s)y(s)\Delta s \\ &\quad + \int_{t_0}^{t} L(t,\sigma(s)) \left[\int_{t_0}^{s} K(s,\tau)y(\tau)\Delta\tau\right] \Delta s + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s \\ &= -\int_{t_0}^{t} L(t,\sigma(s))Z(s)\Delta s, \end{aligned}$$

which implies $Z(t) \equiv 0$, by the uniqueness of the solution of Volterra integral equations [16]. Hence $y(t)$ is a solution of (1.1). ■

As a straightforward consequence of Theorem 3.1 we obtain Lemma 2.1 of [24]. Also, it is to be noted that, if $L(t,s)$ is the differentiable resolvent corresponding to the kernel $K(t,s)$, then the equations (3.1), (3.2) and (3.3) give the usual variation of constants formula (2.4).

Corollary 3.2 *Let $L(t,s)$ be a $n \times n$ matrix function on $\Omega_{(q,h)}$. Then (1.2) is equivalent to the following system*

$$\begin{cases} \Delta_{(q,h)} y(t) = B(t)y(t) + \sum\limits_{s\in[t_0,t)} G(t,s)y(s)\mu(s) + H(t), \\ y(t_0) = y_0, \end{cases} \tag{3.7}$$

where

$$\begin{aligned} B(t) &= A(t) - L(t,t), \\ H(t) &= F(t) + L(t,t_0)x_0 + \sum_{s\in[t_0,t)} L(t,\sigma(s))F(s)\mu(s) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} G(t,s) &= K(t,s) + L(t,\sigma(s))A(n) + \frac{L(t,\sigma(s)) - L(t,s)}{\mu(s)} \\ &\quad + \sum_{s\in[\sigma(s),t)} L(t,\sigma(\tau))K(\tau,s)\sigma(\tau). \end{aligned}$$

Our next result is about the estimation of the solution of (1.1). For this result we assume that matrix B commutes with its integral, so B commutes with its matrix exponential, that is, $B(t)e_B(t,s) = e_B(t,s)B(t)$, [11, 12].

Theorem 3.3 *Let $B \in C(\mathbb{T}, M_n(\mathbb{R}))$ and M, $\alpha > 0$. Assume that matrix B commutes with its integral. If*

$$\|e_B(t,s)\| \leq Me_\alpha(s,t), t, s \in \Omega, \tag{3.9}$$

then every solution $x(t)$ of (1.1) satisfies

$$\begin{aligned} \|x(t)\| &\leq M\,\|x_0\|\,e_\alpha(t_0,t) + M\int_{t_0}^{t} e_\alpha(\sigma(s),t)\,\|H(s)\|\,\Delta s \\ &+M\int_{t_0}^{t}\left[\int_{\sigma(s)}^{t} e_\alpha(\sigma(\tau),t)\,\|G(\tau,s)\|\,\Delta\tau\right]\|x(s)\|\,\Delta s. \end{aligned} \tag{3.10}$$

Proof. Let $x(t)$ be the solution of (3.1) and define $q(t) = e_B(t_0,t)x(t)$. Then

$$q^\Delta(t) = -B(t)e_B(t_0,\sigma(t))x(t) + e_B(t_0,\sigma(t))x^\Delta(t).$$

Substituting for $x^\Delta(t)$ from (3.1) and integrating from t_0 to t, we obtain

$$q(t)-q(t_0) = \int_{t_0}^{t} e_B(t_0,\sigma(s))H(s)\Delta s + \int_{t_0}^{t} e_B(t_0,\sigma(s))\left[\int_{t_0}^{s} G(s,\tau)x(\tau)\Delta\tau\right]\Delta s.$$

Using Theorem 1.8, we obtain

$$\begin{aligned} x(t) &= e_B(t,t_0)x_0 + \int_{t_0}^{t} e_B(t,\sigma(s))H(s)\Delta s \\ &+ \int_{t_0}^{t}\left[\int_{\sigma(s)}^{t} e_B(t,\sigma(\tau))G(\tau,s)\Delta\tau\right]x(s)\Delta s. \end{aligned} \tag{3.11}$$

Hence, using (3.9) and applying norm on (3.11), we obtain (3.10), which completes the proof. ■

The continuous version ($\mathbb{T} = \mathbb{R}$) of the Theorem 3.3 can be found in [24, Lemma 2.3].

Corollary 3.4 *Let $B : \mathbb{T}^r_{(q,h)} \to M_n(\mathbb{R})$ and M, $\alpha > 0$. If*

$$\left\|\prod_{r\in[s,t)} (I + \mu(r)B)\right\| \leq \prod_{r\in[s,t)} \frac{M}{(1+\mu(r)\alpha)} \tag{3.12}$$

then every solution x of (1.2) satisfies

$$\begin{aligned}\|x(t)\| &\leq \prod_{r\in[t_0,t)} \frac{M\|x_0\|}{(1+\mu(r)\alpha)} + M \sum_{s\in[t_0,t)} \prod_{r\in[\sigma(s),t)} \frac{\|H(t)\|\,\mu(s)}{(1+\mu(r)\alpha)} \\ &+M \sum_{s\in[t_0,t)} \left[\sum_{\tau\in[\sigma(s),t)} \prod_{r\in[\sigma(\tau),t)} \frac{\|G(\tau,s)\|\,\mu(\tau)}{(1+\mu(r)\alpha)} \right] \|x(s)\|\,\mu(s).\end{aligned} \tag{3.13}$$

In the next theorem we present sufficient conditions for asymptotic stability.

Theorem 3.5 *Let $L(t,s)$ be an $n\times n$ continuously differentiable matrix function on Ω, such that*

(*a*) *the assumptions of Theorem 3.3 holds,*

(*b*) $\|L(t,s)\| \leq \dfrac{L_0 e_\gamma(s,t)}{(1+\mu(t)\alpha)(1+\mu(t)\gamma)}$,

(*c*) $\displaystyle\sup_{t_0\leq s\leq t<\infty} \int_{\sigma(s)}^t e_\alpha(\sigma(\tau),t)\,\|G(\tau,s)\|\,\Delta\tau \leq \alpha_0$,

(*d*) $F(t)\equiv 0$,

where L_0, $\gamma>\alpha$, α_0, are positive real constants.
If $\alpha\ominus M\alpha_0>0$, then every solution $x(t)$ of (1.1) tends to zero exponentially as $t\to+\infty$.

Proof. In view of Theorem 3.1 and the fact that $L(t,s)$ satisfies (*a*), it is enough to show that every solution of (3.1) tends to zero as $t\to+\infty$. From (*a*) and using (3.10) we obtain the following inequality

$$\begin{aligned}e_\alpha(t,0)\,\|x(t)\| &\leq M\|x_0\|\,e_\alpha(t_0,0) + M\int_{t_0}^t e_\alpha(\sigma(s),0)\,\|H(s)\|\,\Delta s \\ &+M\int_{t_0}^t \left[\int_{\sigma(s)}^t e_\alpha(\sigma(\tau),0)\,\|G(\tau,s)\|\,\Delta\tau\right]\|x(s)\|\,\Delta s.\end{aligned} \tag{3.14}$$

Since

$$\int_{t_0}^t e_\alpha(\sigma(s),0)\,\|H(s)\|\,\Delta s \leq L_0\|x_0\|\,e_\gamma(t_0,0)\int_{t_0}^t \frac{e_\alpha(\sigma(s),0)e_\gamma(0,s)}{(1+\mu(s)\alpha)(1+\mu(s)\gamma)}\Delta s,$$

then by Lemma 1.5 and the fact that $\gamma > \alpha$, we obtain

$$\int_{t_0}^{t} e_\alpha(\sigma(s), 0) \left\| H(s) \right\| \Delta s \le \frac{L_0 \left\| x_0 \right\| e_\alpha(t_0, 0)}{\gamma - \alpha}.$$

Using (3.14), (b), (c) and (d) we have

$$\begin{aligned} e_\alpha(t, 0) \left\| x(t) \right\| \;\; &\le \;\; M \left\| x_0 \right\| e_\alpha(t_0, 0) + M L_0 \left\| x_0 \right\| \frac{e_\alpha(t_0, 0)}{\gamma - \alpha} \\ &\quad + M \int_{t_0}^{t} \alpha_0 e_\alpha(s, 0) \left\| x(s) \right\| \Delta s, \end{aligned}$$

which implies

$$\begin{aligned} e_\alpha(t, 0) \left\| x(t) \right\| \;\; &\le M \left\| x_0 \right\| \Big(1 + \frac{L_0}{\gamma - \alpha}\Big) e_\alpha(t_0, 0) \\ &\quad + M \int_{t_0}^{t} \alpha_0 e_\alpha(s, 0) \left\| x(s) \right\| \Delta s. \end{aligned} \tag{3.15}$$

Lemma 1.7 yields that

$$e_\alpha(t, 0) \left\| x(t) \right\| \le M \left\| x_0 \right\| \Big(1 + \frac{L_0}{\gamma - \alpha}\Big) e_\alpha(t_0, 0) e_{M\alpha_0}(t, t_0).$$

Using [6, Theorem 2.36], we obtain

$$\left\| x(t) \right\| \le M \left\| x_0 \right\| \Big(1 + \frac{L_0}{\gamma - \alpha}\Big) e_{\alpha \ominus M\alpha_0}(t_0, 0) e_{\alpha \ominus M\alpha_0}(0, t).$$

By Lemma 1.6 we have $e_{\alpha \ominus M\alpha_0}(0, t) \le \dfrac{1}{1 + (\alpha \ominus M\alpha_0)t}$, so we obtain

$$\left\| x(t) \right\| \le \frac{M \left\| x_0 \right\| \Big(1 + \frac{L_0}{\gamma - \alpha}\Big) e_{\alpha \ominus M\alpha_0}(t_0, 0)}{1 + (\alpha \ominus M\alpha_0)t}.$$

Hence, in view of $\alpha \ominus M\alpha_0 > 0$, we obtain the required result. ∎

Theorem 3.5 generalizes the continuous version ($\mathbb{T} = \mathbb{R}$) of [24, Theorem 2.5].

Corollary 3.6 *Let $L(t, s)$ be a $n \times n$ matrix function on $\Omega_{(q,h)}$, such that* (*a*) *all the assumptions of Corollary 3.4 holds,*

(*b*) $\|L(t,s)\| \leq \prod_{r\in[s,t)} \frac{L_0}{(1+\alpha\mu(t))(1+\gamma\mu(t))(1+\mu(r)\gamma)}$,

(*c*) $\sup_{t_0\leq s\leq t<\infty} \sum_{\tau\in[\sigma(s),t)} \prod_{r\in[\sigma(\tau),t)} (1+\mu(r)\alpha)\,\|G(\tau,s)\|\,\mu(\tau) \leq \alpha_0$,

(*d*) $F(n) \equiv 0$,

where L_0, $\gamma > \alpha$, α_0, *are positive real constants. If* $\alpha \ominus M\alpha_0 > 0$, *then every solution* $x(t)$ *of (1.2) tends to zero exponentially as* $t \to +\infty$.

Example 3.7 *Let us consider the following Volterra integro-dynamic equation*

$$\begin{cases} x^{\Delta}(t) = \ominus 2x(t) + \displaystyle\int_0^t e_{\ominus 2}(t,s)x(s)\Delta s, \\ x(0) = 1, \end{cases} \tag{3.16}$$

where $A(t) = \ominus 2$ *and* $K(t,s) = e_{\ominus 2}(t,s)$. *Now consider* $L(t,s) = 0$ *then* $B(t) = \ominus 2$. *The matrix function* $G(t,s)$ *given in (3.3) becomes*

$$G(t,s) = e_{\ominus 2}(t,s). \tag{3.17}$$

In the following we have to check that the assumptions of Theorem 3.5 hold for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ respectively.

Let $\mathbb{T} = \mathbb{R}$. Then we have

$$|e_B(t,s)| = |e_{-2}(t,s)| = e^{2(s-t)} \leq Me^{2(s-t)}, \;\; M = 2,$$

and

$$0 = |L(t,s)| < L_0 e^{3(s-t)}, \;\; L_0 = 1.$$

Here the constants are $\alpha = 2$ and $\gamma = 3$. From (3.17) it follows that

$$G(t,s) = e^{-2(t-s)}. \tag{3.18}$$

Then from (3.18), we obtain that $G(t,s)$ is a positive function, and

$$\int_s^t e^{2(\tau-t)}\,|G(\tau,s)|\,d\tau \;\; = \;\; \int_s^t e^{2(\tau-t)} e^{-2(\tau-s)} d\tau$$

$$
\begin{aligned}
&= e^{2(s-t)}(t-s) \\
&\le \frac{(t-s)}{1+2(t-s)} \\
&< \frac{1}{2},
\end{aligned}
$$

from which it follows that

$$\sup_{0\le s\le t<\infty} \int_s^t e^{\frac{1}{2}(\tau-t)} |G(\tau,s)|\, d\tau \le \frac{1}{2}.$$

Since $\alpha_0 = \dfrac{1}{2}$, then we have that $\alpha - M\alpha_0 > 0$. Therefore, since all the assumptions of Theorem 3.5 hold for the system (3.16), it follows that the solution of (3.16) tends to zero exponentially as $t \to +\infty$.

Now we consider $\mathbb{T} = \mathbb{N}$. Then we have

$$|e_B(t,s)| = \left|e_{-\frac{2}{3}}(t,s)\right| = \left(\frac{1}{3}\right)^{t-s} \le M\,(3)^{s-t}, \quad M = 2,$$

$$0 = |L(t,s)| < \frac{L_0\,(4)^{s-t}}{8}, \quad L_0 = 1.$$

Here the constants are $\alpha = 2$ and $\gamma = 3$. From (3.17) it follows that

$$G(t,s) = \left(\frac{1}{3}\right)^{t-s}.$$

Now we have to calculate

$$
\begin{aligned}
\sum_{\tau\in[s+1,t)} 3^{\tau+1-t} |G(\tau,s)| &= \sum_{\tau\in[s+1,t)} 3^{\tau+1-t} \left(\frac{1}{3}\right)^{\tau-s} \\
&= 3^{s-t+1}(t-s-2) < 3^{s-t+1}(t-s-1) \\
&< \frac{1}{2},
\end{aligned}
$$

from which it follows that

$$\sup_{0\le s\le t<\infty} \sum_{\tau\in[s+1,t)} 2^{\tau+1-t} |G(\tau,s)|\, d\tau \le \frac{1}{2}.$$

Since $\alpha_0 = \dfrac{1}{2}$, then we that $\dfrac{\alpha - M\alpha_0}{1+M\alpha_0} > 0$. Therefore, since all the assumptions of Theorem 3.5 hold for the system(3.16), it follows that the solution of (3.16) tends to zero exponentially as $t \to +\infty$.

Theorem 3.8 *Let $L \in C(\Omega, M_n(\mathbb{R}))$ such that $\Delta_s L(t,s) \in C(\Omega, M_n(\mathbb{R}))$ for $(t,s) \in \Omega$ and*

(i) *the assumptions (a), (b) and (d) of Theorem 3.5 hold,*

(ii) $\|\Delta_s L(t,s)\| \le N_0 e_\delta(s,t)$ *and* $\|K(t,s)\| \le K_0 e_\theta(s,t)$,

(iii) $\|A(t)\| \le A_0$ *for* $t_0 \le t < \infty$,

(iv) $\sup\limits_{t_0 \le s \le t < \infty} \int_{\sigma(s)}^{t} \left[(K_0 + N_0)(1 + \mu(\tau)\alpha) + \dfrac{A_0 L_0 + (\tau - \sigma(s)) L_0 K_0}{\mu(\tau)\alpha} \right] \Delta\tau \le \alpha_0^\star$, *for some* $\alpha_0^\star > 0$,

where A_0, N_0, K_0, δ and θ are positive real numbers such that $\gamma > \delta > \alpha$, $\theta > \alpha$. If $\alpha \ominus M\alpha_0^\star > 0$, then every solution $x(t)$ of (1.1) tends to zero exponentially as $t \to +\infty$.

Proof. From (3.3) we obtain

$$\begin{aligned} \|G(t,s)\| &\le \|K(t,s)\| + \|\Delta_s L(t,s)\| + \|L(t,\sigma(s))\| \, \|A(s)\| \\ &\quad + \int_{\sigma(s)}^{t} \|L(t,\sigma(u))\| \, \|K(u,s)\| \, \Delta u, \end{aligned}$$

which implies

$$\begin{aligned} \|G(t,s)\| &\le K_0 e_\theta(s,t) + N_0 e_\delta(s,t) + \frac{L_0 e_\gamma(s,t)}{(1+\mu(t)\alpha)(1+\mu(t)\gamma)} A_0 \\ &\quad + \int_{\sigma(s)}^{t} \frac{L_0 K_0 e_\gamma(u,t) e_\theta(s,u)}{(1+\mu(t)\alpha)(1+\mu(t)\gamma)} \Delta u. \end{aligned} \tag{3.19}$$

Since $\lambda > \delta > \alpha$, $\theta > \alpha$, then from (i), (ii) and (iii), (3.19) becomes

$$\begin{aligned} \|G(t,s)\| &\le K_0 e_\alpha(s,t) + N_0 e_\alpha(s,t) \\ &\quad + \frac{L_0 e_\alpha(s,t)}{(1+\mu(t)\alpha)(1+\mu(t)\gamma)} A_0 + \frac{(\tau - \sigma(s)) L_0 K_0 e_\alpha(s,t)}{(1+\mu(t)\alpha)(1+\mu(t)\gamma)} \end{aligned} \tag{3.20}$$

and

$$e_\alpha(\sigma(t),0) \, \|G(t,s)\| \le \Big[(K_0 + N_0)(1 + \mu(\tau)\alpha)$$

$$+\frac{A_0L_0 + (\tau - \sigma(s))L_0K_0}{\mu(\tau)\alpha}\Bigg] e_\alpha(s,0).$$

Integrating the above inequality and using (iv), we obtain the following

$$\int_{\sigma(s)}^{t} e_\alpha(\sigma(\tau),0)\,\|G(\tau,s)\|\,\Delta\tau \le \alpha_0^\star e_\alpha(s,0). \tag{3.21}$$

Substituting (3.21) in (3.14) we obtain the following

$$\begin{aligned} e_\alpha(t,0)\,\|x(t)\| \;&\le\; M\,\|x_0\|\Big(1+\frac{L_0}{\gamma-\alpha}\Big)e_\alpha(t_0,0) \\ &\quad + M\int_{t_0}^{t} \alpha_0^\star e_\alpha(s,0)\,\|x(s)\|\,\Delta s. \end{aligned}$$

Lemma 1.7 yields that

$$e_\alpha(t,0)\,\|x(t)\| \le M\,\|x_0\|\Big(1+\frac{L_0}{\gamma-\alpha}\Big)e_\alpha(t_0,0)e_{M\alpha_0^\star}(t,t_0).$$

Using [6, Theorem 2.36], we obtain

$$\|x(t)\| \le M\,\|x_0\|\Big(1+\frac{L_0}{\gamma-\alpha}\Big)e_{\alpha\ominus M\alpha_0^\star}(t_0,0)e_{\alpha\ominus M\alpha_0^\star}(0,t).$$

Then by Lemma 1.6, we have

$$\|x(t)\| \le \frac{M\,\|x_0\|\Big(1+\frac{L_0}{\gamma-\alpha}\Big)e_{\alpha\ominus M\alpha_0^\star}(t_0,0)}{1+(\alpha\ominus M\alpha_0^\star)t}.$$

Hence, in view of (i) and $\alpha \ominus M\alpha_0^\star > 0$, we obtain the required result. ■

The continuous version ($\mathbb{T} = \mathbb{R}$) of the above theorem can be found in [24, Corollary 2.6].

Corollary 3.9 *Let $L(t,s)$ and $L(t,\sigma(s)) \in \Omega_{(q,h)}$, such that*

(i) *the assumptions (a), (d) of Corollary 3.6 hold,*

(ii) $\|L(t,\sigma(s))\| \le \prod_{\delta\in[s,t)} \frac{N_0}{(1+\mu(r)\delta)}$ *and* $\|K(n,m)\| \le \prod_{\theta\in[s,t)} \frac{K_0}{(1+\mu(r)\theta)}$, *(iii)* $\|B(t)\| \leq B_0$ *for* $t_0 \leq t < \infty$*, where* B_0*,* N_0*,* K_0*,* δ *and* θ *are positive real numbers such that* $\delta > \alpha$*,* $\theta > \alpha$*,*

(iv) $\sup_{t_0 \leq s \leq t < \infty} \sum_{\tau \in [\sigma(s), t)} (K_0 + N_0)(1 + \mu(\tau)\alpha)\mu(\tau) + A_0 L_0 + (\tau - \sigma(s)) L_0 K_0 \leq \alpha_0^\star$*, for some* $\alpha_0^\star > 0$*.*

If $\alpha \ominus M\alpha_0^\star > 0$*, then every solution* $x(t)$ *of (1.2) tends to zero exponentially as* $t \to +\infty$*.*

4 Boundedness

In the first result of this section, we give sufficient conditions to insure that (1.1) has bounded solutions. Our results are general and apply to (1.1) whether $A(t)$ is stable, identically zero, or completely unstable, and do not require $A(t)$ to be constant nor $K(t,s)$ to be a convolution kernel. Let $C(t)$ and $D(t,s)$ be continuous $n \times n$ matrices, $t_0 \leq s \leq t < \infty$. Let $s \in [t_0, \infty)$ and assume that $C(t)$ is an $n \times n$ regressive matrix. The unique matrix solution of initial valued problem

$$Y^\Delta = C(t)Y, \quad Y(s) = I, \tag{4.1}$$

is called the matrix exponential function (at s) and it is denoted by $e_C(t,s)$ (see [6, Definition 5.18]). Also, if $H(t,s)$ is an $n \times n$ regressive matrix, satisfying

$$\begin{cases} \Delta_t H(t,s) = C(t)H(t,s) + D(t,s), \\ H(s,s) = A(s) - C(s) \end{cases} \tag{4.2}$$

then

$$H(t,s) = e_C(t,s)[A(s) - C(s)] + \int_s^t e_C(t,\sigma(\tau))D(\tau,s)\Delta\tau. \tag{4.3}$$

Theorem 4.1 *Let* $e_C(t,s)$ *be the solution of (4.1), and suppose there are positive constants* N*,* J *and* M *such that*

(i) $\|e_C(t,t_0)\| \leq N$*,*

(ii) $\int_{t_0}^t \left\| e_C(t,s)[A(s) - C(s)] + \int_s^t e_C(t,\sigma(\tau))K(\tau,s)\Delta\tau \right\| \Delta s \leq J < 1$*,*

(iii) $\left\| \int_{t_0}^{t} e_C(t,\sigma(u))[F(u) - G(t)x(t)]\Delta u \right\| \leq M.$

Then all the solutions of (1.1) are uniformly bounded, and the zero solution of corresponding homogenous equation of (1.1) is uniformly stable with initial condition $x(t_0) = 0$.

Proof. Consider the following functional

$$V(t,x(\cdot)) = x(t) - \int_{t_0}^{t} H(t,s)x(s)\Delta s. \tag{4.4}$$

The derivative of $V(t,x(\cdot))$ along a solution $x(t) = x(t,t_0,x_0)$ of (1.1) satisfies

$$V^{\Delta}(t,x(\cdot)) = x^{\Delta}(t) - \Delta_t \int_{t_0}^{t} H(t,s)x(s)\Delta s.$$

From Theorem 1.117 of [6], we obtain

$$\begin{aligned} V^{\Delta}(t,x(\cdot)) &= x^{\Delta}(t) - H(\sigma(t),t)x(t) - \int_{t_0}^{t} \Delta_t H(t,s)x(s)\Delta s \\ &= A(t)x(t) - H(\sigma(t),t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s \\ &\quad - \int_{t_0}^{t} \Delta_t H(t,s)x(s)\Delta s + F(t) \end{aligned}$$

or

$$\begin{aligned} V^{\Delta}(t,x(\cdot)) &= [A(t) - H(\sigma(t),t)]x(t) + F(t) \\ &\quad + \int_{t_0}^{t} \Big[K(t,s) - \Delta_t H(t,s)\Big]x(s)\Delta s. \end{aligned} \tag{4.5}$$

By using (4.3) and Theorems 1.75, 5.21 of [6] we have the following expression

$$\begin{aligned} H(\sigma(t),t) &= e_C(\sigma(t),t)[A(t) - C(t)] + \int_{t}^{\sigma(t)} e_C(\sigma(t),\sigma(\tau))D(\tau,t)\Delta\tau \\ &= (I + \mu(t)C(t))e_C(t,t)[A(t) - C(t)] + \mu(t)e_C(\sigma(t),\sigma(t))D(t,t) \\ &= (I + \mu(t)C(t))[A(t) - C(t)] + \mu(t)D(t,t) \\ &= [A(t) - C(t)] + \mu(t)[C(t)A(t) - C^2(t) + D(t,t)] \end{aligned}$$

which implies that

$$H(\sigma(t),t) = [A(t) - C(t)] + G(t), \tag{4.6}$$

where $G(t) = \mu(t)[C(t)A(t) - C^2(t) + D(t,t)]$. Substituting (4.6) in (4.5) it follows that

$$V^{\Delta}(t,x(\cdot)) = C(t)x(t) - G(t)x(t) + \int_{t_0}^{t} \Big[K(t,s) - \Delta_t H(t,s)\Big] x(s)\Delta s + F(t).$$

By (4.2) and (4.4) we have

$$V^{\Delta}(t,x(\cdot)) = C(t)V(t,x(.)) + \int_{t_0}^{t} [K(t,s) - D(t,s)]x(s)\Delta s + F(t) - G(t)x(t).$$

Thus

$$V(t,x(\cdot)) = e_C(t,t_0)x_0 + \int_{t_0}^{t} e_C(t,\sigma(u))g(u,x(.))\Delta u, \tag{4.7}$$

where

$$g(t,x(\cdot)) = \int_{t_0}^{t} [K(t,s) - D(t,s)]x(s)\Delta s + F(t) - G(t)x(t).$$

Let $D(t,s) = K(t,s)$. Then by (4.3), (ii) is precisely $\int_{t_0}^{t} \|H(t,s)\| \Delta s \le J < 1$. By (4.7) and (i)-(iii),

$$\begin{aligned} |V(t,x(\cdot))| &= \left\| e_C(t,t_0)x_0 + \int_{t_0}^{t} e_C(t,\sigma(u))[F(u) - G(t)x(t)]\Delta u \right\| \\ &\le \|e_C(t,t_0)\| \|x_0\| + \left\| \int_{t_0}^{t} e_C(t,\sigma(u))[F(u) - G(t)x(t)]\Delta u \right\| \\ &\le N\|x_0\| + M. \end{aligned}$$

If $\|x_0\| < B_1$ for some constant, and if $Q = NB_1 + M$, then by (4.4) we obtain

$$\|x(t)\| - \int_{t_0}^{t} \|H(t,s)\| \|x(s)\| \Delta s \le \|V(t,x(.))\| \le Q. \tag{4.8}$$

Now, either there exists $B_2 > 0$ such that $\|x(t)\| < B_2$ for all $t \ge t_0$, and thus $x(t)$ is uniformly bounded, or there exists a monotone sequence $\{t_n\}$ tending to infinity such that $\|x(t_n)\| = \max_{t_0 \le t \le t_n} \|x(t)\|$ and $\|x(t_n)\| \to \infty$ as $t_n \to \infty$. By (ii) and (4.8) we have

$$\|x(t_n)\|\,(1-J) \le \|x(t_n)\| - \int_{t_0}^{t_n} \|H(t_n,s)\|\,\|x(s)\|\,\Delta s \le Q,$$

a contradiction. This complete the proof. ∎

It is noted that the Theorem 4.1 generalizes the continuous version ($\mathbb{T} = \mathbb{R}$) of [21, Theorem 1].

Corollary 4.2 *Suppose that there are positive constants N, J and M such that*

(i) $$\left\| \prod_{r\in[s,t)} (I+\mu(r)C(r)) \right\| \le N,$$

(ii) $$\sum_{s\in[t_0,t)} \left\| \prod_{r\in[s,t)} (I+\mu(r)C(r))[A(s)-C(s)] + \sum_{\tau\in[s,t)} \prod_{r\in[\sigma(\tau),t)} (I+\mu(r)C(r)) \right. \\ \left. \times K(\tau,s)\mu(\tau)\mu(s) \right\| \le J < 1,$$

(iii) $$\left\| \sum_{u\in[t_0,t)} \prod_{r\in[\sigma(u),t)} (I+\mu(r)C(r))[F(u)+G(t)x(t)]\mu(u) \right\| \le M.$$

Then all solutions of (1.2) are uniformly bounded, and the zero solution of corresponding homogenous equation of (1.2) with initial condition $x(t_0) = 0$ is uniformly stable.

In the second part of this section, we consider the system (1.1) with $F(t)$ bounded and suppose that

$$C(t,s) = -\int_t^\infty K(u,s)\Delta u \tag{4.9}$$

is defined and continuous on Ω. The matrix $E(t)$ on $[t_0,\infty)$ is defined by

$$E(t) = A(t) - C(\sigma(t),t). \tag{4.10}$$

Then (1.1) is equivalent to the following system

$$\begin{cases} x^{\Delta}(t) = E(t)x(t) + \Delta_t \int_{t_0}^{t} C(t,s)x(s)\Delta s + F(t),\ t \in \mathbb{T}_0, \\ x(t_0) = x_0. \end{cases} \tag{4.11}$$

Theorem 4.3 *Let $E \in C(\mathbb{T}, M_n(\mathbb{R}))$ and M, $\alpha > 0$. Assume that $E(t)$ commutes with its integral. If*

$$\|e_E(t,s)\| \leq Me_{\alpha}(s,t), \quad t, s \in \Omega, \tag{4.12}$$

then every solution $x(t)$ of (1.1) with $x(t_0) = x_0$ satisfies

$$\begin{aligned} \|x(t)\| &\leq M\|x_0\| e_{\alpha}(t_0,t) + M\int_{t_0}^{t} e_{\alpha}(\sigma(s),t)\|F(s)\|\Delta s \\ &+M\int_{t_0}^{t}\|E(u)\| e_{\alpha}(\sigma(u),t)\left[\int_{t_0}^{u}\|C(u,s)\|\|x(s)\|\Delta s\right]\Delta u \\ &+\int_{t_0}^{t}\|C(t,s)\|\|x(s)\|\Delta s. \end{aligned} \tag{4.13}$$

Proof. Let $x(t)$ be the solution of (1.1) and define $q(t) = e_E(t_0,t)x(t)$. Then

$$q^{\Delta}(t) = -E(t)e_E(t_0,\sigma(t))x(t) + e_E(t_0,\sigma(t))x^{\Delta}(t).$$

Substituting for $x^{\Delta}(t)$ from (4.11) and integrating from t_0 to t, we obtain

$$\begin{aligned} q(t) - q(t_0) &= \int_{t_0}^{t} e_E(t_0,\sigma(s))F(s)\Delta s \\ &\quad + \int_{t_0}^{t} e_E(t_0,\sigma(u))\left[\Delta_u \int_{t_0}^{u} C(u,s)x(s)\Delta s\right]\Delta u. \end{aligned}$$

Applying the integration by parts on the second term of the right hand side [6, Theorem 1.77], we obtain

$$\begin{aligned} x(t) &= e_E(t,t_0)x_0 + \int_{t_0}^{t} e_E(t,\sigma(s))H(s)\Delta s + \int_{t_0}^{t} C(t,s)x(s)\Delta s \\ &+ \int_{t_0}^{t} E(u)e_E(t,\sigma(s))\left[\int_{t_0}^{u} C(u,s)x(s)\Delta s\right]\Delta u. \end{aligned} \tag{4.14}$$

Hence, using (4.12) and applying norm on (4.14), we obtain (4.13), which completes the proof. ■

The continuous version ($\mathbb{T} = \mathbb{R}$) of the Theorem 4.3 can be found in [4, Lemma 2] with $D \equiv 1$

Corollary 4.4 *Let $E : \mathbb{T}^r_{(q,h)} \to M_n(\mathbb{R})$ and M, $\alpha > 0$. If*

$$\left\| \prod_{r\in[s,t)} (I + \mu(r)E(r)) \right\| \le \prod_{r\in[s,t)} \frac{M}{(1+\mu(r)\alpha)}$$

then every solution $x(t)$ of (1.2) satisfies

$$\begin{aligned} \|x(t)\| \;\; &\le \prod_{r\in[t_0,t)} \frac{M\,\|x_0\|}{(1+\mu(r)\alpha)} + M \sum_{s\in[t_0,t)} \prod_{r\in[\sigma(s),t)} \frac{\|F(s)\|\,\mu(s)}{(1+\mu(r)\alpha)} \\ &+ M \sum_{u\in[t_0,t)} \prod_{r\in[\sigma(u),t)} \frac{\|E(u)\|\,\mu(u)}{(1+\mu(r)\alpha)} \left[\sum_{s\in[t_0,u)} \|C(u,s)\|\,\|x(s)\|\,\mu(s) \right] \\ &+ \sum_{s\in[t_0,t)} \|C(t,s)\|\,\|x(s)\|\,\mu(s). \end{aligned}$$

Our next result concerns the boundedness of solutions of (1.1).

Theorem 4.5 *Let $x(t)$ be a solution of (1.1). If $\|E(t)\| \le d$ on $[t_0,\infty)$ for some $d > 0$, $F(t)$ is bounded and $\sup_{t_0\le t<\infty} \int_{t_0}^{t} \|C(t,s)\|\,\Delta s \le \beta$, with β sufficiently small, then $x(t)$ is bounded.*

Proof. For the given t_0 and bounded $F(t)$ there is $C_1 > 0$ with

$$M\,\|x_0\|\,e_\alpha(t_0,t) + M \sup_{t_0\le t<\infty} \int_{t_0}^{t} e_\alpha(\sigma(s),t)\,\|F(s)\|\,\Delta s < C_1. \tag{4.15}$$

Substituting (4.15) in (4.13) we obtain

$$\begin{aligned} \|x(t)\| &\le C_1 + Md \int_{t_0}^{t} e_\alpha(\sigma(u),t) \left[\int_{t_0}^{u} \|C(u,s)\|\,\|x(s)\|\,\Delta s \right] \Delta u \\ &\quad + \int_{t_0}^{t} \|C(t,s)\|\,\|x(s)\|\,\Delta s, \\ &\le C_1 + \frac{Md}{\alpha}\beta \sup_{t_0\le s<\infty} \|x(s)\| + \beta \sup_{t_0\le s<\infty} \|x(s)\| \\ &= C_1 + \beta\Big[1 + \frac{Md}{\alpha}\Big] \sup_{t_0\le s<\infty} \|x(s)\|. \end{aligned}$$

Let β be chosen so that $\beta\Big[1+\dfrac{Md}{\alpha}\Big]=m<1$. Then

$$\|x(t)\| \le C_1 + m \sup_{t_0\le s<t} \|x(s)\|\,.$$

Let $C_2>\|x_0\|$ and $C_1+mC_2<C_2$. If $\|x(t)\|$ is not bounded then there exists a first $t_1>t_0$ with $\|x(t_1)\|=C_2$. Then

$$C_2=\|x(t_1)\|\le C_1+mC_2<C_2,$$

a contradiction. This completes the proof. ■

The Theorem 4.5 generalizes the continuous version ($\mathbb{T}=\mathbb{R}$) of [5, Theorem 2.6.3].

Corollary 4.6 *Let $x(t)$ be a solution of (1.2). If $\|E(t)\|\le d$ on $[t_0,\infty)$ for some $d>0$, $F(t)$ is bounded and $\sup_{t_0\le t<\infty}\sum_{s\in[t_0,t)}\|C(t,s)\|\,\mu(s)\le\beta$, for some sufficiently small β, then $x(t)$ is bounded.*

Example 4.7 *Let us consider the following Volterra integro-dynamic equation*

$$\begin{cases} x^{\Delta}(t)=\frac{\ominus a(1+a^2)}{a^2}x(t)+\displaystyle\int_{t_0}^{t} e_{\ominus a}(\sigma(t),s)x(s)\Delta s+F(t), \\ x(t_0)=1, \end{cases} \tag{4.16}$$

where $A(t)=\frac{\ominus a(1+a^2)}{a^2}$, $K(t,s)=e_{\ominus a}(\sigma(t),s)$ with $a>2$. Assume that $F(t)$ is bounded function.

Since we have that

$$\begin{aligned}\int_t^{\infty} e_{\ominus a}(\sigma(u),s)\Delta u &= \lim_{b\to\infty} -\frac{1}{a}\int_t^b \frac{-a}{e_a(\sigma(u),s)}\Delta u\\ &= \lim_{b\to\infty} -\frac{1}{a}\int_t^b \left(\frac{1}{e_a(u,s)}\right)^{\Delta}\Delta u\\ &= \lim_{b\to\infty} -\frac{1}{a}\left[\frac{1}{e_a(b,s)}-\frac{1}{e_a(t,s)}\right]\\ &= \frac{1}{ae_a(t,s)}.\end{aligned}$$

Using (4.10), we have

$$\begin{aligned}
|E(t)| &= \left| A(t) + \int_{\sigma(t)}^{\infty} e_{\ominus a}(\sigma(u), t)\Delta u \right| \\
&= \left| \frac{\ominus a(1+a^2)}{a^2} + \frac{1}{a e_a(\sigma(t), t)} \right| \\
&= \left| \frac{-(1+a^2)}{a(1+\mu(t)a)} + \frac{1}{a(1+\mu(t)a)} \right| \\
&= |\ominus a| \le a.
\end{aligned}$$

Hence

$$|E(t)| \le a. \tag{4.17}$$

Now, we have to approximate

$$\begin{aligned}
\int_{t_0}^{t} |C(t,s)|\, \Delta s &\le \int_{t_0}^{t} \frac{1}{a e_a(t,s)} \Delta s \\
&= \frac{1}{a}\left[\frac{1}{e_a(t,t)} - \frac{1}{e_a(t,t_0)}\right] \\
&= \frac{1}{a}\left[1 - \frac{1}{e_a(t,t_0)}\right] \\
&\le \frac{1}{a},
\end{aligned}$$

therefore

$$\int_{t_0}^{t} |C(t,s)|\, \Delta s \le \frac{1}{a}, \quad t \ge t_0. \tag{4.18}$$

Finally, by taking the supremum over t in (4.18), over $[t_0, \infty)_{\mathbb{T}}$, we obtain

$$\sup_{t_0 \le t < \infty} \int_{t_0}^{t} |C(t,s)|\, \Delta s \le \frac{1}{a}.$$

Obviously, in this case $d = \alpha = a$, $M \ge 1$ and $\beta = \frac{1}{a}$. If we choose $a > M+1$, then $\beta(1 + \frac{Md}{\alpha}) < 1$. It follows that all the assumptions of Theorem 4.5 are satisfied, hence all solutions of (4.16) are bounded.

Theorem 4.8 *If $F(t) = 0$ in (1.1), $\|E(t)\| \le d$ on $[t_0, \infty)$ for some $d > 0$, and $\int_{t_0}^{t} \|C(t,s)\|\, \Delta s \le \beta$, for β sufficiently small, then the zero solution of (1.1) with initial condition $x(t_0) = 0$ is uniformly stable.*

Proof. Let $\epsilon > 0$ be given. We wish to find $\delta > 0$ such that $t_0 \geq 0$, $\|x_0\| < \delta$, and $t \geq t_0$ implies $\|x(t, x_0)\| < \epsilon$. Let $\delta < \epsilon$ with δ yet to be determined. If $\|x_0\| < \delta$, then $M \|x_0\| \leq M\delta$. From (4.13) with $F(t) = 0$,

$$\begin{aligned} \|x(t)\| &\leq M\delta + \frac{Md}{\alpha}\beta \sup_{t_0 \leq s < t} \|x(s)\| + \beta \sup_{t_0 \leq s < t} \|x(s)\| \\ &= M\delta + \beta\Big[1 + \frac{Md}{\alpha}\Big] \sup_{t_0 \leq s < t} \|x(s)\| . \end{aligned}$$

First take β so that $\beta\Big[1 + \frac{Md}{\alpha}\Big] \leq \frac{3}{4}$ and δ so that $M\delta + \frac{3}{4}\epsilon < \epsilon$. If $\|x_0\| < \delta$ and if there exists $t_1 > t_0$ with $\|x(t_1)\| = \epsilon$, we have

$$\epsilon = \|x(t_1)\| < M\delta + \frac{3}{4}\epsilon < \epsilon,$$

a contradiction. Thus the zero solution is uniformly stable. The proof is complete. ■

The continuous version ($\mathbb{T} = \mathbb{R}$) of the above theorem can be found in [5, Theorem 2.6.4].

Corollary 4.9 *If $F(t) = 0$ in (1.2), $\|E(t)\| \leq d$ on $[t_0, \infty)$ for some $d > 0$ and $\sum_{s \in [t_0, t)} \|C(t, s)\| \mu(s) \leq \beta$, sufficiently small, then the zero solution of (1.2) is uniformly stable with initial condition $x(t_0) = 0$.*

Example 4.10 *If we consider $F(t) = 0$ in Example 4.4, then by (4.17) and (4.18), Theorem 4.5 yields that the zero solution of (**??**) is uniformly stable.*

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