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OSCILLATIONS OF ADVANCED DIFFERENCE EQUATIONS WITH VARIABLE ARGUMENTS

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Abstract. Consider the first-order advanced difference equation of the form

$$\nabla x(n) - p(n)x(\mu(n)) = 0, \ \ n \geq 1, \quad [\Delta x(n) - p(n)x(\nu(n)) = 0, \ \ n \geq 0],$$

where ∇ denotes the backward difference operator $\nabla x(n) = x(n) - x(n-1)$, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$, $\{p(n)\}$ is a sequence of nonnegative real numbers, and $\{\mu(n)\}$ $[\{\nu(n)\}]$ is a sequence of positive integers such that

$$\mu(n) \geq n+1 \ \text{ for all } n \geq 1, \quad [\nu(n) \geq n+2 \ \text{ for all } n \geq 0]\,.$$

Sufficient conditions which guarantee that all solutions oscillate are established. The results obtained essentially improve known results in the literature. Examples illustrating the results are also given.

Keywords: advanced difference equation, variable argument, oscillatory solution, nonoscillatory solution.

1. INTRODUCTION

Differential and difference equations with advanced arguments describe mathematical models in which the present state depends on a future state [7, 8, 11]. Besides its theoretical interest, strong interest in the study of difference equations with advanced arguments is motivated by the fact that they arise in many areas of applied mathematics, such as population dynamics where, for example, a difference equation with constant advanced arguments may represent a mathematical model of species whose k_{th} generation depends on the present and next generations [6]. Presently, there exists an extensive literature on the oscillation theory of advanced type differential and difference equations. See, for example, [1, 5, 9,10, 12−19] and the references cited therein.

Consider the first-order linear difference equation with advanced argument of the form

$$\nabla x(n) - p(n)x(\mu(n)) = 0, \quad n \geq 1, \quad [\Delta x(n) - p(n)x(\nu(n)) = 0, \quad n \geq 0], \qquad \text{(E)}$$

where ∇ denotes the backward difference operator $\nabla x(n) = x(n) - x(n-1)$, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$, $\{p(n)\}$ is a

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sequence of nonnegative real numbers, and $\{\mu(n)\}$ $[\{\nu(n)\}]$ is a sequence of positive integers such that

$$\mu(n) \geq n+1 \quad \text{for all } n \geq 1, \qquad [\nu(n) \geq n+2 \quad \text{for all } n \geq 0]. \tag{1.1}$$

Strong interest in Eq. (E) is motivated by the fact that it represents a discrete analogue of the advanced differential equation

$$x'(t) - p(t)x(\tau(t)) = 0, \qquad t \geq t_0, \tag{E$_1$}$$

where $p, \tau \in C([t_0,\infty), \mathbb{R}^+)$, $\mathbb{R}^+ = [0,\infty)$, $\tau(t)$ is nondecreasing and $\tau(t) > t$ for $t \geq t_0$ [see, for example, 9, 10, 12, 13, 15].

By a solution of Eq. (E), we mean a sequence of real numbers $\{x(n)\}$ which is defined for $n \geq 0$ and satisfies (E) for all $n \geq 1$ $[n \geq 0]$.

As usual, a solution $\{x(n)\}$ of Eq. (E) is said to be *oscillatory* if for every positive integer n_0 there exist $n_1, n_2 \geq n_0$ such that $x(n_1)x(n_2) \leq 0$. In other words, a solution $\{x(n)\}$ is *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, the solution is called *nonoscillatory*.

The oscillatory behavior of Eq. (E) was studied for the first time by Chatzarakis and Stavroulakis in [5] where the following theorems were established:

Theorem 1.1 [5]**.** *Assume that the sequence* $\{\mu(n)\}$ $[\{\nu(n)\}]$ *is nondecreasing. If*

$$\limsup_{n\to\infty} \sum_{i=n}^{\mu(n)} p(i) \quad \left[\limsup_{n\to\infty} \sum_{i=n}^{\nu(n)-1} p(i)\right] > 1, \tag{1.2}$$

then all solutions of (E) *oscillate.*

Theorem 1.2 [5]**.** *Assume that the sequence* $\{\mu(n)\}$ $[\{\nu(n)\}]$ *is nondecreasing, and*

$$\liminf_{n\to\infty} \sum_{i=n+1}^{\mu(n)} p\,(i) \quad \left[\liminf_{n\to\infty} \sum_{i=n+1}^{\nu(n)-1} p\,(i)\right] = \alpha. \tag{1.3}$$

If $0 < \alpha \leq 1$, *and*

$$\limsup_{n\to\infty} \sum_{i=n}^{\mu(n)} p(i) \quad \left[\limsup_{n\to\infty} \sum_{i=n}^{\nu(n)-1} p(i)\right] > 1 - \left(1 - \sqrt{1-\alpha}\right)^2, \tag{1.4}$$

then all solutions of (E) *oscillate.*

If $0 < \alpha < (3\sqrt{5}-5)/2$,

$$p(n) \geq 1 - \sqrt{1-\alpha} \quad \textit{for all large } n \tag{1.5}$$

and

$$\limsup_{n\to\infty} \sum_{i=n}^{\mu(n)} p(i) \quad \left[\limsup_{n\to\infty} \sum_{i=n}^{\nu(n)-1} p(i)\right] > 1 - \alpha\left(\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right), \tag{1.6}$$

then all solutions of (E) *oscillate.*

In this paper, our main objective is to improve on the upper bound of the ratio $\frac{x(n-1)}{x(\mu(n))}$ $\left[\frac{x(n-1)}{x(\nu(n))}\right]$ for possible nonoscillatory solutions $\{x(n)\}$ of Eq. (E) and derive new oscillation conditions for all solutions of Eq. (E), when the oscillation condition (1.2) is not satisfied.

2. OSCILLATION CRITERIA

In this section, at first, a new lemma is presented, which will be used in the proof of our main theorem.

Lemma 2.1. *Assume that the sequence* $\{\mu(n)\}$ $[\{\nu(n)\}]$ *is nondecreasing,*

$$\liminf_{n\to\infty} \sum_{i=n+1}^{\mu(n)} p(i) \quad \left[\liminf_{n\to\infty} \sum_{i=n+1}^{\nu(n)-1} p(i)\right] = \alpha \tag{1.3}$$

and $\{x(n)\}$ *is a nonoscillatory solution of* (E).

If $0 < \alpha \leq 1/2$, *then*

$$\liminf_{n\to\infty} \frac{x(n-1)}{x(\mu(n))} \quad \left[\liminf_{n\to\infty} \frac{x(n-1)}{x(\nu(n))}\right] \geq \frac{1}{2}(1-\alpha-\sqrt{1-2\alpha}). \tag{2.1}$$

If $0 < \alpha \leq 6-4\sqrt{2}$, *and*

$$p(n) \geq \frac{\alpha}{2} \quad \textit{for all large } n, \tag{2.2}$$

then

$$\liminf_{n\to\infty} \frac{x(n-1)}{x(\mu(n))} \quad \left[\liminf_{n\to\infty} \frac{x(n-1)}{x(\nu(n))}\right] \geq \frac{1}{4}(2-3\alpha-\sqrt{4-12\alpha+\alpha^2}). \tag{2.3}$$

Proof. The proof below refers to Eq. (E) with the backward difference operator. The proof for (E) with the forward difference operator follows by a similar procedure.

Assume that $\{x(n)\}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $\{-x(n)\}$ is also a solution of (E), we can restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq 0$ be an integer such that $x(n-1) > 0$, $\forall\ n \geq n_1$. Then

$$x(n) > 0,\ x(\mu(n)) > 0 \qquad \forall\ n \geq n_1.$$

Thus, from (E), we have

$$\nabla x(n) = p(n)x(\mu(n)) \geq 0,$$

which means that the sequence $\{x(n)\}$ is eventually nondecreasing.

Consider an arbitrary real number ε with $0 < \varepsilon < \alpha$. Then, in view of (1.3), we can choose an integer $n_0 > n_1$ such that

$$\sum_{i=n+1}^{\mu(n)} p(i) \geq \alpha - \varepsilon \qquad \forall n \geq n_0.$$

Thus

$$\sum_{i=n+1}^{\mu(n)} p\left(i\right) \geq \alpha - \varepsilon \qquad \forall n >> n_0. \tag{2.4}$$

Now, we will show that for each $n >> n_0$, there exists an integer n^* with $n_0 <<$ $n^* \leq n$ such that $\mu(n^*) \geq n+1$, and

$$\sum_{i=n^*+1}^{n} p\left(i\right) < \frac{\alpha - \varepsilon}{2} \qquad \text{and} \qquad \sum_{i=n^*}^{n} p\left(i\right) \geq \frac{\alpha - \varepsilon}{2}. \tag{2.5}$$

Indeed, (2.4) guarantees that

$$\sum_{i=0}^{\infty} p(i) = \infty.$$

In particular, it holds

$$\sum_{i=n^*+1}^{\infty} p(i) = \infty.$$

If $p(n) < (\alpha - \varepsilon)/2$, there always exists an integer n with $n_0 << n^* < n$ so that

$$\sum_{i=n^*+1}^{n} p\left(i\right) < \frac{\alpha - \varepsilon}{2} \qquad \text{and} \qquad \sum_{i=n^*}^{n} p\left(i\right) \geq \frac{\alpha - \varepsilon}{2}.$$

If $p(n) \geq (\alpha - \varepsilon)/2$, then $n = n^* >> n_0$ so that

$$\sum_{i=n^*+1}^{n} p\left(i\right) = \sum_{i=n+1}^{n} p\left(i\right) \text{ (by which we mean) } = 0 < \frac{\alpha - \varepsilon}{2}$$

and

$$\sum_{i=n^*}^{n} p\left(i\right) = \sum_{i=n}^{n} p\left(i\right) = p(n) \geq \frac{\alpha - \varepsilon}{2}.$$

That is, in both cases (2.5) is satisfied.

We will show that $\mu(n^*) \geq n+1$. Indeed, in the case where $p(n) \geq (\alpha - \varepsilon)/2$, since $n^* = n$, we have $\mu(n^*) = \mu(n) \geq n+1$. In the case where $p(n) < (\alpha - \varepsilon)/2$, then $n^* < n$. Assume, for the sake of contradiction, that $\mu(n^*) < n+1$. Then $\mu(n^*) \leq n$, and therefore

$$\sum_{i=n^*+1}^{\mu(n^*)} p\left(i\right) \leq \sum_{i=n^*+1}^{n} p\left(i\right) < \frac{\alpha - \varepsilon}{2}. \tag{2.6}$$

On the other hand, in view of (2.4), we have

$$\sum_{i=n^*+1}^{\mu(n^*)} p\left(i\right) \geq \alpha - \varepsilon > \frac{\alpha - \varepsilon}{2},$$

which contradicts (2.6). Thus, in both cases, we have $\mu(n^*) \geq n+1$.

Furthermore, combining inequalities (2.4) and (2.5), we get

$$\sum_{i=n+1}^{\mu(n^*)} p\left(i\right) = \sum_{i=n^*+1}^{\mu(n^*)} p\left(i\right) - \sum_{i=n^*+1}^{n} p\left(i\right) > (\alpha - \varepsilon) - \frac{\alpha - \varepsilon}{2} = \frac{\alpha - \varepsilon}{2}. \tag{2.7}$$

Summing up (E) from n^* to n, and using the fact that the functions x and μ are nondecreasing, we have

$$x(n) = x(n^* - 1) + \sum_{i=n^*}^{n} p\,(i)\, x(\mu(i)) \geq x(n^* - 1) + \left(\sum_{i=n^*}^{n} p\,(i) \right) x(\mu(n^*))$$

which, in view of (2.5), gives

$$x(n) \geq x(n^* - 1) + \frac{\alpha - \varepsilon}{2} x(\mu(n^*)). \tag{2.8}$$

Similarly, summimg up (E) from $n+1$ to $\mu(n^*)$, we get

$$x(\mu(n^*)) = x(n) + \sum_{i=n+1}^{\mu(n^*)} p\,(i)\, x(\mu(i)) \geq x(n) + \left(\sum_{i=n+1}^{\mu(n^*)} p\,(i) \right) x(\mu(n+1))$$

which, in view of (2.7), gives

$$x(\mu(n^*)) > x(n) + \frac{\alpha - \varepsilon}{2} x(\mu(n+1)). \tag{2.9}$$

Combining inequalities (2.8) and (2.9), we obtain

$$x(n) > x(n^* - 1) + \frac{\alpha - \varepsilon}{2} \left[x(n) + \frac{\alpha - \varepsilon}{2} x(\mu(n+1)) \right]$$

i.e.

$$x(n) > \frac{\left(\frac{\alpha-\varepsilon}{2}\right)^2}{1 - \frac{\alpha-\varepsilon}{2}} x(\mu(n+1)) = \ell_1 x(\mu(n+1)) \qquad \forall n >> n_0, \tag{2.10}$$

where

$$\ell_1 = \frac{\left(\frac{\alpha-\varepsilon}{2}\right)^2}{1 - \frac{\alpha-\varepsilon}{2}}.$$

Let n be an arbitrary integer with $n >> n_0$. We conclude that there exists n^* with $n_0 << n^* \leq n$ such that $\mu(n^*) \geq n+1$, and therefore (2.5) and (2.7) are satisfied. Then (2.8) and (2.9) are also fulfilled. Moreover, in view of (2.10) (for the integer $n^* - 1$), we have

$$x(n^* - 1) > \ell_1 x(\mu(n^*)). \tag{2.11}$$

Using (2.8), (2.11) and (2.9), we obtain

$$\begin{aligned} x(n) &\geq x(n^* - 1) + \frac{\alpha - \varepsilon}{2} x(\mu(n^*)) > \ell_1 x(\mu(n^*)) + \frac{\alpha - \varepsilon}{2} x(\mu(n^*)) \\ &= \left(\ell_1 + \frac{\alpha - \varepsilon}{2} \right) x(\mu(n^*)) > \left(\ell_1 + \frac{\alpha - \varepsilon}{2} \right) \left[x(n) + \frac{\alpha - \varepsilon}{2} x(\mu(n+1)) \right] \end{aligned}$$

i.e.

$$x(n) > \frac{\left(\ell_1 + \frac{\alpha-\varepsilon}{2}\right) \frac{\alpha-\varepsilon}{2}}{1 - \left(\ell_1 + \frac{\alpha-\varepsilon}{2}\right)} x(\mu(n+1)) = \ell_2 x(\mu(n+1)) \qquad \forall n >> n_0,$$

where

$$\ell_2 = \frac{\left(\ell_1 + \frac{\alpha-\varepsilon}{2}\right) \frac{\alpha-\varepsilon}{2}}{1 - \left(\ell_1 + \frac{\alpha-\varepsilon}{2}\right)}.$$

(Clearly, $\ell_2 > \ell_1$.)

Following the above procedure, we can inductively construct a recursive sequence of positive real numbers $\{\ell_m\}, m \geq 1$ with

$$\ell_{m+1} = \frac{\left(\ell_m + \frac{\alpha-\varepsilon}{2}\right)\frac{\alpha-\varepsilon}{2}}{1-\left(\ell_m + \frac{\alpha-\varepsilon}{2}\right)} \quad (m = 1, 2, ...)$$

such that

$$x(n) > \ell_m x(\mu(n+1)) \qquad \forall n >> n_0 \qquad (m = 1, 2, ...). \tag{2.12}$$

Since $\ell_2 > \ell_1$, by induction, we can show that the sequence $\{\ell_m\}$ is strictly increasing. Furthermore, by taking into account the fact that the sequence $\{x(n)\}$ is eventually nondecreasing and, in view of (2.12) we get

$$x(\mu(n+1)) \geq x(n+2) \geq x(n) > \ell_m x(\mu(n+1)) \qquad \forall n >> n_0 \qquad (m = 1, 2, ...).$$

Therefore, for each $m \geq 1$, we have $\ell_m < 1$. This ensures that the sequence $\{\ell_m\}$ is bounded. Since $\{\ell_m\}$ is a strictly increasing and bounded sequence of positive real numbers, it follows that $\lim_{m\to\infty} \ell_m$ exists as a positive real number. Set

$$L = \lim_{m\to\infty} \ell_m.$$

Then (2.12) gives

$$x(n) \geq Lx(\mu(n+1)) \qquad \forall n >> n_0. \tag{2.13}$$

By the definition of $\{\ell_m\}$, we have

$$L = \frac{\left(L + \frac{\alpha-\varepsilon}{2}\right)\frac{\alpha-\varepsilon}{2}}{1-\left(L + \frac{\alpha-\varepsilon}{2}\right)},$$

which gives

$$L = \frac{1}{2}\left[1-(\alpha-\varepsilon) \pm \sqrt{1-2(\alpha-\varepsilon)}\right].$$

In both cases, it holds

$$L \geq \frac{1}{2}\left[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)}\right].$$

Thus, from (2.13), it follows that

$$x(n) \geq \frac{1}{2}\left[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)}\right] x(\mu(n+1)) \qquad \forall n >> n_0 \tag{2.14}$$

Inequality (2.14) gives

$$x(n-1) \geq \frac{1}{2}\left[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)}\right] x(\mu(n)) \qquad \forall n >> n_0 + 1$$

or

$$\frac{x(n-1)}{x(\mu(n))} \geq \frac{1}{2}\left[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)}\right] \qquad \forall n >> n_0 + 1.$$

Consequently,

$$\liminf_{n\to\infty} \frac{x(n-1)}{x(\mu(n))} \geq \frac{1}{2}\left[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)}\right],$$

which, for arbitrarily small values of ε, implies (2.1).

For the rest of the proof, we assume that $0 < \alpha \leq 6 - 4\sqrt{2}$ (which implies that $0 < \alpha < 1/2$) and, in addition, that (2.2) holds. Because of (2.2), we can consider an integer $n_2 >> n_0$ such that $p(n) \geq \frac{\alpha}{2}$ for every $n \geq n_2$. Then

$$p(n) > \frac{\alpha - \varepsilon}{2} \qquad \forall n \geq n_2. \tag{2.15}$$

By (2.14), we have

$$x(n) \geq b_1 x(\mu(n+1)) \qquad \forall n \geq n_2, \tag{2.16}$$

where

$$b_1 = \frac{1}{2}\left[1 - (\alpha - \varepsilon) - \sqrt{1 - 2(\alpha - \varepsilon)}\right].$$

Let us consider an arbitrary integer $n \geq n_2$. By using (2.15) as well as (2.16) (for the integer $n-1$), from (E) we obtain

$$x(n) = x(n-1) + p(n)x(\mu(n)) > b_1 x(\mu(n)) + \frac{\alpha - \varepsilon}{2} x(\mu(n))$$

and consequently

$$x(n) > \left(b_1 + \frac{\alpha - \varepsilon}{2}\right) x(\mu(n)). \tag{2.17}$$

Now, summing up (E) from $n+1$ to $\mu(n)$, and using the fact that the functions x and μ are nondecreasing, we have

$$x(\mu(n)) = x(n) + \sum_{i=n+1}^{\mu(n)} p(i)x(\mu(i)) \geq x(n) + \left(\sum_{i=n+1}^{\mu(n)} p(i)\right) x(\mu(n+1))$$

which, in view of (2.4), gives

$$x(\mu(n)) \geq x(n) + (\alpha - \varepsilon)x(\mu(n+1)). \tag{2.18}$$

Combining inequalities (2.17) and (2.18), we obtain

$$x(n) > \left(b_1 + \frac{\alpha - \varepsilon}{2}\right)\left[x(n) + (\alpha - \varepsilon)x(\mu(n+1))\right],$$

i.e.

$$x(n) > \frac{\left(b_1 + \frac{\alpha-\varepsilon}{2}\right)(\alpha - \varepsilon)}{1 - \left(b_1 + \frac{\alpha-\varepsilon}{2}\right)} x(\mu(n+1)) = b_2 x(\mu(n+1)) \qquad \forall n \geq n_2,$$

where

$$b_2 = \frac{\left(b_1 + \frac{\alpha-\varepsilon}{2}\right)(\alpha - \varepsilon)}{1 - \left(b_1 + \frac{\alpha-\varepsilon}{2}\right)}.$$

(Clearly, $b_2 > b_1$.)

By the arguments applied previously, a recursive sequence of positive real numbers $\{b_m\}, m \geq 1$ can inductively constructed, which satisfies

$$b_{m+1} = \frac{\left(b_m + \frac{\alpha-\varepsilon}{2}\right)(\alpha - \varepsilon)}{1 - \left(b_m + \frac{\alpha-\varepsilon}{2}\right)} \quad (m = 1, 2, ...);$$

this sequence is such that

$$x(n) > b_m x(\mu(n+1)) \qquad \forall n \geq n_2 \qquad (m = 2, 3, ...). \tag{2.19}$$

Since $b_2 > b_1$, by induction, we can show that the sequence $\{b_m\}$ is strictly increasing. Furthermore, by taking into account the fact that x is nondecreasing and by using (for $n = n_2$) inequality (2.19), we obtain

$$x(\mu(n_2+1)) \geq x(n_2) > b_m x(\mu(n_2+1)) \quad (m = 2, 3, ...).$$

Hence, $b_m < 1$ for every $m \geq 2$, which guarantees the boundedness of the sequence $\{b_m\}$. Thus, $\lim_{m\to\infty} b_m$ exists as a positive real number. Set

$$B = \lim_{m\to\infty} b_m.$$

Then it follows from (2.19) that

$$x(n) \geq Bx(\mu(n+1)) \qquad \forall n \geq n_2. \tag{2.20}$$

In view of the definition of $\{b_m\}$, the number B satisfies

$$B = \frac{\left(B + \frac{\alpha-\varepsilon}{2}\right)(\alpha - \varepsilon)}{1 - \left(B + \frac{\alpha-\varepsilon}{2}\right)}$$

which gives

$$B = \frac{1}{4}\left[2 - 3(\alpha-\varepsilon) \pm \sqrt{4 - 12(\alpha-\varepsilon) + (\alpha-\varepsilon)^2}\right].$$

Note that, because of $0 < \alpha - \varepsilon < 6 - 4\sqrt{2}$, it holds

$$4 - 12(\alpha-\varepsilon) + (\alpha-\varepsilon)^2 > 0.$$

We always have

$$B \geq \frac{1}{4}\left[2 - 3(\alpha-\varepsilon) - \sqrt{4 - 12(\alpha-\varepsilon) + (\alpha-\varepsilon)^2}\right]$$

and consequently (2.20) gives

$$x(n) \geq \frac{1}{4}\left[2 - 3(\alpha-\varepsilon) - \sqrt{4 - 12(\alpha-\varepsilon) + (\alpha-\varepsilon)^2}\right] x(\mu(n+1)) \qquad \forall n \geq n_2.$$

Finally, we see that the last inequality can equivalently be written as follows

$$x(n-1) \geq \frac{1}{4}\left[2 - 3(\alpha-\varepsilon) - \sqrt{4 - 12(\alpha-\varepsilon) + (\alpha-\varepsilon)^2}\right] x(\mu(n)), \quad \forall n \geq n_2 + 1,$$

i.e.,

$$\frac{x(n-1)}{x(\mu(n))} \geq \frac{1}{4}\left[2 - 3(\alpha-\varepsilon) - \sqrt{4 - 12(\alpha-\varepsilon) + (\alpha-\varepsilon)^2}\right], \quad \forall n \geq n_2 + 1.$$

Therefore,

$$\liminf_{n\to\infty} \frac{x(n-1)}{x(\mu(n))} \geq \frac{1}{4}\left[2 - 3(\alpha-\varepsilon) - \sqrt{4 - 12(\alpha-\varepsilon) + (\alpha-\varepsilon)^2}\right],$$

which, for arbitrarily small values of ε, implies (2.3).

The proof of the lemma is complete.

Theorem 2.1. *Assume that the sequence* $\{\mu(n)\}$ $[\{\nu(n)\}]$ *is nondecreasing, and*

$$\liminf_{n\to\infty} \sum_{i=n+1}^{\mu(n)} p(i) \quad \left[\liminf_{n\to\infty} \sum_{i=n+1}^{\nu(n)-1} p(i)\right] = \alpha. \tag{1.3}$$

If $0 < \alpha \le 1/2$, *and*

$$\limsup_{n\to\infty} \sum_{i=n}^{\mu(n)} p(i) \left[\limsup_{n\to\infty} \sum_{i=n}^{\nu(n)-1} p(i) \right] > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1-2\alpha}\right) \tag{2.21}$$

then all solutions of (E) *oscillate.*

If $0 < \alpha \le 6 - 4\sqrt{2}$, $p(n) \ge \frac{\alpha}{2}$ *for all large* n, *and*

$$\limsup_{n\to\infty} \sum_{i=n}^{\mu(n)} p(i) \left[\limsup_{n\to\infty} \sum_{i=n}^{\nu(n)-1} p(i) \right] > 1 - \frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right), \tag{2.22}$$

then all solutions of (E) *oscillate.*

Proof. The proof below refers to Eq. (E) with the backward difference operator. The proof for (E) with the forward difference operator follows by a similar procedure.

If $\{x(n)\}$ is a nonoscillatory solution of (E), then it is either eventually positive or eventually negative. As $\{-x(n)\}$ is also a solution of (E), we can restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_0 \ge 0$ be an integer such that $x(n-1) > 0$, $\forall n \ge n_0$. Then

$$x(n) > 0,\ x(\mu(n)) > 0 \qquad \forall n \ge n_0.$$

Thus, from (E) we have

$$\nabla x(n) = p(n)x(\mu(n)) \ge 0 \qquad \forall n \ge n_0,$$

which means that the sequence $\{x(n)\}$ is eventually nondecreasing.

Now, we consider an integer $n_1 >> n_0$ such that $\mu(n) \ge n_0$ for $n \ge n_1$. Furthermore, we choose an integer $N > n_1$ so that $\mu(n) \ge n_1$ for $n \ge N$. Then, by taking into account the fact that the functions x, μ are nondecreasing, from (E) we obtain, for every $n \ge N$,

$$x(\mu(n)) = x(n-1) + \sum_{i=n}^{\mu(n)} p\,(i)\, x(\mu(i)) \ge x(n-1) + \left(\sum_{i=n}^{\mu(n)} p\,(i) \right) x(\mu(n)).$$

Consequently,

$$\sum_{i=n}^{\mu(n)} p\,(i) \le 1 - \frac{x(n-1)}{x(\mu(n))} \qquad \forall n \ge N,$$

which gives

$$\limsup_{n\to\infty} \sum_{i=n}^{\mu(n)} p\,(i) \le 1 - \liminf_{n\to\infty} \frac{x(n-1)}{x(\mu(n))}. \tag{2.23}$$

Assume, first, that $0 < \alpha \le 1/2$. Then, by Lemma 2.1, inequality (2.1) is fulfilled, and so (2.23) leads to

$$\limsup_{n\to\infty} \sum_{i=n}^{\mu(n)} p\,(i) \le 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1-2\alpha}\right),$$

which contradicts condition (2.21).

Next, let us suppose that $0 < \alpha \le 6 - 4\sqrt{2}$ and that (2.2) holds. Then Lemma 2.1 ensures that (2.3) is satisfied. Thus, from (2.23), it follows that

$$\limsup_{n\to\infty} \sum_{i=n}^{\mu(n)} p\,(i) \le 1 - \frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right),$$

which contradicts condition (2.22). The proof of the theorem is complete.

Remark 2.1. In the special case where $\mu(n) = n+k$, $\nu(n) = n+\sigma$ the advanced difference equations (E) takes the form

$$\nabla x(n) - p(n)x(n+k) = 0, \quad n \ge 1, \qquad [\Delta x(n) - p(n)x(n+\sigma) = 0, \quad n \ge 0], \tag{E$'$}$$

where k is a positive integer greater or equal to one and σ is a positive integer greater or equal to two. For this equation, from Theorem 2.1 we derive the following:

Corollary 2.1. *Assume that*

$$\liminf_{n\to\infty} \sum_{i=n+1}^{n+k} p\,(i) \quad \left[\liminf_{n\to\infty} \sum_{i=n+1}^{n+\sigma-1} p\,(i)\right] = \alpha_0. \tag{1.3$'$}$$

If $0 < \alpha_0 \le 1/2$, *and*

$$\limsup_{n\to\infty} \sum_{i=n}^{n+k} p(i) \quad \left[\limsup_{n\to\infty} \sum_{i=n}^{n+\sigma-1} p(i)\right] > 1 - \frac{1}{2}\left(1 - \alpha_0 - \sqrt{1 - 2\alpha_0}\right), \tag{2.21$'$}$$

then all solutions of (E$'$) *oscillate.*

If $0 < \alpha_0 \le 6 - 4\sqrt{2}$, $p(n) \ge \alpha_0/2$ *for all large* n, *and*

$$\limsup_{n\to\infty} \sum_{i=n}^{n+k} p(i) \quad \left[\limsup_{n\to\infty} \sum_{i=n}^{n+\sigma-1} p(i)\right] > 1 - \frac{1}{4}\left(2 - 3\alpha_0 - \sqrt{4 - 12\alpha_0 + \alpha_0^2}\right), \tag{2.22$'$}$$

then all solutions of (E$'$) *oscillate.*

Remark 2.2. A slight modification in the proof of Theorem 2.1 leads to the following corollary about the advanced difference inequalities:

Corollary 2.2. *Assume that all conditions of Theorem* 2.1 *hold. Then we have:*

(i) *The difference inequality*

$$\nabla x(n) - p(n)x(\mu(n)) \ge 0, \ \ n \ge 1, \qquad [\Delta x(n) - p(n)x(\nu(n)) \ge 0, \ \ n \ge 0],$$

has no eventually positive solutions.

(ii) *The difference inequality*

$$\nabla x(n) - p(n)x(\mu(n)) \le 0, \ \ n \ge 1, \qquad [\Delta x(n) - p(n)x(\nu(n)) \le 0, \ \ n \ge 0]$$

has no eventually negative solutions.

Remark 2.3. Observe the following:

When $\alpha \to 0$, then the conditions (2.21) and (2.22) reduce to

$$A := \limsup_{n\to\infty} \sum_{i=n}^{\mu(n)} p(i) \left[\limsup_{n\to\infty} \sum_{i=n}^{\nu(n)-1} p(i) \right] > 1,$$

that is, to the condition (1.2). However, when $0 < \alpha \le 1/2$, then we have

$$\frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha}\right) > \left(1-\sqrt{1-\alpha}\right)^2,$$

which means that the condition (2.21) improves the condition (1.4).

In the case where $0 < \alpha \le 6-4\sqrt{2}$, because $1-\sqrt{1-\alpha} > \alpha/2$, we see that assumption (2.2) is weaker than assumption (1.5), and, moreover, we can show that

$$\frac{1}{4}\left(2-3\alpha-\sqrt{4-12\alpha+\alpha^2}\right) > \alpha\left(\frac{1}{3\sqrt{1-\alpha}+\alpha-2}-1\right),$$

which means that the condition (2.22) improves the condition (1.6).

Remark 2.4. In the case where the sequence $\{\mu(n)\}$ $[\{\nu(n)\}]$ is not assumed to be nondecreasing, define (cf. [2,3,4,5])

$$\sigma(n) = \max\{\mu(s) : 1 \le s \le n,\ s \in \mathbb{N}\}, \qquad [\rho(n) = \max\{\nu(s) : 1 \le s \le n,\ s \in \mathbb{N}\}].$$

Clearly, the sequence of integers $\{\sigma(n)\}$ $[\{\rho(n)\}]$ is nondecreasing. In this case, Theorem 2.1 can be formulated in the following more general form:

Theorem 2.1$_G$. *Assume that*

$$\liminf_{n\to\infty} \sum_{i=n+1}^{\mu(n)} p(i) \left[\liminf_{n\to\infty} \sum_{i=n+1}^{\nu(n)-1} p(i) \right] = \alpha. \tag{1.3}$$

If $0 < \alpha \le 1/2$, *and*

$$\limsup_{n\to\infty} \sum_{i=n}^{\sigma(n)} p(i) \left[\limsup_{n\to\infty} \sum_{i=n}^{\rho(n)-1} p(i) \right] > 1 - \frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha}\right) \tag{2.21$_G$}$$

then all solutions of (E) *oscillate.*

If $0 < \alpha \le 6-4\sqrt{2}$, $p(n) \ge \alpha/2$ *for all large* n, *and*

$$\limsup_{n\to\infty} \sum_{i=n}^{\sigma(n)} p(i) \left[\limsup_{n\to\infty} \sum_{i=n}^{\rho(n)-1} p(i) \right] > 1 - \frac{1}{4}\left(2-3\alpha-\sqrt{4-12\alpha+\alpha^2}\right), \tag{2.22$_G$}$$

then all solutions of (E) *oscillate.*

3. EXAMPLES

We illustrate the significance of our results by the following examples.

Example 3.1. Consider the difference equation

$$\nabla x(n) - p(n)x(n+1+[n^\alpha])) = 0, \tag{3.1}$$

with

$$p(n) = \begin{cases} \frac{\alpha}{(n+1)\ln(n+1)}, & n \in \mathbb{N}_0 \backslash B \\ \\ d, & n \in B \end{cases},$$

where α is a positive real number with $0 < \alpha \le 1/2$, $[n^\alpha]$denotes the integer part of n^α, d is a positive real number such that

$$1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1-2\alpha}\right) < \alpha + d < 1 - \left(1 - \sqrt{1-\alpha}\right)^2$$

and

$$B = \{\text{terms of the sequence } \{b(n)\}\},$$

$$b(n) = \left[(b(n-1)+1)^{1/\alpha} + 1\right], \quad n \ge 1 \quad \text{and } b(0) = 0$$

where $\left[(b(n-1)+1)^{1/\alpha} + 1\right]$ denotes the integer part of $(b(n-1)+1)^{1/\alpha} + 1$.

Equation (3.1) is of type (E) with $\mu(n) = n + 1 + [n^\alpha]$. Here, $\{p(n)\}$ is a sequence of positive real numbers, and $\{\mu(n)\}$ is a sequence of positive integers such that $\mu(n) \ge n + 1$ for all $n \ge 1$. Moreover, we note that the sequence $\{\mu(n)\}$ is nondecreasing.

We will first show that

$$\lim_{n\to\infty} \sum_{i=n+1}^{n+1+[n^\alpha]} \frac{\alpha}{(i+1)\ln(i+1)} = \alpha. \tag{3.2}$$

Since $\frac{\alpha}{(i+1)\ln(i+1)}$ is nonincreasing, and taking into account the fact that

$$\int_{c-1}^{c} f(x)dx \ge f(c) \ge \int_{c}^{c+1} f(x)dx,$$

where $f(x)$ is a nonincreasing positive function, we have

$$\begin{aligned} \sum_{i=n+1}^{n+1+[n^\alpha]} \frac{\alpha}{(i+1)\ln(i+1)} &\ge \alpha \sum_{i=n+1}^{n+1+[n^\alpha]} \int_i^{i+1} \frac{ds}{(s+1)\ln(s+1)} \\ &= \alpha \int_{n+1}^{n+2+[n^\alpha]} \frac{ds}{(s+1)\ln(s+1)} \\ &= \alpha \ln \frac{\ln(n+3+[n^\alpha])}{\ln(n+2)} \end{aligned}$$

and

$$\begin{aligned}\sum_{i=n+1}^{n+1+[n^\alpha]} \frac{\alpha}{(i+1)\ln(i+1)} &\leq \alpha \sum_{i=n+1}^{n+1+[n^\alpha]} \int_{i-1}^{i} \frac{ds}{(s+1)\ln(s+1)} \\ &= \alpha \int_{n}^{n+1+[n^\alpha]} \frac{ds}{(s+1)\ln(s+1)} \\ &= \alpha \ln \frac{\ln(n+2+[n^\alpha])}{\ln(n+1)}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\lim_{n\to\infty} \left(\alpha \ln \frac{\ln(n+3+[n^\alpha])}{\ln(n+2)}\right) &= \lim_{n\to\infty} \left(\alpha \ln \frac{\ln(n+2+[n^\alpha])}{\ln(n+1)}\right) \\ &= \alpha \cdot 1 = \alpha.\end{aligned}$$

From the above it is obvious that (3.2) holds true.

In particular, since $b(n)+1 \leq b(n)+1+[(b(n))^\alpha]$ it follows from (3.2) that

$$\lim_{n\to\infty} \sum_{i=b(n)+1}^{b(n)+1+[(b(n))^\alpha]} \frac{\alpha}{(i+1)\ln(i+1)} = \alpha. \tag{3.3}$$

Observe (it is a matter of elementary calculations to find) that

$$b(n) < b(n)+1 \leq b(n)+1+[(b(n))^\alpha] < b(n+1) \qquad \text{for large } n. \tag{3.4}$$

Now, in view of (3.4), we get

$$\sum_{i=b(n)+1}^{b(n)+1+[(b(n))^\alpha]} p(i) = \sum_{i=b(n)+1}^{b(n)+1+[(b(n))^\alpha]} \frac{\alpha}{(i+1)\ln(i+1)} \qquad \text{for all large } n$$

and consequently, because of (3.3)

$$\lim_{n\to\infty} \sum_{i=b(n)+1}^{b(n)+1+[(b(n))^\alpha]} p(i) = \alpha. \tag{3.5}$$

Furthermore, since $d \geq \frac{\alpha}{(i+1)\ln(i+1)}$ for all large i, we obtain

$$\sum_{i=n+1}^{n+1+[n^\alpha]} p(i) \geq \sum_{i=n+1}^{n+1+[n^\alpha]} \frac{\alpha}{(i+1)\ln(i+1)} \qquad \text{for all large } n,$$

which, by virtue of (3.2), gives

$$\liminf_{n\to\infty} \sum_{i=n+1}^{n+1+[n^\alpha]} p(i) \geq \alpha. \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$\liminf_{n\to\infty} \sum_{i=n+1}^{n+1+[n^\alpha]} p(i) = \alpha. \tag{3.7}$$

Next, we shall prove that

$$\limsup_{n\to\infty} \sum_{i=n}^{n+1+[n^\alpha]} p(i) = \alpha + d. \tag{3.8}$$

Observe that

$$\sum_{i=b(n)}^{b(n)+1+[(b(n))^\alpha]} p(i) = d + \sum_{i=b(n)+1}^{b(n)+1+[(b(n))^\alpha]} p(i) \qquad \text{for all large } n,$$

and so, because of (3.5),

$$\lim_{n\to\infty} \sum_{i=b(n)}^{b(n)+1+[(b(n))^\alpha]} p(i) = d + \alpha. \tag{3.9}$$

But it is easy to prove that, for each large n, there exists at most one integer n^* so that

$$n + 1 \le b(n^*) \le n + 1 + [n^\alpha].$$

By taking into account this fact, we obtain

$$\begin{aligned} \sum_{i=n}^{n+1+[n^\alpha]} p(i) &\le \sum_{i=n}^{n+1+[n^\alpha]} \frac{\alpha}{(i+1)\ln(i+1)} + d \\ &= \frac{\alpha}{(n+1)\ln(n+1)} + \sum_{i=n+1}^{n+1+[n^\alpha]} \frac{\alpha}{(i+1)\ln(i+1)} + d \end{aligned}$$

for all large n. Thus, by using (3.2), we derive

$$\limsup_{n\to\infty} \sum_{i=n}^{n+1+[n^\alpha]} p(i) \le \alpha + d. \tag{3.10}$$

From (3.9) and (3.10) we conclude that (3.8) is always valid. Thus,

$$1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1-2\alpha}\right) < \limsup_{n\to\infty} \sum_{i=n}^{\mu(n)} p(i) = \alpha + d < 1 - \left(1 - \sqrt{1-\alpha}\right)^2 < 1$$

that is, condition (2.21) of Theorem 2.1 is satisfied and therefore all solutions of (3.1) oscillate. Observe, however, that none of the conditions 1.4 and 1.2 is satisfied.

Example 3.2. Consider the equation

$$\nabla x(n) - p(n)x(n+2) = 0, \qquad n \ge 1 \tag{3.11}$$

where

$$p(3n-2) = p(3n-1) = 3 - 2\sqrt{2}, \;\; p(3n) = \frac{1001}{1000}\left(\sqrt{2} - 1\right), \qquad n \ge 1.$$

Equation (3.11) is of type (E′) with $k = 2$. We have

$$\alpha_0 = \liminf_{n\to\infty} \sum_{i=n+1}^{n+2} p(i) = 2\left(3 - 2\sqrt{2}\right) = 6 - 4\sqrt{2}$$

and

$$\limsup_{n\to\infty}\sum_{i=n}^{n+2} p(i) = 2\left(3-2\sqrt{2}\right) + \frac{1001}{1000}\left(\sqrt{2}-1\right) = 0.757773526.$$

Also,

$$p(n) \geq \alpha_0/2 \qquad \forall n \geq 1.$$

Observe that

$$0.757773526 > 1 - \frac{1}{4}\left(2 - 3\alpha_0 - \sqrt{4 - 12\alpha_0 + \alpha_0^2}\right) \simeq 0.757359312,$$

that is, condition (2.22′) of Corollary 2.1 is satisfied and therefore all solutions of (3.11) oscillate. Observe, however, that

$$0.757773526 < 1 - \frac{1}{2}\left(1 - \alpha_0 - \sqrt{1-2\alpha_0}\right) \simeq 0.951621308,$$

$$0.757773526 < 1 - \left(1 - \sqrt{1-\alpha_0}\right)^2 \simeq 0.964076655$$

$$0.757773526 < 1,$$

and therefore none of the conditions (2.21′), (1.4) and (1.2) is satisfied.

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